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Stability and Stabilization of Aperiodic Sampled-Data Control Systems Using Robust Linear Matrix Inequalities

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Stability and Stabilization of Aperiodic Sampled-Data Control Systems Using Robust Linear Matrix Inequalities*

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Stability analysis of an aperiodic sampled-data control system is discussed for application to network and embedded control. The stability condition is described in a linear matrix inequality to be satisfied for all possible sampling intervals. Although this condition is numerically intractable, a tractable sufficient condition can be constructed with the mean value theorem. Special care is paid on tightness of the sufficient condition for less conservative stability analysis. Asymptotic exactness of the approach is discussed and a technique of adaptive division is presented for computational efficiency. Extension to stabilization is also discussed. Examples show the efficacy of the approach.

Keywords: sampled-data control, robust linear matrix inequality, semidefinite programming, conservatism, asymptotic exactness, adaptive division.

1. Introduction

Sampled-data control is a matured research area and established methodology is available both for analysis and design [3]. However, most of the existing results assume constant sampling interval and cannot be applied to network and embedded control systems, whose sampling interval is uncertain and varying with time.

For analysis and design of such an aperiodic sampled-data control system, several approaches have been proposed. Some of them are based on the continuous-time or hybrid framework [6, 14, 15]. The stability conditions presented there are rather conservative though applicable to general systems. Recent approaches such as [7, 8, 10, 22] provide less conservative stability conditions in the discrete-time framework. Hetel *et al.* [10] gave a stability condition by approximately evaluating the effect of aperiodic sampling with a polynomial. Increase of the degree of the polynomial reduces conservatism of the result. On the other hand, Fujioka [7, 8] and Suh [22] gave a stability condition based on division of the region where the uncertain sampling interval takes a value. Here, increase of the resolution of the division leads us to a less conservative result. Skaf and Boyd [20] used such division to evaluate degradation of the optimal quadratic performance of an aperiodic sampled-data control system.

The approach to be presented in this paper inherits some ideas from Fujioka [7, 8] and Suh [22] but incorporates the following three techniques for reduction of conservatism. First, the stability condition to be used converges to the continuous-time stability condition as the sampling interval goes to zero. Due to this property, less conservative stability analysis is possible without numerical difficulty even when the sampling interval can be small. Second, the effect of aperiodic sampling is modeled as parametric uncertainty rather than matrix uncertainty. Third, the region where the

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sampling interval varies is divided into several subregions. Asymptotic exactness of the approach is discussed and adaptive division is considered for computational efficiency. In this paper, the stability condition is in the form of a linear matrix inequality (LMI, in short) to be satisfied for all possible sampling intervals. In general, a parameter-dependent LMI to be satisfied for all possible parameter values is called a *robust LMI*. Recently, intensive investigation has been made on a robust LMI whose parameter dependence is polynomial or rational ([1, 5, 12, 18, 19] for example). Since our stability condition exponentially depends on the sampling interval, we took a different approach based on the mean value theorem. This is an adaptation of the technique of Chesi–Hung [4].

This paper is organized as follows. In Section 2, our problem is provided. In Section 3, the stability condition is presented in a robust LMI and its tractable sufficient condition is constructed. Section 4 introduces a region-dividing technique for less conservative stability analysis. Asymptotic exactness as well as adaptive division is discussed there. Section 5 provides extensions of the approach. After illustrating examples are presented in Section 6, the paper is concluded in Section 7.

Notation is standard. The symbols O and I denote the zero matrix and the identity matrix of appropriate size. For a matrix A, the symbol $\overline{\sigma}(A)$ expresses its maximum singular value. For a symmetric matrix A, the symbols $\overline{\lambda}(A)$ and $\underline{\lambda}(A)$ stand for its maximum and minimum eigenvalues, respectively. For symmetric matrices A and B, the inequalities $A \succ B$ and $A \succeq B$ mean $\underline{\lambda}(A-B) > 0$ and $\underline{\lambda}(A-B) \geq 0$, respectively.

2. Problem

We consider a continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with the n-dimensional state $x(t)$ and its stabilization by state-feedback control with a constant gain F. The state is measured only at discrete time instants $0 = t_0 < t_1 < t_2 < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$ and the input is a piecewise signal determined as

$$u(t) = Fx(t_k) \qquad (t_k \leq t < t_{k+1})$$

for each $k = 0, 1, 2, \ldots$. We refer to this control system by S henceforth.

The control system S is different from a conventional sampled-data control system in that the sampling interval $t_{k+1} - t_k$ is not necessarily constant but may vary with k. We assume availability of its bounds $\underline{h}$ and $\overline{h}$ such that

$$\underline{h} \leq t_{k+1} - t_k \leq \overline{h} \qquad (k = 0, 1, 2, \ldots).$$

Our problem is to verify stability of the control system S. We present an approach using a robust LMI in the following sections.

3. Proposed Stability Condition

3.1. Formulation into a robust LMI

We use the criterion of quadratic stability for stability analysis of the control system S. Note first in S that the states at two adjacent sampling instants are related by

$$x(t_{k+1}) = \Phi(t_{k+1} - t_k)x(t_k) \qquad (k = 0, 1, 2, \ldots)$$

with

$$\Phi(h) = \mathrm{e}^{Ah} + \int_0^h \mathrm{e}^{At}\mathrm{d}t\, BF = I + \int_0^h \mathrm{e}^{At}\mathrm{d}t(A + BF).$$

Hence, the exponential stability of S follows if there exists a symmetric matrix Q such that

$$Q \succ O, \qquad Q - \Phi(h)Q\Phi(h)^{\mathrm{T}} \succ O \quad (\underline{h} \le h \le \overline{h}).$$

This stability condition has two drawbacks to be utilized for stability analysis. First, the condition is in the form of a robust LMI and we need to find Q that satisfies the inequality for infinitely many values of h. Second, the matrix $\Phi(h)$ is close to identity when h is small, which makes the condition difficult to handle numerically. In the following, we consider the second drawback first and then the first.

In order to avoid the numerical drawback of $\Phi(h)$, we use the following stability condition equivalent to the previous one.

Proposition 1. *The control system S is exponentially stable if there exists a symmetric matrix Q such that*

$$\begin{pmatrix} -\Psi(h)Q - Q\Psi(h)^{\mathrm{T}} & \sqrt{h}\Psi(h)Q \\ \sqrt{h}Q\Psi(h)^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad (\underline{h} \le h \le \overline{h}), \tag{1}$$

where

$$\Psi(h) = \frac{1}{h}(\Phi(h) - I) = \frac{1}{h}\int_0^h \mathrm{e}^{At}\,\mathrm{d}t(A + BF). \tag{2}$$

Proof. Decompose (1) into $Q \succ O$ and its Schur complement. Substitution of the definition of $\Psi(h)$ shows the equivalence to the quadratic stability condition given above. The proposition hence follows. □

In the limit of $h \to 0$, the matrix $\Psi(h)$ converges to $A + BF$, which is the system matrix of the continuous-time control system where a continuous-time state-feedback control $u(t) = Fx(t)$ is applied to $\dot{x}(t) = Ax(t) + Bu(t)$. In the same limit, the stability condition (1) assures stability of this continuous-time control system. Hence, this condition has no numerical drawback discussed above even for small h.

