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Robust nonparametric confidence intervals and tests for the median in the presence of (c, γ)-contamination

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Abstract The problem of constructing robust nonparametric confidence intervals and tests for the median is considered when the data distribution is unknown and the data may be contaminated. A new form of the (c, γ)-neighborhood is proposed and it is used in order to describe the contaminaton of the data. The (c, γ)-neighborhood is a generalization of the neighborhoods defined in terms of ε-contamination and total variation distance. A modification of the sign test and its associated confidence intervals are proposed, and their robustness and efficiency are studied under the (c, γ)-neighborhood of a continuous distribution. The derived results are natural extensions of those in the case of ε-contamination. Some tables and figures of coverage probability and maximum asymptotic length for the confidence intervals are also given.

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1 Introduction

Huber (1965) introduced censored probability ratio tests to robustify the Neyman-Pearson optimal test, and showed that they are minimax for test problems between two composite hypotheses described in terms of ε-contamination or total variation neighborhoods. Huber (1968) considered a class of neighborhoods including ε-contamination and total variation neighborhoods, and derived robust confidence intervals for a location parameter which cover the target parameter with the nominal probability for all distributions in the neighborhoods. Many contributions have been made to robust testing

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and interval inference since these Huber's pioneer works appeared. Among them, there are Huber and Strassen (1973), Rieder (1977, 1978, 1981, 1982), Bednarski (1981, 1982), Hettmansperger and Sheather (1986), Morgenthaler (1986), He, Simpson and Portnoy (1990), Friman, Yohai and Zamar (2001) and others.

Recently, Yohai and Zamar (2004) considered the problem of constructing robust non-parametric confidence intervals and tests for the median when the data distribution is unknown and the data may contain a small fraction of contamination. They proposed a confidence interval associated with the sign test which attains the nominal significance level for any distribution in the ε-contamination neighborhood of a continuous distribution. They also defined some measures of robustness and efficiency for confidence intervals and tests under ε-contamination, and computed these measures for the proposed procedures.

The purpose of this paper is to propose a new form of the (c,γ)-neighborhood and to extend Yohai and Zamar's (2004) results to the case that the contamination in the data is described by the (c,γ)-neighborhood. The (c,γ)-neighborhood was introduced by Ando and Kimura (2003) as a generalization of the neighborhoods defined in terms of ε-contamination, total variation distance and Rieder's (1977) (ε,δ)-contamination.

Let F° be a continuous distribution on the real line R and let $\mathcal{M}$ be the set of all distributions on R. For $0 \le \gamma < 1/2,\ 1-\gamma \le c < \infty$ the (c,γ)-neighborhood of F° is defined as

$$\mathcal{P}_{c,\gamma}(F^\circ) = \{G \in \mathcal{M} \mid G(B) \le cF^\circ(B) + \gamma,\ \forall B \in \mathcal{B}\}, \tag{1.1}$$

where $\mathcal{B}$ is the Borel σ -field of subsets of R. Note that G (F°) is used as both distribution function and probability measure for convenience. Thus the (c,γ)-neighborhood is generated by the special capacity v defined as $v(B) = \min\{cF^\circ(B)+\gamma,\ 1\}$ for $\phi \ne B \in \mathcal{B}$ and $v(B) = 0$ for $B = \phi$. Therefore the (c,γ) -neighborhood has good properties such that there exist the least favorable pairs of distributions between two (c,γ) -neighborhoods, which are applicable to constructing the maximin tests (see Bednarski, 1981 for special capacities). As easily seen, for $\varepsilon \ge 0,\ \delta \ge 0$ and $\varepsilon + \delta < 1$ we have the neighborhoods $\mathcal{P}_{1-\varepsilon,\varepsilon}(F^\circ), \mathcal{P}_{1,\delta}(F^\circ)$ and $\mathcal{P}_{1-\varepsilon,\varepsilon+\delta}(F^\circ)$ defined in terms of ε-contamination, total variation and Rieder's (1977) (ε,δ)-contamination, respectively. Applications of the (c,γ)-neighborhood to bias-robustness of estimates are found in Ando and Kimura (2003, 2004, 2005).

A new form of the (c,γ)-neighborhood we propose in this paper is defined as follows: Let f° be the density function of F° (with respect to Lebesgue measure) and let $\mathcal{F}_{c,\gamma}(F^\circ)$ be the set of all continuous distributions F whose densities f satisfy $0 \le f \le (\frac{c}{1-\gamma})f^\circ$, that is,

$$\mathcal{F}_{c,\gamma}(F^\circ) = \left\{F \in \mathcal{M}^* \;\middle|\; 0 \le f \le \left(\frac{c}{1-\gamma}\right) f^\circ \right\}, \tag{1.2}$$

where $0 \le \gamma < 1/2,\ 1-\gamma \le c < \infty$ and $\mathcal{M}^*$ is the set of all continuous distributions on R. Then we define the (c,γ)-neighborhood of F° as

$$\mathcal{P}_{c,\gamma}(F^\circ) = \{G = (1-\gamma)F + \gamma K \mid F \in \mathcal{F}_{c,\gamma}(F^\circ),\ K \in \mathcal{M}\}. \tag{1.3}$$

The following equivalence result is obtained from Proposition 1.2, which gives a characterization of the (c,γ)- neighborhood.

Proposition 1.1 The two definitions (1.1) and (1.3) of the (c,γ)-neighborhood are equivalent.

Proposition 1.2 (Theorem 2.1 of Ando and Kimura, 2003). The following characterization of the (c,γ)-neighborhood holds:

$$\mathcal{P}_{c,\gamma}(F^\circ) = \{G = c(F^\circ - W) + \gamma K \mid W \in \mathcal{W}_\tau(F^\circ),\ K \in \mathcal{M}\},$$

where $\mathcal{W}_\tau(F^\circ)$ is the set of all measures W on $\mathcal{B}$ such that $W(B) \le F^\circ(B)$ for $\forall B \in \mathcal{B}$ and $W(R) = \tau = (c+\gamma-1)/c$.

Let $x_{(1)} \le x_{(2)}, \cdots, \le x_{(n)}$ be the order statistics of a random sample $\boldsymbol{X}_n = (x_1, \cdots, x_n)$ with common distribution G in $\mathcal{P}_{c,\gamma}(F^\circ)$, where F° is unknown. We assume the following two conditions:

A1. F° is continuous with a unique median $\theta(F^\circ) = (F^\circ)^{-1}(1/2)$.

A2. $1 \le \dfrac{c}{1-\gamma} < 2$.

The first inequality in A2 is a restriction in the definition of the (c,γ)-neighborhood, and the second one implies that $0 < F(\theta) < 1$ holds for all F in $\mathcal{F}_{c,\gamma}(F^\circ)$, that is, the median θ lies inside the support of F.

Consider the null hypothesis $H_0 : \theta = \theta_0$ and the sign test statistic

$$T_{n,\theta}(\boldsymbol{X}_n) = \sum_{i=1}^{n} \chi_{(0,\infty)}(x_i - \theta), \tag{1.4}$$

where $\chi_{(0,\infty)}$ denotes the indicator function of $(0,\infty)$.

The confidence interval for θ

$$I(\boldsymbol{X}_n) = [x_{(k+1)}, x_{(n-k)}) \tag{1.5}$$

is obviously obtained by inverting the acceptance region of the sign test

$$k < T_{n,\theta}(\boldsymbol{X}_n) < n-k, \tag{1.6}$$

where k is an integer determined by a level α $(0 < \alpha < 1)$.

