NANZAN-TR-2004-07

On the Maximum Asymptotic Bias of Robust Regression Estimates over Certain Contamination Neighborhoods

Masakazu Ando and Miyoshi Kimura

March 2005

Technical Report of the Nanzan Academic Society
Mathematical Sciences and Information Engineering

On the Maximum Asymptotic Bias of Robust Regression Estimates over Certain Contamination Neighborhoods

Masakazu Ando [1] and Miyoshi Kimura [2]
JSPS Research Fellow and Nanzan University

Abstract

A certain broad class of robust regression estimates in the linear model are considered. Lower and upper bounds for the maximum asymptotic bias of the regression estimates over the neighborhoods (called (c, γ)-neighborhoods) defined by certain special capacities are derived without imposing zero-intercept and elliptical regressors. The (c, γ)-neighborhoods are a generalization of Rieder's (ε, δ)-neighborhoods and include ε-contamination and total variation neighborhoods, as special cases. In the case of Gaussian regressors, the lower and upper bounds for the maximum asymptotic bias of τ-estimates are obtained. Tables of their upper bounds are also given for three (Huber, Tukey, Deninis-Welsh) score functions .

AMS 2000 Subject classifications: Primary 62F35, Secondary 62H12, 62J05
Key words: maximum asymptotic bias, robust regression estimates, S-estimates, τ-estimates, contamination neighborhoods

1. Introduction

The maximum asymptotic bias $B_T(\varepsilon)$ of an estimate T when the underlying distribution ranges over the ε-contamination neighborhood of some central model distribution, was first introduced by Huber (1964) for the location model. The $B_T(\varepsilon)$ is one of the most informative global quantitative measures to assess robustness of T, because $B_T(\varepsilon)$ shows the whole performance of T from $\varepsilon = 0$ (the central model distribution) to the breakdown point and under some regularity conditions its derivative $B_T(0)'$ equals the gross error sensitivity as a local robustness measure. Huber (1964) established that the median minimizes $B_T(\varepsilon)$ among translation equivariant location estimates. Martin and Zamer (1989, 1993) obtained minimax bias robust scale estimates. Adrover (1998) derived minimax bias robust dispersion matrix estimates.

As for the linear regression model, in the case of the zero-intercept and elliptical regressors, Martin, Yohai and Zamer (1989) obtained the minimax bias estimates in the classes of M-estimates with general scale and GM- estimates of regression. In particular, they showed that the least median of square estimate (LMS) introduced by Rousseeuw (1984) is nearly minimax. Yohai and Zamer (1993) extended this result to the larger

[1] Research Fellow of the Japan Society for the Promotion of Science. Graduate School of Economics, Nagoya City University, 1 Yamanohata, Mizuho-cho, Mizuho-ku, Nagoya 467-8501, JAPAN. E-mail address: andomasa@econ.nagoya-cu.ac.jp.

[2] Department of Mathematical Sciences, Nanzan University, 27 Seirei-cho, Seto, Aichi, 489-0863, JAPAN. E-mail address: kimura@ms.nanzan-u.ac.jp

class of residual admissible estimates. Berrendero and Zamer (2001) obtained maximum asymptotic bias of robust regression estimates in a broad class, which includes S-estimates, τ-estimates, R-estimates and so on, without requiring zero-intercept and/or elliptical regressors. Their results argues against criticisms that the maxbias theory applies only to regression models with the zero-intercept and elliptical regressors. All the authors mentioned above adopt ε-contamination neighborhoods to express deviation from the central model.

On the other hand, Ando and Kimura (2003) introduced a neighborhood, called a (c,γ)-neighborhood, to express deviation from the central model. The (c,γ)-neighborhood is defined by a certain special capacity and includes ε-contamination, total variation and Rieder's (1977) (ε,δ)-neighborhoods as special cases. They characterized the (c,γ)-neighborhoods and gave their applications to bias-robustness of estimates. Among them, there are the extensions of Huber's (1964) and He and Simpson's (1993) results. The former states that the median minimizes the maximum asymptotic bias $B_T(c,\gamma)$ over (c,γ)-neighborhoods among translation equivariant location estimates. Ando and Kimura (2004) derived the lower and upper bounds for $B_S(c,\gamma)$ of regression S-estimates over (c,γ)-neighborhoods in the zero-intercept linear model with elliptical regressors. In the case of Rieder's neighborhood, the lower and upper bounds coincide and become $B_S(c,\gamma)$.