The condition (1) is again a robust LMI to be satisfied for infinitely many values of h. Since it is difficult to find Q satisfying the robust LMI, we consider its sufficient condition expressed by finitely many LMIs. We can solve those LMIs using a standard interior-point method. Once a solution Q is found, the same Q serves as a solution of the original robust LMI. In this approach, it is critical to use a tight sufficient condition. To this aim, we assume availability of the real Jordan canonical form of A and consider a sufficient condition based on it.

We consider the case of $\underline{h} > 0$ in Section 3.2 and the case of $\underline{h} = 0$ in Section 3.3. The equality $\underline{h} = 0$ means that a positive lower bound is not available for the sampling interval. This latter case can be considered because our stability condition (1) is usable with small h.

3.2. The case of $\underline{h} > 0$

The function $\Psi(h)$ in the condition (1) has exponential dependence on the uncertain parameter h. Since exponential parameter dependence is difficult to handle, we replace $\Psi(h)$ with a new function dependent on not only h but also additional uncertain parameters in a multi-affine way. This multi-affinity leads us to the desired sufficient condition expressed by finitely many LMIs.

Definition of the new function uses the real Jordan canonical form of A [11, Section 3.4]. For the real $n \times n$ matrix A, there exists a real nonsingular matrix T such that

$$A = T \begin{pmatrix} A^{(1)} & & & \\ & A^{(2)} & & \\ & & \ddots & \\ & & & A^{(r)} \end{pmatrix} T^{-1}$$

(the elements not presented are equal to zero). Here, each $A^{(i)}$ is a matrix having one of the following two forms:

$$\begin{pmatrix} \lambda^{(i)} & 1 & & \\ & \lambda^{(i)} & \ddots & \\ & & \ddots & 1 \\ & & & \lambda^{(i)} \end{pmatrix}, \qquad \begin{pmatrix} P^{(i)} & I_2 & & \\ & P^{(i)} & \ddots & \\ & & \ddots & I_2 \\ & & & P^{(i)} \end{pmatrix}, \tag{3}$$

where $\lambda^{(i)}$ is some real number, $P^{(i)}$ is a 2×2 real matrix of the form $P^{(i)} = \begin{pmatrix} p^{(i)} & q^{(i)} \\ -q^{(i)} & p^{(i)} \end{pmatrix}$, and I_2 is the 2×2 identity matrix.

The function for replacement of $\Psi(h)$ is now defined as

$$\Psi_{\widehat{h}}(h, \theta) = \frac{1}{h} \Big[\int_0^{\widehat{h}} \mathrm{e}^{At} \, \mathrm{d}t + (h - \widehat{h}) T E(\theta) T^{-1} \Big] (A + BF) \tag{4}$$

with $\widehat{h}$ being any fixed number such that $\underline{h} \leq \widehat{h} \leq \overline{h}$. Here, $E(\theta)$ is an affine function of a newly introduced n-dimensional uncertain parameter θ and has the block-diagonal form

$$E(\theta) = \begin{pmatrix} E^{(1)}(\theta) & & & \\ & E^{(2)}(\theta) & & \\ & & \ddots & \\ & & & E^{(r)}(\theta) \end{pmatrix}$$

consistent with the real Jordan canonical form of A. More precisely, when $A^{(i)}$ is an $n^{(i)} \times n^{(i)}$ matrix of the left form of (3), the corresponding $E^{(i)}(\theta)$ is

$$\begin{pmatrix} \theta_1^{(i)} & \theta_2^{(i)} & \cdots & \theta_{n^{(i)}}^{(i)} \\ & \theta_1^{(i)} & \ddots & \vdots \\ & & \ddots & \theta_2^{(i)} \\ & & & \theta_1^{(i)} \end{pmatrix}$$

with $n^{(i)}$ uncertain parameters $\theta_j^{(i)}$ $(j = 1, 2, \ldots, n^{(i)})$, which are elements of θ. When $A^{(i)}$ is a $2m^{(i)} \times 2m^{(i)}$ matrix of the right form of (3), the corresponding $E^{(i)}(\theta)$ is

$$\begin{pmatrix} \Xi_1^{(i)} & \Xi_2^{(i)} & \cdots & \Xi_{m^{(i)}}^{(i)} \\ & \Xi_1^{(i)} & \ddots & \vdots \\ & & \ddots & \Xi_2^{(i)} \\ & & & \Xi_1^{(i)} \end{pmatrix},$$

where $\Xi_j^{(i)} = \begin{pmatrix} \xi_j^{(i)} & \eta_j^{(i)} \\ -\eta_j^{(i)} & \xi_j^{(i)} \end{pmatrix}$ and $\xi_j^{(i)}$, $\eta_j^{(i)}$ $(j = 1, 2, \ldots, m^{(i)})$ are $2m^{(i)}$ uncertain parameters contained again in θ. Hence, θ is a vector consisting of the parameters $\theta_j^{(i)}$, $\xi_j^{(i)}$, and $\eta_j^{(i)}$, whose dimension sums up to n. The domain of the parameters $\underline{\theta}_j^{(i)} \leq \theta_j^{(i)} \leq \overline{\theta}_j^{(i)}$, $\underline{\xi}_j^{(i)} \leq \xi_j^{(i)} \leq \overline{\xi}_j^{(i)}$, and $\underline{\eta}_j^{(i)} \leq \eta_j^{(i)} \leq \overline{\eta}_j^{(i)}$ are defined as

$$\begin{aligned} \underline{\theta}_j^{(i)} &= \min_{\underline{h} \leq h \leq \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{\lambda^{(i)} h}, & \overline{\theta}_j^{(i)} &= \max_{\underline{h} \leq h \leq \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{\lambda^{(i)} h}, \\ \underline{\xi}_j^{(i)} &= \min_{\underline{h} \leq h \leq \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \cos q^{(i)} h, & \overline{\xi}_j^{(i)} &= \max_{\underline{h} \leq h \leq \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \cos q^{(i)} h, \\ \underline{\eta}_j^{(i)} &= \min_{\underline{h} \leq h \leq \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \sin q^{(i)} h, & \overline{\eta}_j^{(i)} &= \max_{\underline{h} \leq h \leq \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \sin q^{(i)} h \end{aligned}$$

with the convention $0! = 1$. The domain of θ is hence a box-shaped set, which will be denoted by Θ.