In Section 2 we are concerned with confidence intervals (1.5) based on the sign test statistic and extend Theorems 1 and 2 in Yohai and Zamar (2004) to the case of the (c,γ)-contamination. These extensions (Theorems 2.1 and 2.2) of robustness and efficiency are natural. Theorem 2.1 shows that Yohai and Zamar's nonparametric ε- robusut confidence interval has coverage probability $1-\alpha$ over not only the ε-contamination neighborhood but also (c,γ)-neighborhoods for all c and γ such that $\varepsilon = c+2\gamma-1$. Theorem 2.2 establishes robustness and efficiency of confidence intervals over γ-contamination neighborhoods of F which does not need to be unimodal and symmetric, whereras Theorem 2 considers only under ε-contamination neighborhoods of a symmetric and unimodal F°. We also present some tables and figures of coverage probability and maximum asymptotic length for the confidence intervals. In Section 3 we consider a modified sign test and give a natural extension (Theorem 3.2) of Theorems 3 in Yohai and Zamar (2004). All the proofs of our results are collected in Section 4. We should point out that our proof of Theorem 2.2 is essentially different from that of Theorem 2 due to Yohai and Zamar.

2 Robust nonparametric confidence intervals

We begin with the definition of a nonparametric robust confidence interval which we try to construct.

Definition 2.1 A confidence interval $I_n = [a_n(\boldsymbol{X}_n), b_n(\boldsymbol{X}_n))$ is said to have (c,γ)-robust coverage $1-\alpha$ at F° if

$$\inf_{G\in\mathcal{P}_{c,\gamma}(F^\circ)} P_G\{a_n(\boldsymbol{X}_n) \le \theta < b_n(\boldsymbol{X}_n)\} = 1-\alpha. \tag{2.1}$$

Definition 2.2 A confidence interval $I = [a_n(\boldsymbol{X}_n), b_n(\boldsymbol{X}_n))$ is said to have nonparametric (c,γ)-robust coverage $1-\alpha$ if it has (c,γ)-robust coverage at F° for all F° satisfying A1.

The following theorem is an extension of Theorem 1 of Yohai and Zamar (2004), which enables us to construct robust nonparametric confidence intervals.

Theorem 2.1 Let $\boldsymbol{X}_n = (X_1,\cdots,X_n)$ be a random sample from $G\in\mathcal{P}_{c,\gamma}(F^\circ)$ with F° satisfying A1. Then

(i) $I_n = [x_{(k+1)}, x_{(n-k)})$ has nonparametric (c,γ)-robust coverage $1-\alpha^*$, that is,

$$\inf_{G\in\mathcal{P}_{c,\gamma}(F^\circ)} P_G\{x_{(k+1)} \le \theta < x_{(n-k)}\} = 1-\alpha^*, \tag{2.2}$$

where

$$\alpha^* = \alpha^*(n,k,c,\gamma) = 1 - P(k < Z_n < n-k), \tag{2.3}$$

with Z_n distributed as Binomial$\left(n, \frac{1-\lambda}{2}\right)$ and $\lambda = c+2\gamma-1$ $(0\le\lambda<1)$.

(ii) The infimum in (2.2) is achieved for any γ -contaminating distribution of F_L° (or F_R°) which places all its mass to the left (or the right) of θ, where F_L° and F_R° are the stochastically smallest and largest distributions in $\mathcal{F}_{c,\gamma}(F^\circ)$, respectively, that is,

$$\begin{aligned} F_L^\circ(x) &= \min\left\{\left(\tfrac{c}{1-\gamma}\right)F^\circ(x), 1\right\}, \quad x\in R, \\ F_R^\circ(x) &= \max\left\{\left(\tfrac{c}{1-\gamma}\right)F^\circ(x) - \left(\tfrac{c}{1-\gamma}-1\right), 0\right\}, \quad x\in R. \end{aligned} \tag{2.4}$$

Remark 2.1 Theorem 2.1 states that the nonparametric (c,γ) - robust confidence interval $I_n = [x_{(k+1)}, x_{(n-k)})$ with coverage probability $1-\alpha^*$ is determined by c and γ through $\lambda = c+2\gamma-1$. Therefore, when $\lambda=\varepsilon$, the interval I_n is the same as the nonparametric ε-robust confidence interval with coverage probability $1-\alpha^*$ in Yohai and

Zamar (2004). This fact implies that the Yohai and Zamar's confidence interval has (c,γ)-robust coverage $1-\alpha^*$ for all c and γ such that $\varepsilon = c+2\gamma-1$. Since the class of such (c,γ)-neighborhoods of F° includes the ε-contamination neighborhood of F° as a special case (i.e., $c = 1-\varepsilon, \gamma = \epsilon$), Theorem 2.1 strengthens the robustness property of their confidence interval. We should also point out that $0 \le \lambda < 1$ whereas $0 \le \varepsilon < \frac{1}{2}$.

Tables 1 and 2 exhibit the minimum coverage probabilities of the interval $I(\boldsymbol{X}_n) = [x_{(k+1)}, x_{(n-k)})$ for α=0.05 and α=0.10, respectively, which are calculated using Theorem 2.1. The values in Tables 1 and 2 are very low for large n. This fact shows that the confidence interval $I(\boldsymbol{X}_n)$ with confidence coefficient $1-\alpha$ in the uncontaminated case ($\lambda = 0$) is inappropreate in the contaminated case ($\lambda > 0$). Note that the results for λ=0.00, 0.05, 0.10 and 0.15 are the same as Table 1 in Yohai and Zamar (2004). Although their ε takes the value in the interval $[0, \frac{1}{2})$, our λ does in the wider interval [0, 1).

Using Theorem 2.1, for given contamination sizes c and γ we can construct a confidence interval $I_n = [x_{(k+1)}, x_{(n-k)})$ with (c,γ)-robust coverage $1-\alpha^*$. Hereafter we denote the real contamination sizes by $\tilde{c}$ and $\tilde{\gamma}$, and distinguish them from the design contamination sizes c and γ which are used to construct (c,γ)-robust coverage $1-\alpha$ confidence intervals.

Table 1: Minimum coverage probability for contaminated sample, $\alpha = 0.05$.

	$1-\alpha \approx 0.95$									
	λ									
n	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.60
20	0.959	0.954	0.938	0.912	0.873	0.821	0.754	0.584	0.383	0.196
40	0.962	0.952	0.922	0.868	0.788	0.684	0.559	0.297	0.103	0.019
100	0.943	0.912	0.815	0.655	0.457	0.266	0.125	0.012	0	0
200	0.944	0.881	0.689	0.414	0.174	0.047	0.008	0	0	0
500	0.946	0.789	0.376	0.074	0.005	0	0	0	0	0
1000	0.946	0.636	0.108	0.002	0	0	0	0	0	0
2000	0.948	0.385	0.006	0	0	0	0	0	0	0

Let us consider a sequence $\{I_n\}$ of intervals $I_n = [a_n(\boldsymbol{X}_n), b_n(\boldsymbol{X}_n))$. The maximum asymptotic length $L\{I_n, F^\circ, (\tilde{c},\tilde{\gamma})\}$ of $\{I_n\}$ under contamination of size $(\tilde{c},\tilde{\gamma})$ at F° is defined by

$$L\{I_n, F^\circ, (\tilde{c},\tilde{\gamma})\} = \sup_{G \in \mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ)} \operatorname{essup} \limsup_{n\to\infty} (b_n(\boldsymbol{X}_n) - a_n(\boldsymbol{X}_n)), \tag{2.5}$$

where essup stands for essential supremum. The following length breakdown point is the confidence interval counterpart of breakdown point of a point estimate.