In this paper, following Berrendero and Zamer (2001), without imposing the zero-intercept and/or elliptical regressors , we derive the lower and upper bounds for $B_T(c,\gamma)$ of estimates in the larger class. In the case of ε-contamination neighborhoods, the lower and upper bounds coincide and this result is reduced to Theorem 1 of Berrendero and Zamar (2001). As an important special case, we obtain the lower and upper bounds for the maximum asymptotic bias $B_\tau(c,\gamma)$ of τ-estimates under Gaussian regressors. We also give some tables of the upper bounds for τ-estimates based on three (Huber, Tukey, Dennis-Welsch) score functions . The use of the characterization of the (c,γ)-neighborhoods is indispensable to the derivation of our results in the paper.

The paper is organized as follows. Section 2 presents basic definitions and preliminary results. Section 3 gives auxiliary results for verifying the main theorem. Section 4 derives the lower and upper bounds for $B_\tau(c,\gamma)$ which is the main result. Section 5 obtains the lower and upper bounds for $B_\tau(c,\gamma)$ of τ-estimates under Gaussian error and regressors, and evaluates the upper bounds for three types of τ-estimates. Their tables are also given.

2. Preliminaries

We consider the linear regression model

$$y \quad = \quad \alpha_0 + \boldsymbol{\theta}_0'\boldsymbol{x} + u,$$

where $\boldsymbol{x} = (x_1,\ldots,x_p)'$ is a random vector in R^p, $\boldsymbol{\theta}_0 = (\theta_{10},\ldots,\theta_{p0})'$ is the vector in R^p of the true regression parameters, α_0 is the true intercept parameter in R and the error u is a random variable independent of $\boldsymbol{x}$. Let F_0 be the nominal distribution function of u and G_0 the nominal distribution function of $\boldsymbol{x}$. Then the nominal distribution function H_0 of $(y,\boldsymbol{x})$ is

$$H_0(y,\boldsymbol{x}) = \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_p} F_0(y-\alpha_0-\boldsymbol{\theta}_0'\boldsymbol{s})dG_0(\boldsymbol{s}). \tag{2.1}$$

Let $\mathcal{M}$ be the set of all distribution functions H on $(R^{p+1}, \mathcal{B}^{p+1})$, where $\mathcal{B}^{p+1}$ is the Borel σ-field on R^{p+1}. Let $\boldsymbol{T}$ be a R^p-valued functional defined on $\mathcal{M}$. Given a sample of independent observations $(y_1, \boldsymbol{x}_1), \cdots, (y_n, \boldsymbol{x}_n)$ of size n from H, we define the corresponding estimate of $\boldsymbol{\theta}_0$ as $\boldsymbol{T}(H_n)$, where H_n is the empirical distribution of the sample.

The asymptotic bias of $\boldsymbol{T}$ at H is defined by

$$b_{\boldsymbol{A}}(\boldsymbol{T}, H) = [(\boldsymbol{T}(H) - \boldsymbol{\theta}_0)' \boldsymbol{A} (\boldsymbol{T}(H) - \boldsymbol{\theta}_0)]^{\frac{1}{2}},$$

where $\boldsymbol{A}$ is an affine equivariant covariance functional of $\boldsymbol{x}$ under G_0. Since we only work with regression and affine equivariant estimates and $b_{\boldsymbol{A}}(\boldsymbol{T}, H)$ is invariant under regression and affine equivariant transformations, we can assume without loss of generality that $\boldsymbol{\theta}_0 = 0$ and $\boldsymbol{A} = \boldsymbol{I}_p$ (the identity matrix). Therefore the asymptotic bias $b_{\boldsymbol{A}}(\boldsymbol{T}, H)$ is given by

$$b(\boldsymbol{T}, H) = \|\boldsymbol{T}(H)\|, \tag{2.2}$$

where $\|\cdot\|$ denotes the Euclidean norm. We assume that $\boldsymbol{T}$ is Fisher consistent at H_0, i.e., $\boldsymbol{T}(H_0) = \boldsymbol{0}$.