Example 2. When A is 2×2, its real Jordan canonical form is one of the three forms:

$$T \begin{pmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{pmatrix} T^{-1}, \qquad T \begin{pmatrix} \lambda^{(1)} & 1 \\ 0 & \lambda^{(1)} \end{pmatrix} T^{-1}, \qquad T \begin{pmatrix} p^{(1)} & q^{(1)} \\ -q^{(1)} & p^{(1)} \end{pmatrix} T^{-1}. \tag{5}$$

Correspondingly, the function $E(\theta)$ in (4) is defined as

$$\begin{pmatrix} \theta_1^{(1)} & 0 \\ 0 & \theta_1^{(2)} \end{pmatrix}, \qquad \begin{pmatrix} \theta_1^{(1)} & \theta_2^{(1)} \\ 0 & \theta_1^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} \xi_1^{(1)} & \eta_1^{(1)} \\ -\eta_1^{(1)} & \xi_1^{(1)} \end{pmatrix}.$$

In each case, the number of the new parameters is two and thus θ is a 2-dimensional vector. □

Properties of $\Psi_{\widehat{h}}(h, \theta)$ and Θ are summarized in the next lemma.

Lemma 3. *Let $\widehat{h}$ be any number such that $\underline{h} \leq \widehat{h} \leq \overline{h}$. The function $\Psi_{\widehat{h}}(h, \theta)$ and the domain Θ defined above has the following properties: (i) for any $\underline{h} \leq h \leq \overline{h}$, there exists $\theta \in \Theta$ such that $\Psi(h) = \Psi_{\widehat{h}}(h, \theta)$; (ii) $h\Psi_{\widehat{h}}(h, \theta)$ is multi-affine in h and θ; (iii) $\Psi_{\widehat{h}}(h, \theta)$ is independent of θ at $h = \widehat{h}$.*

Proof. We prove the property (i) in the special case that A is a 2×2 matrix of the leftmost form in (5). The proof in the general case is similar.

For notational convenience, we write

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}.$$

From the definition (2) of $\Psi(h)$ it follows that

$$
\begin{aligned}
\Psi(h) &= \frac{1}{h}\int_0^h \mathrm{e}^{At}\,\mathrm{d}t(A+BF) \\
&= \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + T\begin{pmatrix} \int_{\widehat{h}}^h \mathrm{e}^{\lambda_1 t}\,\mathrm{d}t & 0 \\ 0 & \int_{\widehat{h}}^h \mathrm{e}^{\lambda_2 t}\,\mathrm{d}t \end{pmatrix} T^{-1}\Big](A+BF) \\
&= \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + T\begin{pmatrix} (h-\widehat{h})\mathrm{e}^{\lambda_1 h_1} & 0 \\ 0 & (h-\widehat{h})\mathrm{e}^{\lambda_2 h_2} \end{pmatrix} T^{-1}\Big](A+BF). \qquad (6)
\end{aligned}
$$

The last equality is implied by the mean value theorem with h_1 and h_2 being some numbers between $\widehat{h}$ and h. Since $h_1, h_2 \in [\underline{h}, \overline{h}]$, we have

$$
\min_{\underline{h}\le h\le\overline{h}} \mathrm{e}^{\lambda_i h} \le \mathrm{e}^{\lambda_i h_i} \le \max_{\underline{h}\le h\le\overline{h}} \mathrm{e}^{\lambda_i h} \qquad (i=1,2).
$$

The property (i) hence follows.

The properties (ii) and (iii) are obvious from the definition of $\Psi_{\widehat{h}}(h,\theta)$. $\square$

We replace $\Psi(h)$ by $\Psi_{\widehat{h}}(h,\theta)$ in the robust LMI (1). We will see below that the resulting LMI has the vertex property due to the multi-affinity of $h\Psi_{\widehat{h}}(h,\theta)$ (Lemma 3 (ii)). That is, the LMI holds for all $\underline{h} \le h \le \overline{h}$ and $\theta \in \Theta$ if and only if the same LMI holds only at the vertices. We hence obtain a sufficient condition consisting of a finite number of LMIs, which makes the stability analysis tractable. Let $\mathrm{ver}\,\Theta$ denote the set of the vertices of the box-shaped set Θ.

Theorem 4. *Suppose $\underline{h} > 0$. Let $\widehat{h}$ be any number such that $\underline{h} \le \widehat{h} \le \overline{h}$. The control system S is exponentially stable if there exists a symmetric matrix Q such that*

$$
\begin{pmatrix} -\Psi_{\widehat{h}}(h,\theta)Q - Q\Psi_{\widehat{h}}(h,\theta)^{\mathrm{T}} & \sqrt{h}\Psi_{\widehat{h}}(h,\theta)Q \\ \sqrt{h}Q\Psi_{\widehat{h}}(h,\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad (h = \underline{h}, \overline{h};\ \theta \in \mathrm{ver}\,\Theta). \qquad (7)
$$

Proof. Multiplication of $\sqrt{h}$ to the first row and to the first column of the matrix in (7) gives an equivalent inequality

$$
\begin{pmatrix} -h\Psi_{\widehat{h}}(h,\theta)Q - hQ\Psi_{\widehat{h}}(h,\theta)^{\mathrm{T}} & h\Psi_{\widehat{h}}(h,\theta)Q \\ hQ\Psi_{\widehat{h}}(h,\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O.
$$

Thanks to the multi-affinity of $h\Psi_{\widehat{h}}(h,\theta)$, this inequality is multiconvex in h and θ. Hence, the inequality holds for all $\underline{h} \le h \le \overline{h}$ and $\theta \in \Theta$ if and only if the same inequality holds only at the vertices $h = \underline{h}, \overline{h}$ and $\theta \in \mathrm{ver}\,\Theta$. When the inequality holds for all $\underline{h} \le h \le \overline{h}$ and $\theta \in \Theta$, the desired exponential stability follows by Lemma 3 (i) together with Proposition 1. $\square$

Some remarks are to follow.