Definition 2.3 The sequence $\{I_n\}$ of intervals $I_n = [a_n(\boldsymbol{X}_n), b_n(\boldsymbol{X}_n)), n \ge n_0$, is said to have $(\tilde{c},\tilde{\gamma})$-robust length at F° if $L\{I_n, F^\circ, (\tilde{c},\tilde{\gamma})\} < \infty$. The length breakdown point of the sequence $\{I_n\}$ at F° given $\tilde{c}$ is defined as

$$\tilde{\gamma}^*\{I_n, F^\circ, \tilde{c}\} = \sup\{\tilde{\gamma} \ : \ L\{I_n, F^\circ, (\tilde{c},\tilde{\gamma})\} < \infty\}. \tag{2.6}$$

Table 2: Minimum coverage probability for contaminated sample, $\alpha = 0.10$.

	$1-\alpha \approx 0.90$									
	λ									
n	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.5	0.6
20	0.885	0.876	0.849	0.804	0.744	0.668	0.582	0.392	0.214	0.087
40	0.919	0.904	0.859	0.784	0.681	0.559	0.428	0.193	0.054	0.008
100	0.911	0.872	0.755	0.578	0.377	0.204	0.088	0.007	0	0
200	0.896	0.811	0.582	0.307	0.110	0.025	0.003	0	0	0
500	0.902	0.702	0.279	0.043	0.002	0	0	0	0	0
1000	0.906	0.537	0.068	0.001	0	0	0	0	0	0
2000	0.897	0.273	0.002	0	0	0	0	0	0	0

If $\tilde{\gamma}^*\{I_n, F^\circ, \tilde{c}\}$ does not depend on $\tilde{c}$, then it is called the breakdown point of $\{I_n\}$ at F° and denoted by $\tilde{\gamma}^*\{I_n, F^\circ\}$.

For any $\alpha \in (0,1)$ we consider the sequence $\{I_n\}$ of intervals $I_n = [x_{(k_n+1)}, x_{(n-k_n)})$ given in Theorem 2.1, where k_n are the integers defined by

$$k_n = k_n(n, \alpha, \lambda) = \arg\min |\alpha^*(n, k, \lambda) - \alpha|, \tag{2.7}$$

which satisfies

$$\lim_{n\to\infty} \alpha^*(n, k_n, \lambda) = \alpha.$$

The following lemma, which is Lemma 2 in Yohai and Zamar (2004) with ε replaced by λ, is used to prove Theorem 2.1.

Lemma 2.1 Let $\boldsymbol{X}_n = (x_1, \cdots, x_n)$ be *i.i.d.* random variables with distribution G. Consider the sequence $\{I_n\}$ of intervals $I_n = [x_{(k_n+1)}, x_{(n-k_n)})$ with length $l_n(\boldsymbol{X}_n) = x_{(n-k_n)} - x_{(k_n+1)}$ and levels $\alpha^*(n, k_n, \lambda) \to \alpha$, $0 < \alpha < 1$. Then

$$\lim_{n\to\infty} l_n(\boldsymbol{X}_n) = G^{-1}\left(\frac{1+\lambda}{2}\right) - G^{-1}\left(\frac{1-\lambda}{2}\right).$$

The following theorem is an extension of Theorem 2 of Yohai and Zamar (2004), and states the asymptotic length-robustness of the modified intervals based on sign tests.

Theorem 2.2 Suppose that F° is continuous and has a symmetric (around θ) and unimodal density. Let $0 < \alpha < 1$ and consider the sequence $\{I_n\}$ of confidence intervals $I_n = [x_{(k_n+1)}, x_{(n-k_n)})$ with k_n given by (2.7). Then the following results hold:

(i) For $0 \le \tilde{\gamma} < (1-\lambda)/2$,

$$L\{I_n, F^\circ, (\tilde{c}, \tilde{\gamma})\} = (F^\circ)^{-1}\left(\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}}\right) - (F^\circ)^{-1}\left(\frac{1-\lambda}{2\tilde{c}}\right).$$

(ii) $\tilde{\gamma}^*\{I_n, F^\circ, \tilde{c}\} = \tilde{\gamma}^*\{I_n, F^\circ\} = \dfrac{1-\lambda}{2}$.

(iii) The sequence $\{I_n\}$ has (c,γ)-robust length if and only if $c + 4\gamma < 2$.

(iv) Let $\{I_n\}$ be a sequence of confidence intervals $I_n = [A_n(\boldsymbol{X}_n), B_n(\boldsymbol{X}_n))$ such that

$$\inf_{G \in \mathcal{P}_{c,\gamma}(G_0)} P_G\{A_n(\boldsymbol{X}_n) \le G_0^{-1}(1/2) < B_n(\boldsymbol{X}_n)\} = 1 - \alpha$$

for any continuous distribution G_0. Suppose that $\lim_{n\to\infty} A_n(\boldsymbol{X}_n) = A_0$ and $\lim_{n\to\infty} B_n(\boldsymbol{X}_n) = B_0$ almost surely when the sample comes from F°. Then

$$B_0 \ge (F^\circ)^{-1}\left(\frac{1+\lambda}{2}\right) \quad \text{and} \quad A_0 \le (F^\circ)^{-1}\left(\frac{1-\lambda}{2}\right).$$

Remark 2.2

1. When $c = 1-\varepsilon$, $\gamma = \varepsilon$, $\tilde{c} = 1-\delta$ and $\tilde{\gamma} = \delta$, it is clear that Theorem 2.2 reduces to Theorem 2 of Yohai and Zamar (2004). Although Theorem 2 considers the robustness and efficiency of $\{I_n\}$ under ε-contamination neighborhoods of a symmetric and unimodal F°, Theorem 2.2 does so under γ-contamination neighborhoods of any $F \in \mathcal{F}_{c,\gamma}(F^\circ)$. Therefore the latter treats γ-contamination neighborhoods of F which does not need to be unimodal and symmetric.
2. The length breakdown point $\tilde{\gamma}^*\{I_n, F^\circ, \tilde{c}\} = \frac{1-\lambda}{2}$ does not depend on $\tilde{c}$.
3. Assertion (iv) of Theorem 2.2 shows that in the case of uncontaminated data (i.e., $\tilde{c} = 1$, $\tilde{\gamma} = 0$, and hence $\lambda = 0$), the interval I_n is efficient in that it has the smallest asymptotic length among all nonparametric (c,γ) -robust confidence intervals for the median whose upper and lower limits converge.

Tables 3 and 4 exhibit the exact minimum coverage probabilities (CP) and expected lengths (EL) for two confidence intervals $I_n = [x_{(k_n+1)}, x_{(n-k_n)})$ with k_n given by (2.7) which have approximate (c,γ)-robust coverages 0.95 and 0.90. The expected lengths are computed under two cases: uncontaminated sample (ELU) and under contaminated sample (ELC). The real contamination sizes $\tilde{c}$ and $\tilde{\gamma}$ are chosen to be equal to the design contamination sizes c and γ, where $\lambda = c + 2\gamma - 1$ and $\tilde{\gamma} = 0.05$. As shown in the proof of Theorem 2.2, the least favorable distribution in $\mathcal{P}_{c,\gamma}(F^\circ)$ is given by $(1-\gamma)F^\circ_{LR} + \gamma\delta_y$ with $y \to \pm\infty$, where F°_{LR} is defined by $F^\circ_{LR}(x) = F^\circ_L(x)$ for $x \le (F^\circ)^{-1}(\frac{1-\gamma}{2c})$, $F^\circ_{LR}(x) = \frac{1}{2}$ for $(F^\circ)^{-1}(\frac{1-\gamma}{2c}) \le x \le (F^\circ)^{-1}(1 - \frac{1-\gamma}{2c})$, $F^\circ_{LR}(x) = F^\circ_R(x)$ for $x \ge (F^\circ)^{-1}(1 - \frac{1-\gamma}{2c})$ and F°_L and F°_R are given in (2.4). The expected lengths were computed using 8000 replications and the contaminated distribution $(1-\gamma)\Phi_{LR}(x) + \gamma\Phi(x - 10000)$ was used for the calculation of ELC, where Φ denotes the standard normal distribution.