In order to express deviation from the nominal distribution H_0 we adopt the following neighborhood of H_0 introduced by Ando and Kimura (2003):

$$\mathcal{P}_{H_0}(c, \gamma) = \{H \in \mathcal{M} \ : \ H(B) \leq c\, H_0(B) + \gamma, {}^{\forall} B \in \mathcal{B}^{p+1}\}, \tag{2.3}$$

where $0 \leq \gamma < 1$ and $1 - \gamma \leq c < \infty$. Note that $H_0(H)$ is used as both a distribution function and a probability measure for convenience. The neighborhood $\mathcal{P}_{H_0}(c, \gamma)$, which is called a (c, γ)-neighborhood, is a generalization of ε-contamination and total variation neighborhoods: Let ε and δ be some given constants such that $\varepsilon \geq 0, \delta \geq 0$ and $\varepsilon + \delta < 1$. Then we have the ε-contamination neighborhood $\mathcal{P}_{H_0}(1 - \varepsilon, \varepsilon)$ for $c = 1 - \varepsilon$ and $\gamma = \varepsilon$, the total variation neighborhood $\mathcal{P}_{H_0}(1, \delta)$ for $c = 1$ and $\gamma = \delta$, and Rieder's (1977) (ε, δ)-neighborhood $\mathcal{P}_{H_0}(1 - \varepsilon,\ \varepsilon + \delta)$ for $c = 1 - \varepsilon$ and $\gamma = \varepsilon + \delta$. We should notice that $\mathcal{P}_{H_0}(c, \gamma)$ is also generated by a special capacity (see Bednarski, 1981). Ando and Kimura (2003) gives the following useful characterization of $\mathcal{P}_{H_0}(c, \gamma)$.

Proposition 2.1. *For $0 \leq \gamma < 1$ and $1 - \gamma \leq c < \infty$ it holds that*

$$\mathcal{P}_{H_0}(c,\ \gamma) = \{H = c\,(H_0 - W) + \gamma\, K \ : \ W \in \mathcal{W}_{H_0, \lambda},\ K \in \mathcal{M}\},$$

where $\mathcal{W}_{H_0, \lambda}$ is the set of all measures W such that $W(B) \leq H_0(B)$ holds for ${}^{\forall} B \in \mathcal{B}^{p+1}$ and $W(R^{p+1}) = \lambda = (c + \gamma - 1)/c$.

Corollary 2.1. *For $\varepsilon \geq 0$, $\delta \geq 0$ and $\varepsilon + \delta < 1$ it holds that*

$$\mathcal{P}_{H_0}(1 - \varepsilon, \varepsilon + \delta) = \{H = (1 - \varepsilon)\,(H_0 - W) + (\varepsilon + \delta)\, K \ : \ W \in \mathcal{W}_{H_0, \lambda},\ K \in \mathcal{M}\},$$

where $\lambda = \delta/(1 - \varepsilon)$.

The maximum asymptotic bias of $\boldsymbol{T}$ over $\mathcal{P}_{H_0}(c,\gamma)$ is defined as

$$B_{\boldsymbol{T}}(c,\gamma) = \sup\{\|\boldsymbol{T}(H)\| \;:\; H \in \mathcal{P}_{H_0}(c,\gamma)\}. \tag{2.4}$$

We consider the following class of robust estimates defined as

$$(T_0(H), \boldsymbol{T}(H)) = \arg\min_{\alpha,\boldsymbol{\theta}} J(F_{H,\alpha,\boldsymbol{\theta}}), \tag{2.5}$$

where $J(\cdot)$ is a robust loss functional defined on the set of all distributions on the real line and $F_{H,\alpha,\theta}$ is the distribution of the absolute residual $r(\alpha,\boldsymbol{\theta}) = |y - \alpha - \boldsymbol{\theta}'\boldsymbol{x}|$ under H. This class of estimates includes the well-known robust estimates such as S-estimates, τ-estimates and R-estimates. We assume that J, F_0 and G_0 satisfy the following conditions A1 and A2 corresponding to Berrendero and Zamar (2001).

Let $\mathcal{L}^+$ be the set of all distributions F on $[0,\infty)$ and let $\mathcal{L}_c^+$ be the subset of $\mathcal{L}^+$ whose elements are continuous on $(0,\infty)$.

A1. (a) J is weakly continuous.

(b) Let $F \in \mathcal{L}^+$ and $G \in \mathcal{L}^+$. If $F(v) \le G(v)$ ($F(v) < G(v)$) for every $v \ge 0$, then $J(F) \ge J(G)$ ($J(F) > J(G)$).