Remark 5. The choice of $\widehat{h}$ is up to the user. In particular, the choice $\widehat{h} = \underline{h}$ or $\widehat{h} = \overline{h}$ is computationally attractive. Indeed, by Lemma 3 (iii), the inequality (7) is independent of θ either at $h = \underline{h}$ or $h = \overline{h}$ with this choice, which decreases the number of LMIs. When $\underline{h}$ is close to zero, the choice $\widehat{h} = \underline{h}$ is preferable because the choice $\widehat{h} = \overline{h}$ makes $(h-\widehat{h})/h$ large at $h = \underline{h}$, which results in large effect of θ in $\Psi_{\widehat{h}}(\underline{h},\theta)$. $\square$

Remark 6. The coordinates of $\theta \in \operatorname{ver} \Theta$ consist of $\underline{\theta}_j^{(i)}$, $\overline{\theta}_j^{(i)}$, $\underline{\xi}_j^{(i)}$, and others, whose computation needs minimization and maximization of some functions. When it is difficult, we can use a box-shaped set larger than Θ by computing lower and upper bounds of the functions. This simplification however introduces additional conservatism into the stability analysis. $\square$

Remark 7. The number of LMIs in the stability condition (7) is $2^n + 1$ with $\widehat{h} = \underline{h}$ or $\widehat{h} = \overline{h}$. When this number is large, it may be better to use its further sufficient condition consisting of $2n+2$ LMIs. Such a sufficient condition is given by Ben-Tal–Nemirovski [2]. The associated conservatism has been evaluated in their work. $\square$

3.3. The case of $\underline{h} = 0$

We next consider the case of $\underline{h} = 0$. Although we could take the limit of the condition (7), we follow a different path with higher-order expansion of $\Psi(h)$.

The function for the replacement of $\Psi(h)$ is in this case

$$\Psi_0(\theta) = [I + TE(\theta)T^{-1}A](A + BF)$$

with the same $E(\theta)$ as the previous subsection. The parameter θ is again an n-dimensional vector consisting of $\theta_j^{(i)}$, $\xi_j^{(i)}$, $\eta_j^{(i)}$. The lower and upper bounds of these parameters are

$$\begin{aligned}
\underline{\theta}_j^{(i)} &= 0, & \overline{\theta}_j^{(i)} &= \overline{h} \max_{0 \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{\lambda^{(i)} h}, \\
\underline{\xi}_j^{(i)} &= \overline{h} \min_{0 \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \cos q^{(i)} h, & \overline{\xi}_j^{(i)} &= \overline{h} \max_{0 \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \cos q^{(i)} h, \\
\underline{\eta}_j^{(i)} &= \overline{h} \min_{0 \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \sin q^{(i)} h, & \overline{\eta}_j^{(i)} &= \overline{h} \max_{0 \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \sin q^{(i)} h.
\end{aligned}$$

The domain of θ is hence a box-shaped set, denoted by Θ_0. Note that $\overline{\theta}_j^{(i)}$, $\overline{\xi}_j^{(i)}$, $\overline{\eta}_j^{(i)}$ in this section are different from their counterparts in the previous subsection by factor of $\overline{h}$. Hence, the parameter set Θ_0 tends to be small when $\overline{h}$ is close to zero. In such a case, the following stability condition is expected to give a less conservative result.

Lemma 8. *The function $\Psi_0(\theta)$ and the domain Θ_0 has the following properties: (i) for any $0 \le h \le \overline{h}$, there exists $\theta \in \Theta_0$ such that $\Psi(h) = \Psi_0(\theta)$; (ii) $\Psi_0(\theta)$ is affine in θ.*

Proof. We again consider the special case that

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$$

to prove the property (i). Proof in the general case is similar.

In the definition (2) of $\Psi(h)$, the integral satisfies

$$\int_0^h \mathrm{e}^{At} \,\mathrm{d}t = hI + \int_0^h (\mathrm{e}^{At} - I) \,\mathrm{d}t = hI + \int_0^h \int_0^t \mathrm{e}^{Au} \,\mathrm{d}u \,\mathrm{d}t \, A$$

$$= hI + T \begin{pmatrix} \int_0^h \int_0^t \mathrm{e}^{\lambda_1 u}\,\mathrm{d}u\,\mathrm{d}t & 0 \\ 0 & \int_0^h \int_0^t \mathrm{e}^{\lambda_2 u}\,\mathrm{d}u\,\mathrm{d}t \end{pmatrix} T^{-1}A.$$

Application of the mean value theorem twice shows that it is equal to

$$hI + T \begin{pmatrix} hh_1\mathrm{e}^{\lambda_1 h_1'} & 0 \\ 0 & hh_2\mathrm{e}^{\lambda_2 h_2'} \end{pmatrix} T^{-1}A$$

for some $0 \le h_1' \le h_1 \le h$ and $0 \le h_2' \le h_2 \le h$. Here,

$$0 \le h_i\mathrm{e}^{\lambda_i h_i'} \le \overline{h} \max_{0\le h\le\overline{h}} \mathrm{e}^{\lambda_i h} \qquad (i = 1, 2).$$

Using these results in the definition of $\Psi(h)$, we have the property (i).

The property (ii) is obvious from the definition. □

We replace $\Psi(h)$ by $\Psi_0(\theta)$ in the condition (1). The affinity of $\Psi_0(\theta)$ implies the vertex property.

Theorem 9. *Suppose $\underline{h} = 0$. The control system S is exponentially stable if there exists a symmetric matrix Q such that*

$$\begin{pmatrix} -\Psi_0(\theta)Q - Q\Psi_0(\theta)^{\mathrm{T}} & \sqrt{h}\Psi_0(\theta)Q \\ \sqrt{h}Q\Psi_0(\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad (h = 0, \overline{h};\ \theta \in \mathrm{ver}\,\Theta_0). \tag{8}$$

Proof. Affinity of $\Psi_0(\theta)$ implies multiconvexity of the inequality (8) in θ and $\sqrt{h}$, which means that this inequality holds for all $0 \le h \le \overline{h}$ and $\theta \in \Theta_0$ if and only if it holds only at the vertices. Hence, under the condition of the theorem, exponential stability of S follows from Lemma 8 (i) and Proposition 1. □

4. A Region-Dividing Technique

4.1. Division of the region of the sampling interval

The proposed condition can be considerably conservative when the region of the sampling interval, $[\underline{h}, \overline{h}]$, is large because the condition is based on the mean value theorem there. In such a case, we can reduce conservatism of the condition by division of the region of the sampling interval.

In the sequel, we mean by a *division* a set of subregions $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1, 2, \ldots, J\}$ such that

$$\underline{h} = \underline{h}^{[1]} < \overline{h}^{[1]} = \underline{h}^{[2]} < \overline{h}^{[2]} = \underline{h}^{[3]} < \cdots < \overline{h}^{[J]} = \overline{h}.$$

For a given division Δ, the following stability condition is considered.