Tables 5 and 6 give the maximum asymptotic length of the nonparametric robust confidence intervals for the $(\tilde{c},\tilde{\gamma})$-neighborhoods of $F^\circ = \Phi$. The results for $(\tilde{c},\tilde{\gamma}) = (1,0)$, $(1-\tilde{\gamma},\tilde{\gamma})$ show the asymptotic length under Φ and under the least favorable contamination distribution in $\mathcal{P}_{1-\tilde{\gamma},\tilde{\gamma}}(\Phi)$, respectively, which coincide with the values in Table 4 of Yohai and Zamar (2004).

Figures 1 through 6 are the graphs of maximum asymptotic length of nonparametric robust intervals when one of λ , $\tilde{c}$ and $\tilde{\gamma}$ varies for the others fixed. They show that the maximum asymptotic length is concave in $\tilde{c}$, convex in $\tilde{\gamma}$ and nearly linear in λ. These features come from the different roles of λ , $\tilde{c}$ and $\tilde{\gamma}$.

Table 3: Coverage probability (CP) and expected length (EL) for robust confidence interval with approximate 95% coverage probability, $\lambda = c + 2\gamma - 1$, $c = \tilde{c}$, $\gamma = \tilde{\gamma} = 0.05$.

	$\lambda = 0$		$\lambda = 0.05$			$\lambda = 0.1$			$\lambda = 0.3$		
n	CP	ELU	CP	ELU	ELC	CP	ELU	ELC	CP	ELU	ELC
20	0.959	1.16	0.954	1.17	1.25	0.938	1.17	1.33	0.956	1.90	2.28
40	0.962	0.79	0.952	0.79	0.88	0.960	0.94	1.12	0.936	1.43	1.85
60	0.948	0.57	0.961	0.67	0.76	0.955	0.76	0.95	0.964	1.41	1.83
80	0.943	0.47	0.949	0.54	0.64	0.955	0.68	0.87	0.939	1.22	1.66
100	0.943	0.41	0.941	0.46	0.56	0.957	0.62	0.82	0.944	1.18	1.63
200	0.944	0.26	0.947	0.34	0.44	0.949	0.47	0.68	0.957	1.07	1.53
500	0.946	0.15	0.947	0.23	0.32	0.952	0.35	0.57	0.950	0.92	1.41
1000	0.946	0.10	0.947	0.18	0.27	0.948	0.29	0.51	0.948	0.86	1.35
2000	0.948	0.06	0.949	0.15	0.23	0.950	0.25	0.47	0.952	0.82	1.31

Table 4: Coverage probability (CP) and expected length (EL) for robust confidence interval with approximate 90% coverage probability, $\lambda = c + 2\gamma - 1$, $c = \tilde{c}$, $\gamma = \tilde{\gamma} = 0.05$.

	$\lambda = 0$		$\lambda = 0.05$			$\lambda = 0.1$			$\lambda = 0.3$		
n	CP	ELU	CP	ELU	ELC	CP	ELU	ELC	CP	ELU	ELC
20	0.885	0.87	0.876	0.87	0.95	0.938	1.17	1.33	0.882	1.52	1.92
40	0.919	0.65	0.904	0.64	0.74	0.922	0.79	0.97	0.879	1.25	1.70
60	0.908	0.49	0.883	0.48	0.58	0.923	0.67	0.86	0.890	1.18	1.62
80	0.907	0.41	0.918	0.47	0.57	0.891	0.54	0.73	0.903	1.14	1.59
100	0.911	0.36	0.912	0.41	0.51	0.904	0.51	0.71	0.915	1.11	1.57
200	0.896	0.22	0.908	0.29	0.38	0.912	0.41	0.62	0.897	0.97	1.44
500	0.902	0.13	0.895	0.19	0.28	0.904	0.31	0.53	0.898	0.88	1.36
1000	0.906	0.08	0.903	0.16	0.24	0.904	0.27	0.48	0.902	0.83	1.32
2000	0.897	0.05	0.899	0.13	0.21	0.900	0.24	0.45	0.902	0.79	1.29

Table 5: Maximum asymptotic length of nonparametric robust intervals, $F^\circ = \Phi$.

	$\lambda = 0.05$					
	$\tilde{\gamma}$					
$\tilde{c}$	0.00	0.01	0.05	0.10	0.15	0.20
0.85	—	—	—	—	0.151	0.310
0.90	—	—	—	0.141	0.286	0.439
0.95	—	—	0.132	0.267	0.407	0.555
0.99	—	0.127	0.229	0.359	0.495	0.640
1.00	0.125	0.151	0.252	0.381	0.516	0.660
1.10	0.343	0.367	0.461	0.582	0.709	0.846
1.20	0.528	0.550	0.638	0.753	0.874	1.006

Table 6: Maximum asymptotic length of nonparametric robust intervals, $F^\circ = \Phi$.

	$\lambda = 0.15$					
	$\tilde{\gamma}$					
$\tilde{c}$	0.00	0.01	0.05	0.10	0.15	0.20
0.85	—	—	—	—	0.458	0.629
0.90	—	—	—	0.425	0.578	0.744
0.95	—	—	0.399	0.539	0.687	0.849
0.99	—	0.382	0.487	0.623	0.768	0.926
1.00	0.378	0.404	0.508	0.643	0.787	0.945
1.10	0.578	0.601	0.699	0.826	0.963	1.114
1.20	0.748	0.771	0.863	0.984	1.116	1.261

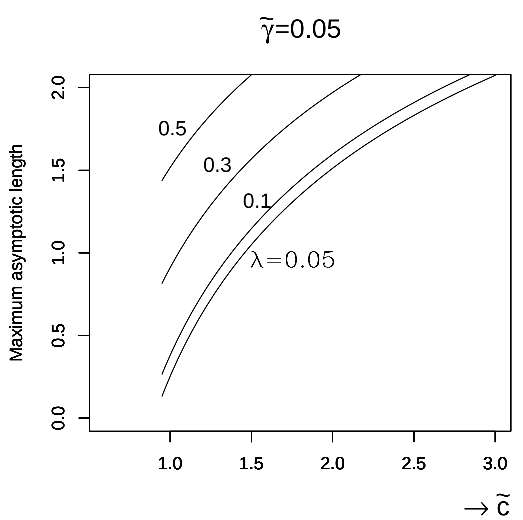

Figure 1: Maximum asymptotic length of nonparametric robust intervals for $\tilde{c}$ $(\lambda = 0.05, 0.1, 0.3, 0.5)$

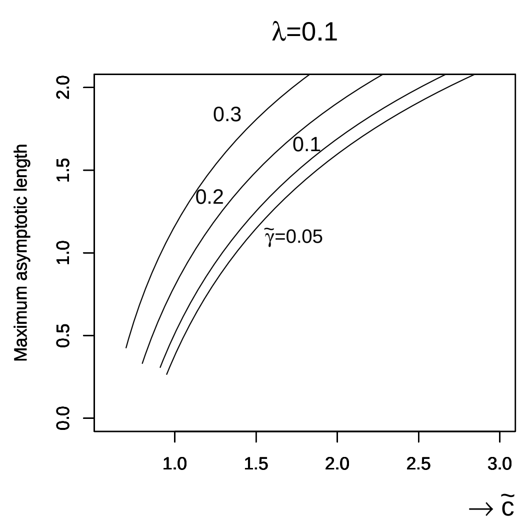

Figure 2: Maximum asymptotic length of nonparametric robust intervals for $\tilde{c}$ $(\tilde{\gamma} = 0.05, 0.1, 0.2, 0.3)$

