The smallest and largest distributions of the absolute difference and their applications to robust scale estimation

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Abstract  A certain class of symmetric and unimodal continuous distributions is considered. The smallest and largest distributions of the absolute difference of two independent random variables with a common distribution in the class are derived. The results are extended to the case that any distribution in the class may be contaminated. As an application of the results to robust estimation, the implosion and explosion biases of an Q-estimate for scale over a class of contaminated distributions are derived. This contaminated distribution class is related to \((c, \gamma)\)-contamination.

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1 Introduction and basic results

For a given symmetric and unimodal continuous function we consider a certain class of symmetric unimodal continuous distributions and derive some basic results (Proposition, Corollary). From these results we obtain the stochastically smallest and largest distributions of the absolute difference of two independent random variables with a common distribution in the class (Theorem 1). Next we consider the case that any distribution in the class may be contaminated and extend the results to a broader class which consists of all the \(\varepsilon\)-contaminated distributions (Theorem 2). There are various statistics based on the absolute difference and the results have large applicability. As an application of Theorem 2 to robust estimation, we use the contaminated distribution class to describe the departure from the model distribution and derive the implosion and explosion biases of an Q-estimate for scale (Theorems 3 and 4). The Q-estimate was proposed as an alternative to MAD (median absolute deviation) by Rousseeuw and Croux (1993). We notice that the class of contaminated distributions is related to \((c, \gamma)\)-contamination introduced by Ando and Kimura (2003). See Ando and Kimura (2004, 2005, 2006) for applications of \((c, \gamma)\)-contamination.

Let \(g\) be nonnegative continuous unimodal function defined on the real line \(R\) which is even and satisfies

\[
1 \leq \int_{-\infty}^{\infty} g(x) \, dx < \infty.
\]
For $g$ we define $\mathcal{F}_g$ as the class of all continuous distributions $F$ which has an even unimodal continuous (on the support) density $f$ such that $0 \leq f \leq g$. In what follows, for any $F$ in $\mathcal{F}_g$ we use its even unimodal continuous density $f$. Let $X$ and $Y$ be independent and identically distributed with a common $F$ in $\mathcal{F}_g$. There are a lot of important statistics based on the absolute difference $|X - Y|$ and we are interested in its smallest and largest distributions.

Let $\hat{F}$ and $\bar{F}$ have the densities $\hat{f}$ and $\bar{f}$ defined as follows:

\begin{equation}
\hat{f}(x) = \begin{cases} 
g(x) & \text{if } |x| \leq a, \\
0 & \text{if } |x| > a,
\end{cases}
\end{equation}

\begin{equation}
\bar{f}(x) = \begin{cases} 
g(b) & \text{if } |x| \leq b, \\
g(x) & \text{if } |x| > b,
\end{cases}
\end{equation}

where $a$ and $b$ are the positive constants such that $\hat{f}$ and $\bar{f}$ are densities. Note that $\hat{F}$ and $\bar{F}$ belong to $\mathcal{F}_g$.

The following basic results are used to derive main theorems.

**Proposition.** The following hold:

(i) \( \sup_{F \in \mathcal{F}_g} \int_{-\infty}^\infty \{F(x + t) - F(x - t)\}dF(x) = \int_{-\infty}^\infty \{\hat{F}(x + t) - \hat{F}(x - t)\}d\hat{F}(x), \quad t \geq 0, \)

(ii) \( \inf_{F \in \mathcal{F}_g} \int_{-\infty}^\infty \{F(x + t) - F(x - t)\}dF(x) = \int_{-\infty}^\infty \{\bar{F}(x + t) - \bar{F}(x - t)\}d\bar{F}(x), \quad t \geq 0. \)

**Proof.** (i) Let $F$ be any element of $\mathcal{F}_g$. Since $F(x + t) - F(x - t)$ is nonincreasing in $|x|$ and $f \leq \hat{f}$ on $[-a, a]$, it follows that

\begin{equation}
\int_{-\infty}^\infty \{F(x + t) - F(x - t)\}f(x)dx \leq \int_{-\infty}^\infty \{F(x + t) - F(x - t)\}\hat{f}(x)dx.
\end{equation}

Since $f$ and $\hat{f}$ are even, it follows that

\[ \int_{-\infty}^\infty \{F(x + t) - F(x - t)\}\hat{f}(x)dx = \int_{-\infty}^\infty \{F(x + t) - F(-x - t)\}\hat{f}(x)dx. \]