Theorem 10. *Let $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1, 2, \ldots, J\}$ be a division of $[\underline{h}, \overline{h}]$ and $\widehat{h}^{[j]}$ be any number in the subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$ for each j. For each j with $\underline{h}^{[j]} > 0$, let $\Theta^{[j]}$ be Θ with their $\underline{h}$, $\overline{h}$, $\widehat{h}$ replaced by $\underline{h}^{[j]}$, $\overline{h}^{[j]}$, $\widehat{h}^{[j]}$, respectively. For j with $\underline{h}^{[j]} = 0$, if any, let $\Theta_0^{[j]}$ be Θ_0 with the same replacement. Then, the control system S is exponentially stable if there exists a symmetric matrix Q such that*

$$\begin{pmatrix} -\Psi_{\widehat{h}^{[j]}}(h,\theta)Q - Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} & \sqrt{h}\Psi_{\widehat{h}^{[j]}}(h,\theta)Q \\ \sqrt{h}Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad (h = \underline{h}^{[j]}, \overline{h}^{[j]};\ \theta \in \mathrm{ver}\,\Theta^{[j]}) \tag{9}$$

for each j with $\underline{h}^{[j]} > 0$ and

$$\begin{pmatrix} -\Psi_0(\theta)Q - Q\Psi_0(\theta)^{\mathrm{T}} & \sqrt{h}\Psi_0(\theta)Q \\ \sqrt{h}Q\Psi_0(\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad (h = 0, \overline{h}^{[j]};\ \theta \in \mathrm{ver}\,\Theta_0^{[j]}) \tag{10}$$

for j with $\underline{h}^{[j]} = 0$, if any.

Proof. For each subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$, the discussion in the proof of Theorem 4 or 9 is applicable depending on whether $\underline{h}^{[j]} > 0$ or $\underline{h}^{[j]} = 0$. Consequently, the inequality (1) holds for any h in $\cup_{j=1,2,\ldots,J}[\underline{h}^{[j]}, \overline{h}^{[j]}] = [\underline{h}, \overline{h}]$. Proposition 1 then implies the exponential stability of S. □

4.2. Asymptotic exactness

The stability condition of Theorem 10 is asymptotically exact when the choice $\widehat{h}^{[j]} = \underline{h}^{[j]}$ is adopted for all the subregions. Namely, if there exists Q satisfying the original intractable stability condition (1), the same Q satisfies the condition of Theorem 10 for a sufficiently fine division Δ. Hence, conservatism of our tractable stability condition can be reduced to any degree at the cost of increased computational complexity. For more precise statement, we call $|\overline{h}^{[j]} - \underline{h}^{[j]}|$ the *radius* of the subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$ and $\max_{j=1,2,\ldots,J} |\overline{h}^{[j]} - \underline{h}^{[j]}|$ the *maximum radius* of the division $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1, 2, \ldots, J\}$, which is denoted by $\overline{\mathrm{rad}}\,\Delta$.

Theorem 11. *Suppose that there exists Q satisfying the original stability condition* (1)*. Then, the same Q satisfies the condition of Theorem 10 for a division Δ having sufficiently small $\overline{\mathrm{rad}}\,\Delta$ when the choice $\widehat{h}^{[j]} = \underline{h}^{[j]}$ is adopted for any j.*

Proof. See Appendix A. □

4.3. Formulation into an SDP problem

To prepare for the next subsection, we show here that our stability condition can be restated equivalently in a semidefinite programming (SDP, in short) problem. Here, an SDP problem is an optimization problem having LMIs as its constraints. This fact is important also because it enables us to test our stability condition with the softwares for an SDP problem.

Theorem 12. *Let $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1, 2, \ldots, J\}$ be a division of $[\underline{h}, \overline{h}]$. The control system S is exponentially stable if the following SDP problem is feasible with a positive x:*

$$\begin{aligned} &\text{maximize} \quad x \\ &\text{subject to} \quad Q \succeq I, \\ &\qquad -\Psi_{\widehat{h}^{[j]}}(h,\theta)Q - Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} - h\Psi_{\widehat{h}^{[j]}}(h,\theta)Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} \succeq xI \\ &\qquad\quad (h = \underline{h}^{[j]}, \overline{h}^{[j]};\ \theta \in \mathrm{ver}\,\Theta^{[j]}) \\ &\qquad \text{for each } j \text{ with } \underline{h}^{[j]} > 0, \text{ and} \\ &\qquad -\Psi_0(\theta)Q - Q\Psi_0(\theta)^{\mathrm{T}} - h\Psi_0(\theta)Q\Psi_0(\theta)^{\mathrm{T}} \succeq xI \\ &\qquad\quad (h = 0, \overline{h}^{[j]};\ \theta \in \mathrm{ver}\,\Theta_0^{[j]}) \\ &\qquad \text{for } j \text{ with } \underline{h}^{[j]} = 0, \text{ if any.} \end{aligned}$$

Proof. Suppose that the SDP problem above is feasible with a positive x. Then, all the left-hand side matrices of the inequalities are positive definite. This implies the existence of Q that satisfies the condition of Theorem 10. Hence, the exponential stability of S follows. $\square$

Some remarks are necessary on the SDP problem above. First, its LMI constraints have been reduced in size from those in Theorem 10 by considering the Schur complements. This is beneficial for computational efficiency. Second, when the SDP problem is feasible for a positive x, its maximum value is unbounded because its LMI constraints are linear in x and Q. Finally, no conservatism is introduced with the restatement as an SDP problem. To see this, assume the existence of Q satisfying the condition of Theorem 10. Then, decompose each inequality (9) or (10) into $Q \succ O$ and its Schur complement. Since $Q \succ O$, there exists a positive number c such that $cQ \succeq I$. Then, with this cQ and some positive x, the SDP problem in Theorem 12 is feasible.

4.4. Adaptive division

While fine division gives a less conservative stability condition, it increases the number of LMIs and then computational cost. It is hence desirable that fine division is made only in an important subregion of h. Although such an important subregion is difficult to find *a priori*, the following technique of *adaptive division* is often effective. The corresponding technique is used in [16, 17].

Suppose that we construct the SDP problem of Theorem 12 for some division and obtain the nonpositive maximum value. Since the stability of the control system S is not assured in this case, we are to refine the division. We here notice an *active constraint*, which is an LMI constraint such that the discrepancy between its two sides has a zero eigenvalue with the obtained maximum solution. Since an active constraint prevents the maximum value from being improved, we may be able to make improvement by subdividing a subregion having an active constraint. Based on this idea, we have the following algorithm for adaptive division, which is expected to produce an efficient division, that is, a division that gives a less conservative result with small amount of computation. We here mean by an *active subregion* a subregion having an active constraint. When the maximum value is not attained, an active constraint or an active subregion is not defined.

Algorithm 13.