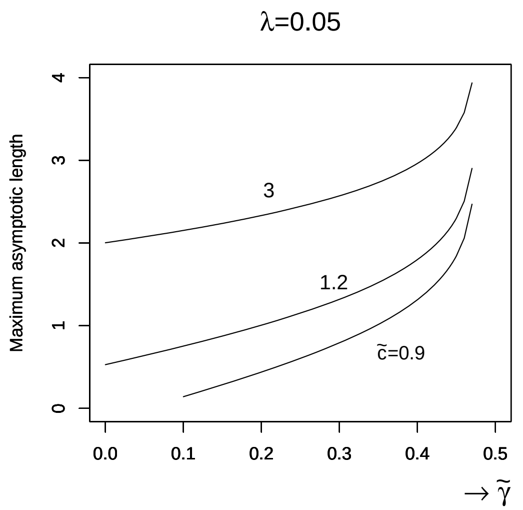

Figure 3: Maximum asymptotic length of nonparametric robust intervals for $\tilde{\gamma}$ ($\tilde{c} = 0.9, 1.2, 3$)

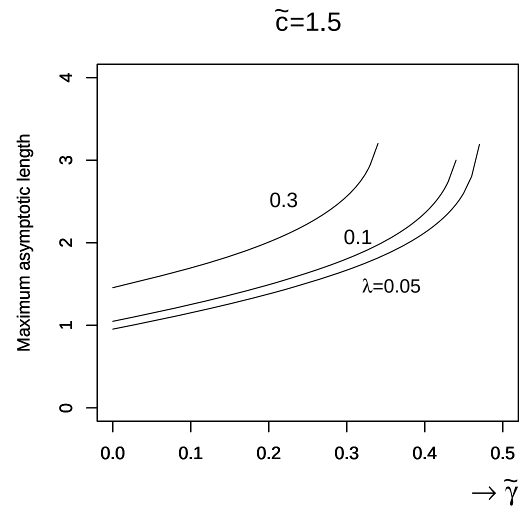

Figure 4: Maximum asymptotic length of nonparametric robust intervals for $\tilde{\gamma}$ ($\lambda = 0.05, 0.1, 0.3$)

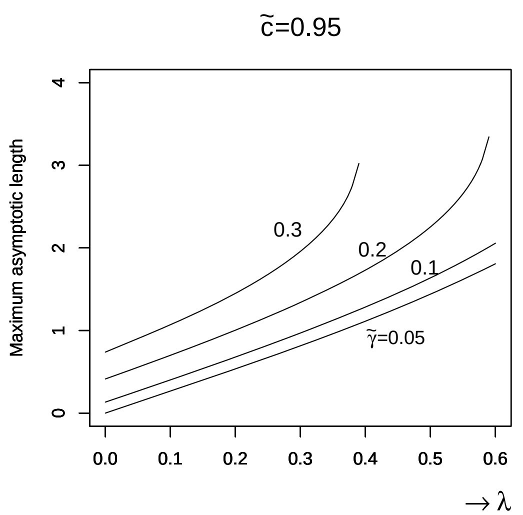

Figure 5: Maximum asymptotic length of nonparametric robust intervals for λ ($\tilde{\gamma} = 0.05, 0.1, 0.2, 0.3$)

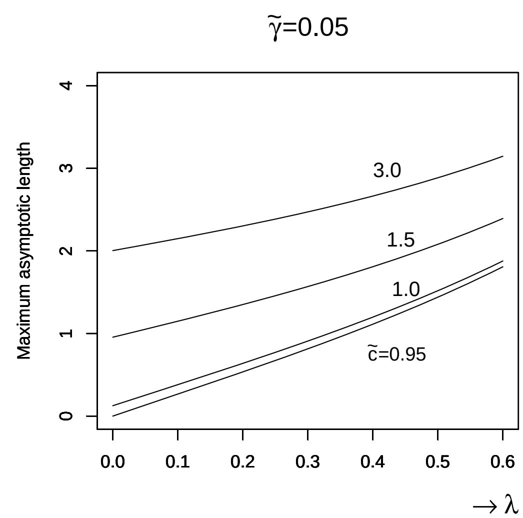

Figure 6: Maximum asymptotic length of nonparametric robust intervals for λ ($\tilde{c} = 0.95, 1.0, 1.5, 3.0$)

3 Robust nonparametric tests

Let F° be a fixed distribution satisfying A1 with $\theta = \theta_0$ and consider the problem of testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

Definition 3.1 A nonrandomized test φ_{θ_0} for H_0 versus H_1 is said to have (c, γ)-robust level α at F° if

$$\sup_{G\in\mathcal{P}_{c,\gamma}(F^\circ)} P_G\{\varphi_{\theta_0}(\boldsymbol{X}_n)=1\}=\alpha. \tag{3.1}$$

Definition 3.2 A nonrandomized test φ_{θ_0} for H_0 versus H_1 is said to have nonparametric (c,γ)-robust level α if φ_{θ_0} has (c,γ)-robust level α at F° for all F° satisfying A1 with $\theta=\theta_0$.

The following theorem is a direct consequence of Theorem 2.1.

Theorem 3.1 A nonrandomized sign test $\varphi_{\theta_0}(\boldsymbol{X}_n)$ derived from the nonparametric (c,γ)-robust interval $I(\boldsymbol{X}_n)$ in Theorem 2.1 has nonparametric (c,γ)-robust level α at F°, and is given by

$$\varphi_{\theta_0}(\boldsymbol{X}_n) = \begin{cases} 1, & \textit{if} \quad T_{n,\theta_0}(\boldsymbol{X}_n)\le k \textit{ or } T_{n,\theta_0}(\boldsymbol{X}_n)\ge n-k, \\ 0, & \textit{if} \quad k<T_{n,\theta_0}(\boldsymbol{X}_n)<n-k, \end{cases} \tag{3.2}$$

where $T_{n,\theta}(\boldsymbol{X}_n)$ is defined by (1.4) and $\alpha^*(n,k,\lambda)=\alpha$.

Definition 3.3 Let $F^\circ_\eta(x)=F^\circ(x-\eta)$. A sequence (φ_{n,θ_0}), $n\ge n_0$, of nonrandomized tests is said to have $(\tilde{c},\tilde{\gamma})$-robust power at F° if there exists a positive real number M such that

$$\inf_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ_\eta)} \lim_{n\to\infty} P_G\{\varphi_{n,\theta_0}(\boldsymbol{X}_n)=1\}=1, \quad \text{for all } |\eta|>M. \tag{3.3}$$

Definition 3.4 The $(\tilde{c},\tilde{\gamma})$-consistency distance $M\{(\varphi_{n,\theta_0}),F^\circ,(\tilde{c},\tilde{\gamma})\}$ of a sequence (φ_{n,θ_0}), $n\ge n_0$, of tests at F° is the infimum of the set of values M for which (3.3) holds.

Definition 3.5 The power breakdown point $\tilde{\gamma}^*\{(\varphi_{n,\theta_0}),F^\circ,\tilde{c}\}$ of a sequence (φ_{n,θ_0}), $n\ge n_0$ of tests at F° given $\tilde{c}$ is the supremum of the set of values $\tilde{\gamma}$ for which the sequence of tests is $(\tilde{c},\tilde{\gamma})$-robust. If $\tilde{\gamma}^*\{(\varphi_{n,\theta_0}),F^\circ,\tilde{c}\}$ does not depend on $\tilde{c}$, then it is called the power breakdown point and denoted by $\tilde{\gamma}^*\{(\varphi_{n,\theta_0}),F^\circ\}$.