Hence

\begin{equation}
\int_{-\infty}^\infty \{F(x + t) - F(x - t)\}\hat{f}(x)dx = \int_{-\infty}^\infty \{F(x + t) - F(-x - t)\}\hat{f}(x)dx + \int_{-t}^{\infty} \{F(x + t) - F(-x - t)\}\hat{f}(x)dx
\end{equation}

\begin{equation}
= \int_{-t}^{\infty} \{F(x + t) - F(-x - t)\}\hat{f}(x)dx + \int_{-t}^{\infty} \{F(x + t) - F(-x - t)\}\hat{f}(x)dx
\end{equation}

\[ = \int_{-t}^{\infty} \{F(x + t) - F(-x - t)\}\hat{f}(x)dx. \]
Since, for $x > -t$ we have $\hat{f}(x) - \hat{f}(x + 2t) \geq 0$ and

\[(1.6)\]

$$F(x + t) - F(-x - t) \leq \hat{F}(x + t) - \hat{F}(-x - t),$$

it follows that

$$\int_{-t}^{\infty} \{F(x + t) - F(x - t)\} \{\hat{f}(x) - \hat{f}(x + 2t)\} dx$$

$$\leq \int_{-t}^{\infty} \{F(x + t) - F(-x - t)\} \{\hat{f}(x) - \hat{f}(x + 2t)\} dx.$$ 

Therefore the assertion (i) follows from (1.4) and (1.5).

(ii) Let $F$ be any element of $\mathcal{F}_g$. Since $f \geq \bar{f}$ on $[-b, b]$, it follows that

\[(1.7)\]

$$\int_{-\infty}^{\infty} \{F(x + t) - F(x - t)\} f(x) dx \geq \int_{-\infty}^{\infty} \{F(x + t) - F(x - t)\} \hat{f}(x) dx.$$

Since $\bar{f}$ is even and unimodal, for $x > -t$ we have $\bar{f}(x) - \bar{f}(x + 2t) \geq 0$ and

$$\bar{F}(x + t) - \bar{F}(-x - t) \leq F(x + t) - F(-x - t).$$

Hence it follows that

$$\int_{-\infty}^{\infty} \{F(x + t) - F(x - t)\} \bar{f}(x) dx$$

$$= \int_{-t}^{\infty} \{F(x + t) - F(-x - t)\} \{\bar{f}(x) - \bar{f}(x + 2t)\} dx$$

$$\geq \int_{-t}^{\infty} \{\bar{F}(x + t) - \bar{F}(-x - t)\} \{\bar{f}(x) - \bar{f}(x + 2t)\} dx$$

$$= \int_{-\infty}^{\infty} \{\bar{F}(x + t) - \bar{F}(x - t)\} \bar{f}(x) dx.$$

This and (1.7) imply the assertion (ii). \(\Box\)

**Corollary.** The following hold:

(i) $$\sup_{F \in \mathcal{F}_g} \int_{-\infty}^{\infty} F(x + t) dF(x) = \int_{-\infty}^{\infty} \hat{F}(x + t) d\hat{F}(x)$$

(ii) $$\inf_{F \in \mathcal{F}_g} \int_{-\infty}^{\infty} F(x + t) dF(x) = \int_{-\infty}^{\infty} \bar{F}(x + t) d\bar{F}(x)$$

**Proof.** It is easy to see that for any $F$ in $\mathcal{F}_g$

$$\int_{-\infty}^{\infty} F(x - t) dF(x) = 1 - \int_{-\infty}^{\infty} F(x + t) dF(x).$$
This implies that
\[ \int_{-\infty}^{\infty} F(x+t)dF(x) = \frac{1}{2} \int_{-\infty}^{\infty} \{F(x+t) - F(x-t)\}dF(x) + \frac{1}{2}. \]

Therefore the corollary follows from Proposition. \(\square\)

2 Main results

The following useful theorem is readily obtained from Proposition.

Theorem 1. Let \(X\) and \(Y\) be independent random variables distributed with a common \(F\) in \(\mathcal{F}_g\). Then the distribution of \(|X - Y|\) is stochastically smallest and largest under \(\hat{F}\) and \(\bar{F}\), respectively, i.e.,

(i) \(\sup_{F \in \mathcal{F}_g} P_{F \times F}(|X - Y| \leq t) = P_{\hat{F} \times \hat{F}}(|X - Y| \leq t), \quad t \geq 0,\)

(ii) \(\inf_{F \in \mathcal{F}_g} P_{F \times F}(|X - Y| \leq t) = P_{\bar{F} \times \bar{F}}(|X - Y| \leq t), \quad t \geq 0.\)