0. Prepare a coarse division.
1. Solve the SDP problem of Theorem 12 corresponding to the current division.
2. Stop if the problem is feasible with a positive x.
3. If the maximum value is attained, find and subdivide an active subregion. Otherwise, find and subdivide a subregion of the maximum radius.
4. Go back to Step 1 unless the number of subregions exceeds the prescribed number. $\square$

Algorithm 13 appears contradictory with Theorem 11 because the produced non-uniform division is not efficient for reduction of the maximum radius of a division. This contradiction is resolved by the theorem below, which says that reduction of the *maximum active radius* is no worse than reduction of the maximum radius. Here, the maximum active radius of a division $\varDelta$, denoted by $\overline{\text{a-rad}}\,\varDelta$, means the maximum radius over all active subregions in $\varDelta$ when the maximum value is attained in the SDP problem for $\varDelta$. Note that the maximum active radius depends on the maximum solution. When the maximum solution is not unique, the maximum active radius is defined as the minimum among the possible values. It is obvious that $\overline{\text{a-rad}}\,\varDelta \le \overline{\text{rad}}\,\varDelta$. The theorem below says that, when some division $\varDelta$ has the SDP problem that attains the nonpositive maximum value, there exists a finer division $\widetilde{\varDelta}$ that satisfies $\overline{\text{rad}}\,\widetilde{\varDelta} = \overline{\text{a-rad}}\,\varDelta$ but is not superior to $\varDelta$ in the sense that the maximum value of the corresponding SDP problem is not larger than that for $\varDelta$. Hence, nothing is lost with reduction of the maximum active radius, which is what Algorithm 13 aims at.

Theorem 14. *Suppose that the SDP problem of Theorem 12 attains the nonpositive maximum value for a division $\varDelta$. Then, there exists a division $\widetilde{\varDelta}$ such that the maximum radius of $\widetilde{\varDelta}$ is equal to the maximum active radius of $\varDelta$ and the SDP problem for this $\widetilde{\varDelta}$ has the maximum value not larger than that for $\varDelta$.*

Proof. The idea of the proof is essentially the same as Theorem 8 of [16].

Consider the SDP problem for $\varDelta$ and let (x^*, Q^*) be its maximum solution for which the maximum active radius $\overline{\text{a-rad}}\,\varDelta$ is defined. We subdivide each inactive subregion, if necessary, so that each of the created subregion has the radius smaller than or equal to $\overline{\text{a-rad}}\,\varDelta$. The resulting new division $\widetilde{\varDelta}$ satisfies $\overline{\text{rad}}\,\widetilde{\varDelta} = \overline{\text{a-rad}}\,\varDelta$.

The SDP problem for $\widetilde{\varDelta}$ contains the LMI constraints corresponding to the newly created subregions. If (x^*, Q^*) satisfies these constraints, the maximum value for $\widetilde{\varDelta}$ is equal to that for $\varDelta$. Otherwise, it is equal or smaller. In any case, the conclusion of the theorem follows. $\square$

5. Extensions

We extend our approach to stability analysis with matrix uncertainty as well as design of a state-feedback gain.

5.1. Approach with matrix uncertainty

We have assumed so far availability of the real Jordan canonical form of the system matrix A in order to reduce the problem to a robust LMI with parametric uncertainty. This assumption may not be practical when the system dimension n is large. For such a case, we will consider below an approach with matrix uncertainty. This approach does not require the real Jordan canonical form though often more conservative than the previous approach.

For simplification of the description, we present the result only in the case that $[\underline{h}, \overline{h}]$ is not divided. It is, however, straightforward to use the region-dividing technique for reduction of conservatism. We begin with the case of $\underline{h} > 0$. We can show the following lemma, which is a counterpart of Lemma 3.

Lemma 15. *Let $\widehat{h}$ be any number such that $\underline{h} \le \widehat{h} \le \overline{h}$. For any $\underline{h} \le h \le \overline{h}$, there exists $\varOmega$ such that $\varPsi(h) = \varPsi^{\mathrm{m}}_{\widehat{h}}(h, \varOmega)$ and $\overline{\sigma}(\varOmega) \le \omega$, where*

$$\varPsi^{\mathrm{m}}_{\widehat{h}}(h, \varOmega) = \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + (h - \widehat{h})\varOmega \mathrm{e}^{A\widehat{h}}\Big](A + BF)$$

and

$$\omega = \max\Big\{ \max_{\underline{h}\le h\le \widehat{h}} \exp\Big[\underline{\lambda}\big(\frac{A+A^{\mathrm{T}}}{2}\big)(h-\widehat{h})\Big],\ \max_{\widehat{h}\le h\le \overline{h}} \exp\Big[\overline{\lambda}\big(\frac{A+A^{\mathrm{T}}}{2}\big)(h-\widehat{h})\Big]\Big\}. \tag{11}$$

The symbols $\underline{\lambda}$ and $\overline{\lambda}$ stand for the minimum and the maximum eigenvalues, respectively.

Proof. See Appendix B. □

We use in the proof a specific bound on the norm of a matrix exponential function, which has been used in [7, 8]. Similarly to those papers, it is possible to use different bounds instead.

We replace $\Psi(h)$ with $\Psi^{\mathrm{m}}_{\widehat{h}}(h,\Omega)$ in the condition (1) to obtain the following sufficient condition.

Theorem 16. *Suppose $\underline{h} > 0$. Let $\widehat{h}$ be any number such that $\underline{h} \le \widehat{h} \le \overline{h}$. Define ω as (11). The control system S is exponentially stable if there exist a symmetric matrix Q and a nonnegative number s_h, dependent on h, such that*

$$\begin{pmatrix} -\frac{1}{h}E_1Q - \frac{1}{h}QE_1^{\mathrm{T}} - s_hI & \frac{\sqrt{h}}{h}E_1Q & -\frac{\omega(h-\widehat{h})}{h}QE_2^{\mathrm{T}} \\ \frac{\sqrt{h}}{h}QE_1^{\mathrm{T}} & Q & \frac{\omega\sqrt{h}(h-\widehat{h})}{h}QE_2^{\mathrm{T}} \\ -\frac{\omega(h-\widehat{h})}{h}E_2Q & \frac{\omega\sqrt{h}(h-\widehat{h})}{h}E_2Q & s_hI \end{pmatrix} \succ O \qquad (h=\underline{h},\overline{h}),$$

where

$$E_1 = \int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t(A+BF), \qquad E_2 = \mathrm{e}^{A\widehat{h}}(A+BF).$$

Proof. It is known in general [23] that, for real matrices M_1, M_2, and M_3 of appropriate size, the inequality $M_1 + M_2\Omega M_3 + M_3^{\mathrm{T}}\Omega^{\mathrm{T}}M_2^{\mathrm{T}} \succ O$ holds for any matrix Ω such that $\overline{\sigma}(\Omega) \le \omega$ if and only if there exists a nonnegative number s such that

$$\begin{pmatrix} M_1 - sM_2M_2^{\mathrm{T}} & \omega M_3^{\mathrm{T}} \\ \omega M_3 & sI \end{pmatrix} \succ O.$$

We use this result with

$$M_1 = \begin{pmatrix} -\frac{1}{h}E_1Q - \frac{1}{h}QE_1^{\mathrm{T}} & \frac{\sqrt{h}}{h}E_1Q \\ \frac{\sqrt{h}}{h}QE_1^{\mathrm{T}} & Q \end{pmatrix}, \qquad M_2 = \begin{pmatrix} I \\ O \end{pmatrix}, \qquad M_3 = \begin{pmatrix} -\frac{(h-\widehat{h})}{h}E_2Q & \frac{\sqrt{h}(h-\widehat{h})}{h}E_2Q \end{pmatrix}.$$