The following theorem is an extension of Theorem 3 of Yohai and Zamar (2004).

Theorem 3.2 Let $0<\alpha<1$ and $0\le\tilde{\gamma}<\frac{1-\lambda}{2}$, and consider the sequence of tests (φ_{n,θ_0}), $n\ge n_0$, for $H_0:\theta=\theta_0$ versus $H_1:\theta\ne\theta_0$ given by (3.2). Suppose that F° is continuous and has a symmetric (around θ) and unimodal density. Then the following results hold:

(i) The $(\tilde{c},\tilde{\gamma})$-consistency distance for the sequence $(\varphi_{n,\theta_0}), n\ge n_0$, of tests at F° is

$$M\{(\varphi_{n,\theta_0}),F^\circ,(\tilde{c},\tilde{\gamma})\}=(F^\circ)^{-1}\left(\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}}\right).$$

(ii) The power breakdown point of the sequence $(\varphi_{n,\theta_0}), n\ge n_0$, of tests at F° is

$$\tilde{\gamma}^*\{(\varphi_{n,\theta_0}),F^\circ\}=\frac{1-\lambda}{2}.$$

(iii) The sequence $(\varphi_{n,\theta_0}), n\ge n_0$, of tests has (c,γ)-robust power at F° if and only if $c+4\gamma<2$.

4 Proofs

Proof of Proposition 1.1

Let G be any distribution in $\mathcal{P}_{c,\gamma}(F^\circ)$ of (1.3). Then, by Proposition 1.2 there exist a measure $W \in \mathcal{W}_\tau(F^\circ)$ and a distribution $K \in \mathcal{M}$ such that $G = c(F^\circ - W) + \gamma K$.

This can be written as $G = (1-\gamma)F + \gamma K$, where $F = (\frac{c}{1-\gamma})(F^\circ - W)$. Obviously, we see that $0 \le F(B) \le (\frac{c}{1-\gamma})F^\circ(B)$ holds for $\forall B \in \mathcal{B}$ and that $F(R) = 1$, which implies that F belongs to $\mathcal{F}_{c,\gamma}(F^\circ)$. Therefore G is in $\mathcal{P}_{c,\gamma}(F^\circ)$ of (1.2).

Conversely, let $G = (1-\gamma)F + \gamma K$ be any distribution in $\mathcal{P}_{c,\gamma}(F^\circ)$ of (1.2), where $F \in \mathcal{F}_{c,\gamma}(F^\circ)$ and $K \in \mathcal{M}$. Define $W = F^\circ - (\frac{1-\gamma}{c})F$. Then we have $W \in \mathcal{W}_\tau(F^\circ)$. As easily seen, it follows that $G(B) = cF^\circ(B) + \gamma$ holds for $\forall B \in \mathcal{B}$. This implies that G is in $\mathcal{P}_{c,\gamma}(F^\circ)$ of (1.3). $\square$

Proof of Theorem 2.1

For any $G \in \mathcal{P}_{c,\gamma}(F^\circ)$ we have

$$\begin{aligned} P_G(x_{(k+1)} \le \theta < x_{(n-k)}) &= P_G(k < T_{n,\theta}(\boldsymbol{X}_n) < n-k) \\ &= P(k < Z_n < n-k), \end{aligned}$$

where Z_n is distributed as Binomial $(n, 1-G(\theta))$. Since $G(\theta) = (1-\gamma)F(\theta) + \gamma K(\theta)$, $F \in \mathcal{F}_{c,\gamma}(F^\circ)$, $K \in \mathcal{M}$, it follows that

$$G(\theta) \le (1-\gamma)F_L^\circ(\theta) + \gamma = \frac{1+\lambda}{2}$$

and

$$G(\theta) \ge (1-\gamma)F_R^\circ(\theta) = \frac{1-\lambda}{2},$$

where F_L° and F_R° are given in (2.4). This implies that

$$\frac{1-\lambda}{2} \le 1 - G(\theta) \le \frac{1+\lambda}{2}.$$

Noting that

$$h(p) = \sum_{i=k}^{n-k} \binom{n}{i} p^i (1-p)^{n-i}$$

satisfies $h(p) = h(1-p)$ and is nondecreasing on $0 \le p \le \frac{1}{2}$ for all $k = 0, 1, \cdots, [n/2]$ (see Lemma 1 of Yohai and Zamar, 2004), we obtain (i) and (ii) of the theorem. $\square$

Proof of Theorem 2.2

(i) For any $K \in \mathcal{M}$ let

$$\begin{aligned} G_K(x) &= (1-\tilde{\gamma})F^\circ(x) + \tilde{\gamma}K(x), \quad x \in R, \\ G_{L,K}(x) &= (1-\tilde{\gamma})F_L^\circ(x) + \tilde{\gamma}K(x), \quad x \in R, \\ G_{R,K}(x) &= (1-\tilde{\gamma})F_R^\circ(x) + \tilde{\gamma}K(x), \quad x \in R, \end{aligned} \tag{4.1}$$

where F_L° and F_R° are given in (2.4). First we show that for any $K \in \mathcal{M}$

$$\hat{G}_{\hat{K}}^{-1}\left(\frac{1+\lambda}{2}\right) - \hat{G}_{\hat{K}}^{-1}\left(\frac{1-\lambda}{2}\right) \geq G_K^{-1}\left(\frac{1+\lambda}{2}\right) - G_K^{-1}\left(\frac{1-\lambda}{2}\right), \tag{4.2}$$

where $\hat{K}$ is defined by either (4.3) or (4.7), and $\hat{G}_{\hat{K}}$ is given by (4.4).

In order to show (4.2) we first consider Case A and then Case B (=not Case A).

Case A : There exist real numbers x_1 and $a \in \left(\frac{1-\lambda}{2}, \frac{1+\lambda}{2}\right)$ such that $G_{L,K}(x_1) = a$.

In this case, it follows from $0 \leq \tilde{\gamma} < \frac{1-\lambda}{2}$ that

$$0 < a - \tilde{\gamma} K(x_1) < 1 - \tilde{\gamma}.$$

Therefore, by the continuity of $F_R^\circ(x)$ there exists a real number $x_2 (x_2 > x_1)$ such that

$$(1-\tilde{\gamma}) F_R^\circ(x_2) + \tilde{\gamma} K(x_1) = a.$$

Let

$$\hat{K}(x) = K(x_1), \quad x \in R, \tag{4.3}$$

and let

$$\hat{G}_{\hat{K}}(x) = \begin{cases} G_{L,\hat{K}}(x), & \text{if } x \leq x_1, \\ a, & \text{if } x_1 < x < x_2, \\ G_{R,\hat{K}}(x), & \text{if } x \geq x_2, \end{cases} \tag{4.4}$$

where $G_{L,\hat{K}}$ and $G_{R,\hat{K}}$ are $G_{L,K}$ and $G_{R,K}$ with K replaced by $\hat{K}$. Then for any $K \in \mathcal{M}$ we have

$$\begin{array}{ll} \hat{K}(x) \geq K(x), & \text{if } x \leq x_1, \\ \hat{K}(x) \leq K(x), & \text{if } x \geq x_2, \end{array} \tag{4.5}$$

and hence

$$\begin{array}{ll} \hat{G}_{\hat{K}}(x) \geq G_K(x), & \text{if } x \leq x_1, \\ \hat{G}_{\hat{K}}(x) \leq G_K(x), & \text{if } x \geq x_2. \end{array} \tag{4.6}$$

Note that $\hat{K}$ is the distribution with mass $K(x_1)$ and $1-K(x_1)$ at $-\infty$ and ∞, respectively, and hence that $\hat{G}_{\hat{K}}(x)$ is continuous and strictly increasing on $(-\infty, x_1) \cup (x_2, \infty)$.