Proof. For any \(F\) in \(\mathcal{F}_g\) we have

\[ P_{F \times F}(|X - Y| \leq t) = \int_{-\infty}^{\infty} \{F(y+t) - F(y-t)\}dF(y) \]

where \(f\) is a density of \(F\). Therefore the theorem follows from Theorem 1. \(\square\)

Next, we consider the case that \(X\) and \(Y\) may be contaminated. Define the \(\varepsilon\)-contaminated class \(\mathcal{P}_{g,\varepsilon}\) of \(\mathcal{F}_g\) as

\[ (2.1) \quad \mathcal{P}_{g,\varepsilon} = \{H = (1-\varepsilon)F + \varepsilon K : F \in \mathcal{F}_g, K \in \mathcal{M}\}, \]

where \(\mathcal{M}\) is the set of all distributions on \(\mathbb{R}\). The following is an extension of Theorem 1.

Theorem 2. Let \(X\) and \(Y\) be independent random variables distributed with a common \(H\) in \(\mathcal{P}_{g,\varepsilon}\). Then

(i) \(\sup_{H \in \mathcal{P}_{g,\varepsilon}} P_{H \times H}(|X - Y| \leq t) = (1-\varepsilon)^2P_{\hat{F} \times \hat{F}}(|X - Y| \leq t) + 2\varepsilon(1-\varepsilon)\hat{F}(|X| \leq t) + \varepsilon^2, \quad t \geq 0,\)

(ii) \(\inf_{H \in \mathcal{P}_{g,\varepsilon}} P_{H \times H}(|X - Y| \leq t) = (1-\varepsilon)^2P_{\bar{F} \times \bar{F}}(|X - Y| \leq t), \quad 0 \leq t < \infty,\)

where \(\hat{F}\) and \(\bar{F}\) are given by (1.2) and (1.3), and \(\Delta_0\) is the point mass distribution at 0.
Proof. (i) For any $H = (1 - \varepsilon)F + \varepsilon K$ in $P_{g,\varepsilon}$ we have

$$P_{H \times H}(|X - Y| \leq t) = (1 - \varepsilon)^2 P_{F \times F}(|X - Y| \leq t) + 2\varepsilon(1 - \varepsilon)P_{F \times K}(|X - Y| \leq t) + \varepsilon^2 P_{K \times K}(|X - Y| \leq t).$$

It is easy to see that

$$P_{F \times K}(|X - Y| \leq t) \leq P_{\tilde{F} \times \Delta_0}(|X - Y| \leq t) = \hat{F}(|X| \leq t)$$

and

$$P_{K \times K}(|X - Y| \leq t) \leq 1 = P_{\Delta_0 \times \Delta_0}(|X - Y| \leq t).$$

Therefore the assertion (i) follows from (i) of Theorem 1.

(ii) Let $\Delta_n$ be the point mass distribution at $n$. Then, for any $H = (1 - \varepsilon)F + \varepsilon K$ in $P_{g,\varepsilon}$ we have

$$P_{F \times K}(|X - Y| \leq t) \geq \lim_{n \to \infty} P_{\tilde{F} \times K_n}(|X - Y| \leq t) = 0$$

and

$$P_{K \times K}(|X - Y| \leq t) \geq \lim_{n \to \infty} P_{K_n \times K_n}(|X - Y| \leq t) = 0.$$

Therefore the assertion (ii) follows from (ii) of Theorem 2. □.

Remark 1. Let $\mathcal{F}_g^*$ be the set of all continuous densities $f$ such that $f \leq g$. The assertions (i) of Proposition, Corollary and Theorem 1 also hold for this broader $\mathcal{F}_g^*$. These results were proved in a different way by Ando and Kimura (2003). In this case, we note that the symmetry and unimodality of $f$ are not required in the definition of $\mathcal{F}_g^*$. Let $\mathcal{P}_{g,\varepsilon}^*$ be the set defined by replacing $\mathcal{F}_g$ with $\mathcal{F}_g^*$ in the definition (2.1) of $\mathcal{P}_{g,\varepsilon}$. We can see that the assertion (i) of Theorem 2 also holds for $\mathcal{P}_{g,\varepsilon}^*$.