Then, the assumption of the theorem guarantees

$$\begin{pmatrix} -\Psi^{\mathrm{m}}_{\widehat{h}}(h,\Omega)Q - Q\Psi^{\mathrm{m}}_{\widehat{h}}(h,\Omega)^{\mathrm{T}} & \sqrt{h}\Psi^{\mathrm{m}}_{\widehat{h}}(h,\Omega)Q \\ \sqrt{h}Q\Psi^{\mathrm{m}}_{\widehat{h}}(h,\Omega)^{\mathrm{T}} & Q \end{pmatrix} \succ O$$

for $h = \underline{h}, \overline{h}$ and for any matrix Ω with $\overline{\sigma}(\Omega) \le \omega$. Similarly to the proof of Theorem 4, the affinity of $h\Psi^{\mathrm{m}}_{\widehat{h}}(h,\Omega)$ in h implies that the same inequality holds for any $\underline{h} \le h \le \overline{h}$ and any Ω with $\overline{\sigma}(\Omega) \le \omega$. Lemma 15 implies the theorem. □

On the choice of $\widehat{h}$, Remark 5 again applies. That is, $\widehat{h} = \underline{h}$ or $\widehat{h} = \overline{h}$ is a good choice for computational reason. When $\underline{h}$ is close to zero, $\widehat{h} = \underline{h}$ is preferable.

In the case of $\underline{h} = 0$, a sufficient condition can be constructed in a similar way. We present only the results because their derivation is straightforward. The first step is preparation of a function for replacement of $\Psi(h)$.

Table 1. Stability analysis with adaptive division

dividing points	comp. time (s)	max. value
0, 1.7294	0.374	-0.805
0, 0.8647, 1.7294	0.493	-0.147
0, 0.8647, 1.2971, 1.7294	0.540	-0.0353
0, 0.8647, 1.2971, 1.5133, 1.7294	0.624	-0.00870
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.7294	0.658	-0.00214
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7294	0.708	-5.17×10^{-4}
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7024, 1.7294	0.708	-1.11×10^{-4}
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7024, 1.7159, 1.7294	0.727	-9.81×10^{-6}
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7024, 1.7159, 1.7227, 1.7294	0.518	$+\infty$

Lemma 17. *For any $0 \le h \le \overline{h}$, there exists Ω such that $\Psi(h) = \Psi_0^{\mathrm{m}}(\Omega)$ and $\overline{\sigma}(\Omega) \le \omega_0$, where*

$$\Psi_0^{\mathrm{m}}(\Omega) = (I + \Omega A)(A + BF)$$

and

$$\omega_0 = \overline{h} \max_{0 \le h \le \overline{h}} \exp\Big[\overline{\lambda}\big(\frac{A + A^{\mathrm{T}}}{2}\big)h\Big]. \tag{12}$$

We replace $\Psi(h)$ with $\Psi_0^{\mathrm{m}}(\Omega)$ in the condition (1) and remove the matrix uncertainty Ω similarly to the case of $\underline{h} > 0$. The resulting condition is the following.

Theorem 18. *Suppose $\underline{h} = 0$ and define ω_0 as (12). The control system S is exponentially stable if there exists a symmetric matrix Q and a nonnegative number s_h, dependent on h, such that*

$$\begin{pmatrix} -(A+BF)Q - Q(A+BF)^{\mathrm{T}} - s_h I & \sqrt{h}(A+BF)Q & -\omega Q(A+BF)^{\mathrm{T}}A^{\mathrm{T}} \\ \sqrt{h}Q(A+BF)^{\mathrm{T}} & Q & \omega\sqrt{h}Q(A+BF)^{\mathrm{T}}A^{\mathrm{T}} \\ -\omega A(A+BF)Q & \omega\sqrt{h}A(A+BF)Q & s_h I \end{pmatrix} \succ O \qquad (h = 0, \overline{h}).$$

5.2. Design of a state-feedback gain

Our approach can be generalized to design of a state-feedback gain. The stability conditions (7) and (8) contain the products $\Psi_{\widehat{h}}(h,\theta)Q$ and $\Psi_0(\theta)Q$, which include the factor $(A + BF)Q$. If we replace it by $AQ + BG$ and solve the inequalities for Q and G, we can obtain a stabilizing feedback gain by $F = GQ^{-1}$. The region-dividing technique is again effective. Similar extension is possible on the approach with matrix uncertainty.

6. Examples

The proposed approach is applied to the sampled-data control system with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \quad F = (-3.75 \quad -11.5).$$

Stability of this control system has been analyzed with various approaches. For example, the approach of [15] verified stability with the region of the sampling interval h being $(0, 1.1137]$; the approach of

[14] for $h \in (0, 1.3659]$; the approach of [7] for $h \in [0.01, 1.72]$; the approach of [22] for $h \in [0.5, 1.729]$. This system is known to be unstable for the constant sampling interval $h = 1.7295$.

We chose the region of the sampling interval with $\underline{h} = 0$ and $\overline{h} = 1.7294$ (*i.e.*, $h \in (0, 1.7294]$) and successfully verified the stability with the approach of Section 4. This shows efficacy of the present approach because the existing approaches do not allow such a large region of sampling interval. The division was adaptively constructed with Algorithm 13 and consisted of 9 subregions when it assured the stability. The process of the construction is summarized in Table 1. The region near to 1.7294 was divided finely, which was considered important for less conservative analysis. Here, we chose $\widehat{h}^{[j]} = \underline{h}^{[j]}$ for all the subregions. The SDP problems were solved with the SDP solver SeDuMi [21] and the modeling language YALMIP [13]. The used computer was equipped with Intel Core 2 Duo U7500 (1.06 GHz) and memory of 2 GBytes.

The result became even better when we chose $\widehat{h}^{[j]} = \overline{h}^{[j]}$ for all the subregions except for the one including the origin. Namely, the stability was assured for a division consisting of only two subregions $[0, 0.8647]$ and $[0.8647, 1.7294]$. The computational time was 0.306 s.

We next took the approach in Section 5.1 using matrix uncertainty. The bounds $\underline{h}$ and $\overline{h}$ were chosen the same as before and the numbers $\widehat{h}^{[j]}$ equal to $\underline{h}^{[j]}$ for all the subregions. The stability was verified with a division consisting of 17 subregions, which was constructed adaptively. The computational time was 0.911 s. The increase of the number of subregions shows conservatism of this approach. The choice $\widehat{h}^{[j]} = \overline{h}^{[j]}$ was inefficient in this case requiring 31 subregions to assure stability.