Since $\hat{G}_{\hat{K}}^{-1}(\frac{1-\lambda}{2}) < x_1$ and $\hat{G}_{\hat{K}}^{-1}(\frac{1+\lambda}{2}) > x_2$, it follows from (4.6) that $\hat{G}_{\hat{K}}^{-1}(\frac{1-\lambda}{2}) \leq G_K^{-1}(\frac{1-\lambda}{2})$ and $\hat{G}_{\hat{K}}^{-1}(\frac{1+\lambda}{2}) \geq G_K^{-1}(\frac{1+\lambda}{2})$. This implies (4.2).

Case B: There exists a real number x_1 such that

$$G_{L,K}(x_1 - 0) \leq \frac{1-\lambda}{2} < \frac{1+\lambda}{2} \leq G_{L,K}(x_1).$$

In this case, there exist real numbers $a \in \left(\frac{1-\lambda}{2}, \frac{1+\lambda}{2}\right)$ and $\xi \in (0,1)$ such that

$$G_{L,K}(x_1 - 0) + \xi\tilde{\gamma}(K(x_1) - K(x_1 - 0)) = a,$$

that is,

$$(1-\tilde{\gamma})F_L^\circ(x_1) + \tilde{\gamma}\{(1-\xi)K(x_1 - 0) + \xi K(x_1)\} = a.$$

Since it follows from $0 \le \tilde{\gamma} < \frac{1-\lambda}{2}$ that

$$a - \tilde{\gamma}\{(1-\xi)K(x_1 - 0) + \xi K(x_1)\} < 1 - \tilde{\gamma},$$

there exists a real number x_2 ($> x_1$) such that

$$(1-\tilde{\gamma})F_R^\circ(x_2) + \tilde{\gamma}\{(1-\xi)K(x_1 - 0) + \xi K(x_1)\} = a.$$

Let

$$\hat{K}(x) = (1-\xi)K(x_1 - 0) + \xi K(x_1) \tag{4.7}$$

and define $\hat{G}_{\hat{K}}$ by (4.4). Then for any $K \in \mathcal{M}$ we have

$$\begin{array}{ll} \hat{K}(x) \ge K(x), & \text{if } x < x_1, \\ \hat{K}(x) < K(x), & \text{if } x \ge x_2, \end{array} \tag{4.8}$$

and hence

$$\begin{array}{ll} \hat{G}_{\hat{K}}(x) \ge G_K(x), & \text{if } x < x_1, \\ \hat{G}_{\hat{K}}(x) < G_K(x), & \text{if } x \ge x_2. \end{array} \tag{4.9}$$

Since $\hat{G}_{\hat{K}}^{-1}(\frac{1-\lambda}{2}) < x_1$ and $\hat{G}_{\hat{K}}^{-1}(\frac{1+\lambda}{2}) > x_2$, it follows from (4.9) that $\hat{G}_{\hat{K}}^{-1}(\frac{1-\lambda}{2}) \le G_K^{-1}(\frac{1-\lambda}{2})$ and $\hat{G}_{\hat{K}}^{-1}(\frac{1+\lambda}{2}) \ge G_K^{-1}(\frac{1+\lambda}{2})$. This implies (4.2).

Next, let $\eta = \eta(K) = K(x_1)$ in Case A, and $\eta = \eta(K) = (1-\xi)K(x_1 - 0) + \xi K(x_1)$ in Case B. Then we have

$$\begin{array}{rcl} G_{L,\hat{K}}(x) &=& \tilde{c}F^\circ(x) + \tilde{\gamma}\eta, \\ G_{R,\hat{K}}(x) &=& \tilde{c}F^\circ(x) - (\tilde{c} + \tilde{\gamma} - 1) + \tilde{\gamma}\eta. \end{array} \tag{4.10}$$

Therefore it follows that

$$\begin{aligned} &\hat{G}_{\hat{K}}^{-1}\left(\frac{1+\lambda}{2}\right) - \hat{G}_{\hat{K}}^{-1}\left(\frac{1-\lambda}{2}\right) \\ &\qquad = (F^\circ)^{-1}\left(\frac{\lambda - 1 + 2\tilde{\gamma}(1-\eta) + 2\tilde{c}}{2\tilde{c}}\right) - (F^\circ)^{-1}\left(\frac{1 - \lambda - 2\tilde{\gamma}\eta}{2\tilde{c}}\right). \end{aligned} \tag{4.11}$$

We denote the right hand side of (4.11) by $g(\eta)$, i.e.,

$$g(\eta) = (F^\circ)^{-1}\left(\frac{\lambda - 1 + 2\tilde{\gamma}(1-\eta) + 2\tilde{c}}{2\tilde{c}}\right) - (F^\circ)^{-1}\left(\frac{1 - \lambda - 2\tilde{\gamma}\eta}{2\tilde{c}}\right)$$

and find the maximum value of $g(\eta)$ on $[0,1]$.

Differentiating $g(\eta)$ with respect to η, we have

$$g'(\eta) = -\frac{\tilde{\gamma}}{\tilde{c}}\left(\frac{1}{f_0\left((F^\circ)^{-1}\left(\frac{\lambda-1+2\tilde{\gamma}(1-\eta)+2\tilde{c}}{2\tilde{c}}\right)\right)} - \frac{1}{f_0\left((F^\circ)^{-1}\left(\frac{1-\lambda-2\tilde{\gamma}\eta}{2\tilde{c}}\right)\right)}\right).$$

It is easy to see that

$$\frac{\lambda-1+2\tilde{\gamma}(1-\eta)+2\tilde{c}}{2\tilde{c}} = \frac{1}{2} + \frac{\tilde{c}+\lambda-1}{2\tilde{c}} + \frac{\tilde{\gamma}}{\tilde{c}}(1-\eta)$$

and

$$\frac{1-\lambda-2\tilde{\gamma}\eta}{2\tilde{c}} = \frac{1}{2} - \left(\frac{\tilde{c}+\lambda-1}{2\tilde{c}} + \frac{\tilde{\gamma}}{\tilde{c}}\eta\right).$$

Since $f_0(x)$ is unimodal and symmetric about 0, the sign of $g'(\eta)$ is determined by the sizes of $\frac{\tilde{\gamma}}{\tilde{c}}(1-\eta)$ and $\frac{\tilde{\gamma}}{\tilde{c}}\eta$, that is, $g'(\eta) <,=,> 0$ according as $\eta <,=,> \frac{1}{2}$. Therefore the maximum value of $g(\eta)$ is attained at η=0, 1, and it is given by

$$g(0) = (F^\circ)^{-1}\left(\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}}\right) - (F^\circ)^{-1}\left(\frac{1-\lambda}{2\tilde{c}}\right) = g(1).$$

Let $\hat{G}_{\delta_m}$ be defined by (4.4) with $\hat{K}$ replaced by δ_m, where δ_m denotes the point mass distribution at m . Then it is easy to see that $\hat{G}_{\delta_m} \in \mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ)$ and

$$\lim_{m\to\infty}\{\hat{G}_{\delta_m}^{-1}\left(\frac{1+\lambda}{2}\right) - \hat{G}_{\delta_m}^{-1}\left(\frac{1-\lambda}{2}\right)\} = (F^\circ)^{-1}\left(\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}}\right) - (F^\circ)^{-1}\left(\frac{1-\lambda}{2\tilde{c}}\right).$$

This completes the proof of (i).