(c) Let $\{F_n\}$ and $\{G_n\}$ be sequences of $F_n \in \mathcal{L}_c^+$ and $G_n \in \mathcal{L}_c^+$ ($n = 1, 2, \cdots$) such that $F_n(v) \to F(v)$ and $G_n(v) \to G(v)$, where F and G are possibly substochastic and continuous on $(0,\infty)$ with $G(\infty) \ge 1-\gamma$. If $G(v) \ge F(v)$ ($G(v) > F(v)$) for every $v > 0$, then $\lim_{n\to\infty} J(F_n) \ge \lim_{n\to\infty} J(G_n)$ ($\lim_{n\to\infty} J(F_n) > \lim_{n\to\infty} J(G_n)$).

(d) If $F \in \mathcal{L}_c^+$ and $G \in \mathcal{L}^+$, then

$$J((1-\gamma)\,F + \gamma\delta_\infty) = \lim_{n\to\infty} J((1-\gamma)\,F + \gamma U_n) \ge J((1-\gamma)\,F + \gamma G),$$

where U_n stands for the uniform distribution function on $[n - \frac{1}{n}, n + \frac{1}{n}]$.

A2. F_0 has an even and strictly unimodal density f_0 with $f_0(v) > 0$ for every $v \in R$, and $P_{G_0}(\boldsymbol{\theta}'\boldsymbol{x} = a) < 1$, for every $\boldsymbol{\theta} \in R^p$ ($\boldsymbol{\theta} \ne 0$) and $a \in R$.

Remark 2.1 The continuity condition A1(a) is not restrictive. In fact, J corresponding to S-,τ- and R-estimates satisfy A1(a). The ε-monotonicity condition A1(c) guarantees that the corresponding estimate $\boldsymbol{T}$ is residual admissible (see Yohai and Zamar, 1993, for the definition of residual admissible estimates). We should emphasize that A2 does not require ellipticity nor continuity of regressor's distribution.

3. Auxiliary results

Let $\xi_\lambda = \{W_{\alpha,\boldsymbol{\theta},\lambda} : \alpha \in R, \boldsymbol{\theta} \in R^p\}$ be the family of $W_{\alpha,\boldsymbol{\theta},\lambda} \in \mathcal{W}_{H_0,\lambda}$ such that for any $\tilde{\alpha} \in R$ and $\tilde{\boldsymbol{\theta}} \in R^p$,

$$\begin{aligned}(3.1)\qquad \lim_{(\alpha,\boldsymbol{\theta})\to(\tilde{\alpha},\tilde{\boldsymbol{\theta}})} &(H_0 - W_{\alpha,\boldsymbol{\theta},\lambda})(|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}| \le v) \\ &= (H_0 - W_{\tilde{\alpha},\tilde{\boldsymbol{\theta}},\lambda})(|y-\tilde{\alpha}-\tilde{\boldsymbol{\theta}}'\boldsymbol{x}| \le v), \quad {}^\forall v \ge 0.\end{aligned}$$

We let $\mathcal{F}_\lambda$ be the set of all such ξ_λ. In what follows, for simplicity we omit the subscript λ of ξ_λ and $W_{\alpha,\boldsymbol{\theta},\lambda}$.

In order to establish an upper bound we need $\hat{\xi} = \{\hat{W}_{\alpha,\boldsymbol{\theta}}\}$ and $\xi^* = \{W^*_{\alpha,\boldsymbol{\theta}}\}$ defined as follows:

$$(3.2)\qquad \begin{aligned}\hat{W}_{\alpha,\boldsymbol{\theta}}(B) &= H_0\left(B \cap \left\{|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}| \ge a_{\alpha,\boldsymbol{\theta}}\left(\tfrac{c+\gamma-1}{c}\right)\right\}\right), \quad {}^\forall B \in \mathcal{B}^{p+1}, \\ W^*_{\alpha,\boldsymbol{\theta}}(B) &= H_0\left(B \cap \left\{|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}| \le a_{\alpha,\boldsymbol{\theta}}\left(\tfrac{1-\gamma}{c}\right)\right\}\right), \quad {}^\forall B \in \mathcal{B}^{p+1},\end{aligned}$$

where $a_{\alpha,\boldsymbol{\theta}}(\eta)$ $(0 \le \eta < 1)$ denotes the upper $100\eta\%$ point of the distribution of $|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}|$ under H_0 such that

$$H_0\left(|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}| \ge a_{\alpha,\boldsymbol{\theta}}(\eta)\right) = \eta.$$

It is easy to see that $\hat{\xi}$ and ξ^* belong to $\mathcal{F}_\lambda$.