Finally, we designed the state-feedback gain F for the A and B above. We took the approach in Section 5.2 with $\underline{h} = 0$ and $\overline{h} = 10$. We chose $\widehat{h}^{[j]} = \underline{h}^{[j]}$ for all the subregions. As a result, a stabilizing gain $F = (-0.238 \quad -1.674)$ was obtained with the division consisting of $[0, 5]$ and $[5, 10]$. The computational time was 0.406 s. In fact, stabilization was possible for a larger region of sampling interval. In this case, however, the adaptive division did not work well due to numerical difficulty and an efficient division had to be found with an *ad hoc* method.

7. Conclusion

Stability analysis is considered for a sampled-data control system with uncertain sampling interval. Stability condition is presented in a robust LMI and its tractable sufficient condition is obtained with the mean value theorem. The experimental result is satisfactory and shows the efficacy of the proposed approach.

A. Proof of Theorem 11

For the proof of the theorem, we need the following lemma, which states that the maximum error between $\Psi_{\widehat{h}}(h, \theta)$ and $\Psi(h)$ in some subregion can be bounded by a linear function of the radius of the subregion.

Lemma 19. *There exists a positive number C such that the inequality*

$$\max_{\underline{h}' \leq h \leq \overline{h}'} \max_{\theta \in \Theta'} \overline{\sigma}[\Psi_{\widehat{h}'}(h, \theta) - \Psi(h)] \leq C|\overline{h}' - \underline{h}'|$$

holds for any subregion $[\underline{h}', \overline{h}'] \subseteq [\underline{h}, \overline{h}]$ *with* $\underline{h}' > 0$ *when the choice* $\widehat{h}' = \underline{h}'$ *is adopted. Similarly, in the case of* $\underline{h} = 0$*, there exists a positive number* C_0 *such that the inequality*

$$\max_{0 \le h \le \overline{h}'} \max_{\theta \in \Theta_0'} \overline{\sigma}[\Psi_0(\theta) - \Psi(h)] \le C_0 \overline{h}'$$

holds for any subregion $[0, \overline{h}'] \subseteq [0, \overline{h}]$*. Here, the set* Θ' *is an adaptation of* Θ *whose* $\underline{h}$*,* $\overline{h}$*,* $\widehat{h}$ *are replaced by* $\underline{h}'$*,* $\overline{h}'$*,* $\widehat{h}' = \underline{h}'$*, respectively. Similarly for* Θ_0'*.*

Proof. We prove the first statement of the lemma in the special case that

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}.$$

The proof of the remaining part is similar.

The equations (4) and (6) imply

$$\Psi_{\widehat{h}'}(h, \theta) - \Psi(h) = \frac{h - \widehat{h}'}{h} T \begin{pmatrix} \theta_1 - \mathrm{e}^{\lambda_1 h_1} & 0 \\ 0 & \theta_2 - \mathrm{e}^{\lambda_2 h_2} \end{pmatrix} T^{-1}(A + BF).$$

Since θ_1 belongs to $[\min_{\underline{h}' \le h \le \overline{h}'} \mathrm{e}^{\lambda_1 h}, \max_{\underline{h}' \le h \le \overline{h}'} \mathrm{e}^{\lambda_1 h}]$ and so does $\mathrm{e}^{\lambda_1 h_1}$, the discrepancy $|\theta_1 - \mathrm{e}^{\lambda_1 h_1}|$ is bounded from above by $\max_{\underline{h}' \le h \le \overline{h}'} \mathrm{e}^{\lambda_1 h} - \min_{\underline{h}' \le h \le \overline{h}'} \mathrm{e}^{\lambda_1 h}$, which is further bounded by $c_1 |\overline{h}' - \underline{h}'|$ if c_1 is larger than $\max_{\underline{h} \le h \le \overline{h}} |\lambda_1 \mathrm{e}^{\lambda_1 h}|$. Note that c_1 can be chosen independently of the subregion $[\underline{h}', \overline{h}']$. Similar discussion is possible with $|\theta_2 - \mathrm{e}^{\lambda_2 h_2}|$. On the other hand, $|(h - \widehat{h}')/h| \le 1$ because $\underline{h}' = \widehat{h}' \le h \le \overline{h}'$. Hence, the first statement of the lemma follows. □

Proof of Theorem 11. Let $[\underline{h}^{[j]}, \overline{h}^{[j]}]$ be any subregion in the division Δ with $\underline{h}^{[j]} > 0$. From the left-hand side of the inequality (9) with respect to this subregion, subtract the left-hand side of (1) to have

$$\begin{pmatrix} \Psi_{\widehat{h}^{[j]}}(h, \theta) - \Psi(h) \\ O \end{pmatrix} \begin{pmatrix} -Q & \sqrt{h} Q \end{pmatrix} + \begin{pmatrix} -Q \\ \sqrt{h} Q \end{pmatrix} \begin{pmatrix} \Psi_{\widehat{h}^{[j]}}(h, \theta)^{\mathrm{T}} - \Psi(h)^{\mathrm{T}} & O \end{pmatrix}.$$

Due to Lemma 19, the maximum singular value of this matrix can be bounded from above by a number proportional to $|\overline{h}^{[j]} - \underline{h}^{[j]}|$. Hence, if the maximum radius $\overline{\mathrm{rad}}\, \Delta$ is small enough, the maximum singular value of this matrix is small for any subregion in Δ, which establishes the theorem. □

B. Proof of Lemma 15

The claim is obvious for $h = \widehat{h}$. We hence consider the case of $h \ne \widehat{h}$ in the sequel.

By the definition of $\Psi(h)$, we have

$$\begin{aligned} \Psi(h) &= \frac{1}{h} \int_0^h \mathrm{e}^{At}\, \mathrm{d}t (A + BF) \\ &= \frac{1}{h} \Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\, \mathrm{d}t + \int_0^{h - \widehat{h}} \mathrm{e}^{At}\, \mathrm{d}t\, \mathrm{e}^{A\widehat{h}} \Big] (A + BF). \end{aligned}$$

Hence, we are done if the maximum singular value of

$$\frac{1}{h - \widehat{h}} \int_0^{h - \widehat{h}} \mathrm{e}^{At}\, \mathrm{d}t$$

is less than or equal to ω.

Suppose $h - \widehat{h} > 0$ first. Since $\overline{\sigma}(\mathrm{e}^{At}) \leq \exp[\overline{\lambda}((A + A^{\mathrm{T}})/2)t]$ for $t \geq 0$ [9, p. 577], we have

$$\overline{\sigma}\Big(\frac{1}{h - \widehat{h}} \int_0^{h-\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t\Big) \leq \max_{0 \leq t \leq h - \widehat{h}} \exp\Big[\overline{\lambda}\big(\frac{A + A^{\mathrm{T}}}{2}\big)t\Big],$$

which shows the claim. Proof in the case of $h - \widehat{h} < 0$ is similar.

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