3 The explosion and implosion bias of an $Q$-estimate

As an application of the previous results, we consider the explosion and implosion biases of an $Q$-estimate for robust scale estimation. To give a robust scale estimation model we need to define a certain class of distributions. Let $F_0$ be a specified distribution with an even unimodal continuous density $f_0$. For some given constants $c$ and $\gamma$ ($0 < \gamma < 1$ and $c > 1 - \gamma$) we consider the $(c, \gamma)$-symmetric unimodal neighborhood $\mathcal{F}_{c,\gamma}(F_0)$ of $F_0$, defined as the set of all continuous distributions $F$ which has an even unimodal density $f$ and satisfies $f \leq (\frac{c}{1 - \gamma})f_0$. Further we consider the $\gamma$-contamination neighborhood $\mathcal{P}_{c,\gamma}(F_0)$ of $\mathcal{F}_{c,\gamma}(F_0)$, that is

(3.1) $\mathcal{P}_{c,\gamma}(F_0) = \{H = (1 - \gamma)F + \gamma K : F \in \mathcal{F}_{c,\gamma}(F_0), K \in \mathcal{M}\}$. 


The robust scale model we consider here is given as follows: Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $H$. We assume that $H$ belongs to the class
\begin{equation}
\mathcal{P}_{c,\gamma}(F_{\mu,s}) = \{ H : H(x) = (1 - \gamma)F\left(\frac{x - \mu}{s}\right) + \gamma K(x), \ x \in R, \ F \in \mathcal{F}_{c,\gamma}(F_0), K \in M \},
\end{equation}
where $\mu$ is an unknown location parameter and $s > 0$ is an unknown scale parameter to be estimated. For more details of the model (3.2), see Martin and Zamar (1993) and Ando and Kimura (2003). Among various robust estimates for scale $s$, we consider an $Q_n$-estimate given by the $k$th order statistic
\begin{equation}
Q_n = d \cdot \{|X_i - X_j|; \ i < j \}_{(k)},
\end{equation}
where $d$ is a constant factor and $k = \left(\frac{h}{2}\right) \approx \left(\frac{n}{2}\right)/4$, where $h = \lfloor n/2 \rfloor + 1$. The $Q_n$ was proposed as an alternative to $MAD_n$ by Rousseeuw and Croux (1993) and it has 50% breakdown point and higher efficiency than $MAD_n$. Since $Q_n$ is location and scale equivariant, we derive the explosion and implosion biases of $Q_n$ over $\mathcal{P}_{c,\gamma}(F_0)$ ($F_0 = F_{0,1}$). The explosion bias $B^+_Q(c, \gamma)$ and implosion bias $B^-_Q(c, \gamma)$ of the $Q$-estimate $T$ over $\mathcal{P}_{c,\gamma}(F_0)$ are defined by
\begin{equation}
B^+_Q(c, \gamma) = \sup\{ Q(H) : H \in \mathcal{P}_{c,\gamma}(F_0) \},
\end{equation}
\begin{equation}
B^-_Q(c, \gamma) = \inf\{ Q(H) : H \in \mathcal{P}_{c,\gamma}(F_0) \}.
\end{equation}
The asymptotic version of $Q_n$ is given by
\begin{equation}
Q(H) = dG_H^{-1}\left(\frac{1}{4}\right) = dL_H^{-1}\left(\frac{5}{8}\right),
\end{equation}
where $G_H$ and $L_H$ are the distributions of $\left|X - Y\right|$ and $X - Y$, respectively. Note that $L_H$ is symmetric about the origin.

As in Section 1, we define the densities $\hat{f}$ and $\bar{f}$ by
\begin{equation}
\hat{f}(x) = \begin{cases} (\frac{c}{1 - \gamma})f_0(x) & \text{if } |x| \leq a, \\ 0 & \text{if } |x| > a, \end{cases}
\end{equation}
\begin{equation}
\bar{f}(x) = \begin{cases} (\frac{c}{1 - \gamma})f_0(b) & \text{if } |x| \leq b, \\ (\frac{c}{1 - \gamma})f_0(x) & \text{if } |x| > b, \end{cases}
\end{equation}
where $a$ and $b$ are the constants such that
\[ a = F_0^{-1}\left(\frac{c - \gamma + 1}{2c}\right) \quad \text{and} \quad F_0(b) - bf_0(b) = \frac{2c + \gamma - 1}{2c}. \]

The explosion and implosion biases of the $Q$-estimate are obtained by next two theorems. Although the theorems can be readily derived using Theorem 2 and $G_H$, we give the proofs following Rousseeuw and Croux (1993).
**Theorem 3.** Let $F_0$ have an even unimodal continuous density $f_0$. Then

\begin{equation}
B^+_Q(c, \gamma) = \begin{cases} 
  d(\hat{F}^*2)^{-1} \left( \frac{5-8\gamma+4\gamma^2}{8(1-\gamma)^2} \right) & \text{if } 0 \leq \gamma < \frac{1}{2}, \\
  0 & \text{if } \gamma \geq \frac{1}{2}, 
\end{cases}
\end{equation}

where $\hat{F}^*2$ denotes the convolution of $\hat{F}$.