For any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ let

$$(3.3)\qquad F^\xi_{\alpha,\boldsymbol{\theta}}(v) = (H_0 - W_{\alpha,\boldsymbol{\theta}})(|y-\alpha-\boldsymbol{\theta}'\boldsymbol{x}| \le v), \quad {}^\forall v \ge 0.$$

We note that $F^\xi_{\alpha,\boldsymbol{\theta}}$ is used as both function and measure on $(R, \mathcal{B})$. Let

$$(3.4)\qquad d_\xi = J(c\,F^\xi_{0,\mathbf{0}} + \gamma\delta_\infty) \ \text{ and } \ m_\xi(t) = \inf_{\|\boldsymbol{\theta}\|=t}\inf_{\alpha\in R} J(c\,F^\xi_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0),$$

where δ_0 and δ_∞ are the point mass distributions at 0 and ∞, respectively.

Lemma 3.1. *Under A1(c) and A2, for any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ there exists $\alpha(\boldsymbol{\theta}) \in R$ such that*

$$J(c\,F^\xi_{\alpha(\boldsymbol{\theta}),\boldsymbol{\theta}} + \gamma\delta_0) = \inf_{\alpha\in R} J(c\,F^\xi_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0).$$

Moreover, for any $t > 0$ there exists $K_t > 0$ such that $|\alpha(\boldsymbol{\theta})| \le K_t$ for every $\boldsymbol{\theta} \in \{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| = t\}$.

Proof. First we note that by A1(a) and A2 $J(c\,F^{\xi}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0)$ is a continuous function of α and $\boldsymbol{\theta}$. Since, for any $v > 0$, $\lim_{|\alpha|\to\infty} F^{\xi}_{\alpha,\boldsymbol{\theta}}(v) < F^{\xi}_{0,\boldsymbol{\theta}}(v)$, it also follows from A1(c) that

$$\lim_{|\alpha|\to\infty} J(c\,F^{\xi}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0) > J(c\,F^{\xi}_{0,\boldsymbol{\theta}} + \gamma\delta_0).$$

Therefore, for any $\boldsymbol{\theta} \in R^p$ there exists $K_{\boldsymbol{\theta}}$ such that the infimum is attained in the compact set $[-K_{\boldsymbol{\theta}}, K_{\boldsymbol{\theta}}]$. Denoting by $\alpha(\boldsymbol{\theta})$ the value of α which gives the infimum (= the minimum), we obtain the first assertion of the lemma. We note that $\alpha(\boldsymbol{\theta})$ and K_t depend on ξ.

Assume that the second assertion of the lemma is not true. Then, there exist some $t > 0$ and a sequence $\{\boldsymbol{\theta}_n\} \in \{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| = t\}$ such that $\lim_{n\to\infty} |\alpha(\boldsymbol{\theta}_n)| = \infty$. Suppose without loss of generality that $\boldsymbol{\theta}_n \to \tilde{\boldsymbol{\theta}}$. For any $\alpha > 0$ and $v > 0$ we have

$$\lim_{n\to\infty}[c\,F^{\xi}_{\alpha(\boldsymbol{\theta}_n),\boldsymbol{\theta}_n}(v) + \gamma\delta_0(v)] = \gamma \le c\,F^{\xi}_{\alpha,\tilde{\boldsymbol{\theta}}}(v) + \gamma\delta_0(v).$$

Hence

$$\lim_{n\to\infty} J(c\,F^{\xi}_{\alpha(\boldsymbol{\theta}_n),\boldsymbol{\theta}_n} + \gamma\delta_0) \;\ge\; J(c\,F^{\xi}_{\alpha,\tilde{\boldsymbol{\theta}}} + \gamma\delta_0). \tag{3.5}$$

On the other hand, the definition of $\alpha(\boldsymbol{\theta})$ implies that for any $\alpha \in R$,

$$\lim_{n\to\infty} J(c\,F^{\xi}_{\alpha(\boldsymbol{\theta}_n),\boldsymbol{\theta}_n} + \gamma\delta_0) \;\le\; J(c\,F^{\xi}_{\alpha,\tilde{\boldsymbol{\theta}}} + \gamma\delta_0). \tag{3.6}$$

It follows from (3.5) and (3.6) that $J(c\,F^{\xi}_{\alpha,\tilde{\boldsymbol{\theta}}} + \gamma\delta_0)$ does not depend on α. This contradicts $\lim_{|\alpha|\to\infty} J(c\,F^{\xi}_{\alpha,\tilde{\boldsymbol{\theta}}} + \gamma\delta_0) > J(c\,F^{\xi}_{0,\tilde{\boldsymbol{\theta}}} + \gamma\delta_0)$, which implies the second assertion. □