(ii) From (i) it follows that $L\{I_n, F^\circ, (\tilde{c},\tilde{\gamma})\} < \infty$ is equivalent to $\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}} < 1$, that is, $\tilde{\gamma} < \frac{1-\lambda}{2}$, which implies the assertion (ii).

(iii) When $\tilde{\gamma} = \gamma$, the equivalence of $\tilde{\gamma} < \frac{1-\lambda}{2}$ and $c + 4\gamma < 2$ follows from $\lambda = c + 2\gamma - 1$.

(iv) To show $(F^\circ)^{-1}(\frac{1+\lambda}{2}) \le B_0$, let

$$F_R^*(x) = \begin{cases} 0, & \text{if } x < (F^\circ)^{-1}(\gamma), \\ \frac{F^\circ(x)-\gamma}{1-\gamma}, & \text{if } x \ge (F^\circ)^{-1}(\gamma), \end{cases}$$

and

$$H_L^*(x) = \begin{cases} \frac{F^\circ(x)}{\gamma}, & \text{if } x < (F^\circ)^{-1}(\gamma), \\ 1, & \text{if } x \ge (F^\circ)^{-1}(\gamma). \end{cases}$$

Also, let

$$G_0(x) = \left(\frac{1-\gamma}{c}\right) F_R^*(x) + \left(1 - \frac{1-\gamma}{c}\right) U_{(M,M+1)}(x)$$

where $U_{(M,M+1)}$ denotes the uniform distribution on $(M, M+1)$ with a constant M satisfying $M > (F^\circ)^{-1}(\frac{1+\lambda}{2})$. Then, it is clear that G_0 is a continuous distribution and $F_R^* \in \mathcal{F}_{c,\gamma}(G_0)$. Since $F^\circ = (1-\gamma)F_R^* + \gamma H_L^*$, we have $F^\circ \in \mathcal{P}_{c,\gamma}(G_0)$. Let $m = G_0^{-1}(\frac{1}{2})$. Then, from $G_0(m) = \frac{F^\circ(m)-\gamma}{c}$ it follows that $F^\circ(m) = \frac{1+\lambda}{2}$. By the assumptions of (iv), this implies $(F^\circ)^{-1}(\frac{1+\lambda}{2}) = G_0^{-1}(\frac{1}{2}) \in [A_0, B_0]$ and hence $(F^\circ)^{-1}(\frac{1+\lambda}{2}) \le B_0$.

On the other hand, to show $(F^\circ)^{-1}(\frac{1-\lambda}{2}) \ge A_0$, let

$$F_L^*(x) = \begin{cases} \frac{F^\circ(x)}{1-\gamma}, & \text{if } x < (F^\circ)^{-1}(1-\gamma), \\ 1, & \text{if } x \ge (F^\circ)^{-1}(1-\gamma), \end{cases}$$

and

$$H_R^*(x) = \begin{cases} 0, & \text{if } x < (F^\circ)^{-1}(1-\gamma), \\ \frac{F^\circ(x)-(1-\gamma)}{\gamma}, & \text{if } x \ge (F^\circ)^{-1}(1-\gamma). \end{cases}$$

Also, let

$$G_0(x) = \left(\frac{1-\gamma}{c}\right) F_L^*(x) + \left(1 - \frac{1-\gamma}{c}\right) U_{(M-1,M)}(x),$$

where M is a constant satisfying $M < (F^\circ)^{-1}(\frac{1-\lambda}{2})$. Then, it is clear that G_0 is a continuous distribution and $F_L^* \in \mathcal{F}_{c,\gamma}(G_0)$. Since $F^\circ = (1-\gamma)F_L^* + \gamma H_R^*$, we have $F^\circ \in \mathcal{P}_{c,\gamma}(G_0)$. Let $m = G_0^{-1}(\frac{1}{2})$. Then, from $G_0(m) = \frac{F^\circ(m)+c+\gamma-1}{c}$, it follows that $F^\circ(m) = \frac{1-\lambda}{2}$. This implies $(F^\circ)^{-1}(\frac{1-\lambda}{2}) = G_0^{-1}(\frac{1}{2}) \in [A_0, B_0]$ and hence $(F^\circ)^{-1}(\frac{1-\lambda}{2}) \ge A_0$. $\square$

Proof of Theorem 3.2

We assume $\theta = 0$ without loss of generality. First we note that

$$P_G\{\varphi_{n,0}(\boldsymbol{X}_n) = 1\} = P_G\{0 \notin [x_{(k_n)}, x_{(n-k_n)})\}.$$

Since, by lemma 2 we have $x_{(k_n)} \to G^{-1}(\frac{1-\lambda}{2})$ and $x_{(n-k_n)} \to G^{-1}(\frac{1+\lambda}{2})$, it follows that

$$\lim_{n\to\infty} P_G\{\varphi_{n,0}(\boldsymbol{X}_n) = 1\} = \begin{cases} 1, & \text{if } G^{-1}(\frac{1-\lambda}{2}) > 0, \\ 0, & \text{if } G^{-1}(\frac{1-\lambda}{2}) < 0 < G^{-1}(\frac{1+\lambda}{2}), \\ 1, & \text{if } G^{-1}(\frac{1+\lambda}{2}) < 0. \end{cases}$$

Then

$$\inf_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F_\eta^\circ)} \lim_{n\to\infty} P_G\{\varphi_{n,\theta_0}(\boldsymbol{X}_n) = 1\} = 1, \quad \text{for all } |\eta| > M,$$

holds either if

$$\sup_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F_\eta^\circ)} G^{-1}\left(\frac{1+\lambda}{2}\right) = \eta + \sup_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1+\lambda}{2}\right) < 0, \tag{4.12}$$

or

$$\inf_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ_\eta)} G^{-1}\left(\frac{1-\lambda}{2}\right) = \eta + \inf_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1-\lambda}{2}\right) > 0, \tag{4.13}$$

It is easily seen that

$$\sup_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1+\lambda}{2}\right) = (G^\circ_R)^{-1}\left(\frac{1+\lambda}{2}\right) = (F^\circ)^{-1}\left(\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}}\right),$$

and

$$\begin{aligned}\inf_{G\in\mathcal{P}_{\tilde{c},\tilde{\gamma}}(F^\circ)} G^{-1}\left(\frac{1-\lambda}{2}\right) = (G^\circ_L)^{-1}\left(\frac{1-\lambda}{2}\right) &= (F^\circ)^{-1}\left(\frac{1-\lambda-2\tilde{\gamma}}{2\tilde{c}}\right)\\ &= -(F^\circ)^{-1}\left(\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}}\right),\end{aligned}$$

where

$$\begin{aligned} G^\circ_R(x) &= \max\{0, cF^\circ(x)-(c+\gamma-1)\}, \quad x\in R,\\ G^\circ_L(x) &= \min\{cF^\circ(x)+\gamma, 1\}, \quad x\in R.\end{aligned}$$

Therefore, we obtain (4.12) or (4.13) if

$$|\eta| > (F^\circ)^{-1}\left(\frac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}}\right).$$

This implies the assertion (i). The assertion (ii) follows from the assertion (i) and the inequality $\dfrac{1+\lambda+2(\tilde{c}+\tilde{\gamma}-1)}{2\tilde{c}} < 1$. The assertion (iii) is obtained from the assertion (i) and the substitution of $\tilde{\gamma}=\gamma$ into the above inequality. □

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