**Proof.** For any $H = (1 - \gamma)F + \gamma K$ in $\mathcal{P}_{c, \gamma}(F_0)$, $Q(H)$ is the smallest solution of

\begin{equation}
\int_{-\infty}^{\infty} H(y + d^{-1}Q(H))dH(y) \geq \frac{5}{8}.
\end{equation}

Let $X$ be distributed with $F$, and let $Y_1$ and $Y_2$ be distributed with $K$. Then (3.10) is expressed as

\begin{equation}
(1 - \gamma)^2F^2(d^{-1}Q(H)) + \gamma(1 - \gamma)\{1 + P(|X - Y_1| \leq d^{-1}Q(H))\}
\end{equation}

\begin{equation}
+ \gamma^2P(|Y_1 - Y_2| \leq d^{-1}Q(H)) \geq \frac{5}{8}.
\end{equation}

Each term in (3.11) is increasing in $Q(H)$. Hence, $Q(H)$ is maximized when $F^2(d^{-1}Q(H))$, $P(|X - Y_1| \leq d^{-1}Q(H))$ and $P(|Y_1 - Y_2| \leq d^{-1}Q(H))$ are minimized. To do this, by Theorem 1 we need to take $F = \hat{F}$. Let $K = K_n$ be the normal distribution $N(n, n^2)$ with mean $n$ and variance $n^2$. Then, under $H_n = (1 - \gamma)\hat{F} + \gamma K_n$, as $n \to \infty$ we have $P(|X - Y_1| \leq d^{-1}Q(H)) \to 0$ and $P(|Y_1 - Y_2| \leq d^{-1}Q(H)) \to \frac{1}{2}$. Substituting the lower bounds 0 and $\frac{1}{2}$ of $P(|X - Y_1| \leq d^{-1}Q(H))$ and $P(|Y_1 - Y_2| \leq d^{-1}Q(H))$ into (3.11), we obtain Theorem 3. \(\square\)

**Theorem 4.** Let $F_0$ have an even unimodal continuous density $f_0$. Then

\begin{equation}
B^+_Q(c, \gamma) = \begin{cases} 
  Q(\hat{F}) & \text{if } 0 \leq \gamma < \frac{1}{2}, \\
  0 & \text{if } \gamma \geq \frac{1}{2}, 
\end{cases}
\end{equation}

where $Q(\hat{F})$ satisfies the equation

\begin{equation}
(1 - \gamma)^2\hat{F}^2(d^{-1}Q(\hat{F})) + 2\gamma(1 - \gamma)\hat{F}(Q(\hat{F})) + \gamma^2 = \frac{5}{8},
\end{equation}

and $\hat{F}^*2$ denotes the convolution of $\hat{F}$.

**Proof.** By (3.11), $Q(H)$ is minimized when $F^2(d^{-1}Q(H))$, $P(|X - Y_1| \leq d^{-1}Q(H))$ and $P(|Y_1 - Y_2| \leq d^{-1}Q(H))$ are maximized. Hence, by Theorem 1 we choose $F = \hat{F}$ and $K = \Delta_0$, where $K = \Delta_0$ is the point mass distribution at the zero. Then $P(|X - Y_1| \leq d^{-1}Q(\hat{H})) = 2\hat{F}(Q(\hat{H})) - 1$ and $P(|Y_1 - Y_2| \leq d^{-1}Q(\hat{H})) = 1$. Therefore we obtain Theorem 4. \(\square\)

**Remark 2.** Let $\mathcal{P}_{c, \gamma}^*(F_0)$ be the set of defined by replacing $\mathcal{F}_{c, \gamma}$ with $\mathcal{F}_{c, \gamma}^*$, that is

\[ \mathcal{P}_{c, \gamma}^*(F_0) = \{ H = (1 - \gamma)F + \gamma K : F \in \mathcal{F}_{c, \gamma}^*, K \in \mathcal{M} \}. \]
As shown in Ando, Kakiuchi and Kimura (2006), $\mathcal{P}_{c,\gamma}^*(F_0)$ is the $(c, \gamma)$-contamination neighborhood of $F_0$ introduced by Ando and Kimura (2003). We should notice that Theorem 4 holds for $\mathcal{P}_{c,\gamma}^*(F_0)$.

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