Let $\mathcal{F}_{1\lambda}$ be the set of all $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ such that $F^{\xi}_{k\alpha,k\boldsymbol{\theta}}(v)$ is strictly decreasing in $\lambda > 0$ for $0 < F^{\xi}_{k\alpha,k\boldsymbol{\theta}}(v) < (1-\gamma)/c$. The next lemma shows that $\hat{\xi} = \{\hat{W}_{\alpha,\boldsymbol{\theta}}\}$ and $\xi^* = \{W^*_{\alpha,\boldsymbol{\theta}}\}$ belong to $\mathcal{F}_{1\lambda}$.

Lemma 3.2. *Under A2, $F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}(v)$ and $F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}(v)$ are strictly decreasing in $k > 0$ for $0 < F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}(v) < (1-\gamma)/c$ and $0 < F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}(v) < (1-\gamma)/c$, respectively.*

Proof. We note that $F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}$ and $F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}$ are expressed in the form of

$$F^{\hat{\xi}}_{k\alpha,k\boldsymbol{\theta}}(v) = \min\left(F_{H_0,k\alpha,k\boldsymbol{\theta}}(v),\ \frac{1-\gamma}{c}\right),\quad {}^{\forall}v \ge 0,$$

and

$$F^{\xi^*}_{k\alpha,k\boldsymbol{\theta}}(v) = \max\left(F_{H_0,k\alpha,k\boldsymbol{\theta}}(v) - \frac{c+\gamma-1}{c}, 0\right), \quad {}^{\forall}v \geq 0,$$

where $F_{H_0,k\alpha,k\boldsymbol{\theta}}(v)$ is the distribution function of $|y - k\alpha - k\boldsymbol{\theta}'\boldsymbol{x}|$ under H_0. By Lemma 5 of Berrendero and Zamar (2001), $F_{H_0,k\alpha,k\boldsymbol{\theta}}(v)$ is strictly decreasing in $k > 0$. Therefore, $F^{\hat{\xi}}_{\alpha,\boldsymbol{\theta}}(v)$ and $F^{\xi^*}_{\alpha,\boldsymbol{\theta}}(v)$ are strictly decreasing in $k > 0$. $\Box$

Lemma 3.3. *Let $m_\xi(t)$ be as in (3.4). Then, under A1(c) and A2, for any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_{1\lambda}$ the following results hold:*

(a) *There exist $\boldsymbol{\theta}_t \in R^p$ and $\alpha(\boldsymbol{\theta}_t) \in R$ such that $\|\boldsymbol{\theta}_t\| = t$ and $m_\xi(t) = J(c\,F^{\xi}_{\alpha(\boldsymbol{\theta}_t),\boldsymbol{\theta}_t} + \gamma\delta_0)$.*

(b) *$m_\xi(t)$ is strictly increasing.*

Proof. By Lemma 3.1, we have

$$m_\xi(t) = \inf_{\|\boldsymbol{\theta}\|=t} M_\xi(\boldsymbol{\theta}) = \inf_{\|\boldsymbol{\theta}\|=t} \inf_{[-K_t,K_t]} J(c\,F^{\xi}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0),$$

where $J(c\,F^{\xi}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0)$ is uniformly continuous on the compact set $\{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| = t\} \times [-K_t, K_t]$. Therefore, $M_\xi(\boldsymbol{\theta})$ is continuous on the compact set $\{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| = t\}$ and there exists $\|\boldsymbol{\theta}_t\| = t$ such that $M_\xi(\boldsymbol{\theta}_t) = \inf_{\|\boldsymbol{\theta}\|=t} M_\xi(\boldsymbol{\theta})$. This implies the assertion (a).

To show the assertion (b) let t_1 and t_2 be such that $t_1 > t_2$. Define $k = t_2/t_1 < 1$. Applying the assertion (a), there exist $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ such that $m_\xi(t_1) = M_\xi(\boldsymbol{\theta}_1)$ and $m_\xi(t_2) = M_\xi(\boldsymbol{\theta}_2)$. Since, by $\xi \in \mathcal{F}_{1\lambda}$

$$F^{\xi}_{\alpha(\boldsymbol{\theta}_1),\boldsymbol{\theta}_1}(v) < F^{\xi}_{k\alpha(\boldsymbol{\theta}_1),k\boldsymbol{\theta}_1}(v),$$

it follows from A1(b) and the definition of $\alpha(\boldsymbol{\theta})$ that

$$m_\xi(t_1) > J(c\,F^{\xi}_{k\alpha(\boldsymbol{\theta}_1),k\boldsymbol{\theta}_1} + \gamma\delta_0) \geq J(c\,F^{\xi}_{\alpha(k\boldsymbol{\theta}_1),k\boldsymbol{\theta}_1} + \gamma\delta_0). \tag{3.7}$$

Also, by the definition of $m_\xi(t)$ and $\|k\boldsymbol{\theta}_1\| = t_2$

$$m_\xi(t_2) \leq M_\xi(k\boldsymbol{\theta}_1) = J(c\,F^{\xi}_{\alpha(k\boldsymbol{\theta}_1),k\boldsymbol{\theta}_1} + \gamma\delta_0). \tag{3.8}$$

The inequalities (3.7) and (3.8) imply the assertion (b). $\Box$

The following lemma states that $m_{\hat{\xi}}(t)$ is simplified under symmetry and unimodality assumptions on the regressors distribution.

Lemma 3.4. *Assume A1 and A2, and that under G_0 the distribution of $\boldsymbol{\theta}'\boldsymbol{x}$ is symmetric, unimodal and only depends on $\|\boldsymbol{\theta}\|$ for all $\boldsymbol{\theta} \neq \mathbf{0}$. Then, it holds that*

$$\inf_{\alpha \in R} J(c\, F^{\hat{\xi}}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0) = J(c\, F^{\hat{\xi}}_{0,\boldsymbol{\theta}} + \gamma\delta_0) = m_{\hat{\xi}}(\|\boldsymbol{\theta}\|).$$

Proof. It is easy to check that

$$F^{\hat{\xi}}_{\alpha,\boldsymbol{\theta}}(v) = (H_0 - \hat{W}_{\alpha,\boldsymbol{\theta}})(-v + \alpha \leq y - \boldsymbol{\theta}'\boldsymbol{x} \leq v + \alpha), \quad {}^\forall v > 0.$$

By the symmetry and unimodality assumptions on F_0 and G_0 and the definition of $\hat{W}_{\alpha,\boldsymbol{\theta}}$, we have for all $\alpha \in R$,

$$F^{\hat{\xi}}_{\alpha,\boldsymbol{\theta}}(v) \;\; \leq \;\; (H_0 - \hat{W}_{0,\boldsymbol{\theta}})(-v \leq y - \boldsymbol{\theta}'\boldsymbol{x} \leq v) = F^{\hat{\xi}}_{0,\boldsymbol{\theta}}(v), \quad {}^\forall v > 0,$$

and therefore, from A1(b), it follows that

$$J(c\, F^{\hat{\xi}}_{\alpha,\boldsymbol{\theta}} + \gamma\delta_0) \;\; \geq \;\; J(c\, F^{\hat{\xi}}_{0,\boldsymbol{\theta}} + \gamma\delta_0), \quad {}^\forall \alpha \in R.$$

This implies the first equality of the lemma. It is easy to see that $J(c\, F^{\hat{\xi}}_{0,\boldsymbol{\theta}} + \gamma\delta_0)$ only depends on $\boldsymbol{\theta}$ through the value of $\|\boldsymbol{\theta}\|$, because $F^{\hat{\xi}}_{0,\boldsymbol{\theta}}$ is so. $\square$

4. Main results

Let $\mathcal{F}_{2\lambda}$ be the set of all $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_{1\lambda}$ such that for any $\alpha \in R$ and $\boldsymbol{\theta} \in R^p$ it holds that

$$0 < F^{\xi}_{\alpha,\boldsymbol{\theta}}(v) \leq F^{\xi}_{0,\mathbf{0}}(v), \quad {}^\forall v > 0. \tag{4.1}$$

It is easy to check that $\hat{\xi}$ belongs to $\mathcal{F}_{2\lambda}$. Another example of elements of $\mathcal{F}_{2\lambda}$ is $\xi^\circ = \{W^\circ_{\alpha,\boldsymbol{\theta}}\}$ defined as $W^\circ_{\alpha,\boldsymbol{\theta}} = [(c + \gamma - 1)/c]H_0$, ${}^\forall \alpha \in R, {}^\forall \boldsymbol{\theta} \in R^p$. In this case, we note $c\,(H_0 - W^\circ_{\alpha,\boldsymbol{\theta}}) = (1 - \gamma)H_0$. The following theorem gives lower and upper bounds for the maximum asymptotic bias $B_{\boldsymbol{T}}(c, \gamma)$.

Theorem 4.1. *Let $\boldsymbol{T}$ be a regression estimate defined by (2.5). Then*

$$\underline{B}_{\boldsymbol{T}}(c,\, \gamma) \leq B_{\boldsymbol{T}}(c,\, \gamma) \leq \overline{B}_{\boldsymbol{T}}(c,\, \gamma),$$

where

$$\overline{B}_{\boldsymbol{T}}(c,\, \gamma) = m^{-1}_{\hat{\xi}}(d_{\xi^*}) \quad \textit{and} \quad \underline{B}_{\boldsymbol{T}}(c,\, \gamma) = \sup_{\xi \in \mathcal{F}_{2\lambda}} m^{-1}_{\xi}(d_{\xi}).$$

Proof. Let t^* be such that $d_{\xi^*} = m_{\hat{\xi}}(t^*)$. First, we show $B_{\boldsymbol{T}}(c,\gamma) \le t^*$. Let $\tilde{\boldsymbol{\theta}} \in R^p$ be such that $\|\tilde{\boldsymbol{\theta}}\| = t > t^*$. It is enough to show that for any $H \in \mathcal{P}_{H_0}(c,\gamma)$ and any $\alpha \in R$ we have

$$J(F_{H,\alpha,\tilde{\boldsymbol{\theta}}}) > J(F_{H,0,\mathbf{0}}). \tag{4.2}$$

It is clear that for any $H = c\,(H_0 - W) + \gamma K \in \mathcal{P}_{H_0}(c,\gamma)$, $\alpha \in R$ and $v > 0$,

$$F_{H,\alpha,\tilde{\boldsymbol{\theta}}}(v) \;=\; c\,F^{\xi}_{\alpha,\tilde{\boldsymbol{\theta}}}(v) + \gamma F_{K,\alpha,\tilde{\boldsymbol{\theta}}}(v) \le c\,F^{\hat{\xi}}_{\alpha,\tilde{\boldsymbol{\theta}}}(v) + \gamma\delta_0(v), \tag{4.3}$$

where $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ is defined as $W_{\alpha,\boldsymbol{\theta}} = W$ for any $\alpha \in R$ and $\boldsymbol{\theta} \in R^p$. From (4.3), A1(b), the definition of $m_\xi(t)$ and Lemma 3.3 (b) it follows that for any $H \in \mathcal{P}_{H_0}(c,\gamma)$

$$J(F_{H,\alpha,\tilde{\boldsymbol{\theta}}}) \;\ge\; J(c\,F^{\hat{\xi}}_{\alpha,\tilde{\boldsymbol{\theta}}} + \gamma\delta_0) \ge m_{\hat{\xi}}(t) > m_{\hat{\xi}}(t^*). \tag{4.4}$$

The condition $d_{\xi^*} = m_{\hat{\xi}}(t^*)$ and A1(d) imply

$$m_{\hat{\xi}}(t^*) = \lim_{n\to\infty} J(c\,F^{\xi^*}_{0,\mathbf{0}} + \gamma U_n) \ge \lim_{n\to\infty} J(c\,F^{\xi}_{0,\mathbf{0}} + \gamma U_n) \ge J(F_{H,0,\mathbf{0}}). \tag{4.5}$$

Noting $t^* = m_{\hat{\xi}}^{-1}(d_{\xi^*})$, we obtain $B_{\boldsymbol{T}}(c,\gamma) \le \overline{B}_{\boldsymbol{T}}(c,\gamma)$ from (4.4) and (4.5).

Next, we show $B_{\boldsymbol{T}}(c,\gamma) \ge m_\xi^{-1}(d_\xi)$, $^\forall\xi \in \mathcal{F}_{2\lambda}$. Let $t_1 = m_\xi^{-1}(d_\xi)$ and let $t < t_1$. We find a distribution $H \in \mathcal{P}_{H_0}(c,\gamma)$ such that $\|\boldsymbol{T}(H)\| \ge t$. By Lemma 3.3(a), there exist $\boldsymbol{\theta}_t$ and α_t such that $m_\xi(t) = J(c\,F^{\xi}_{\alpha_t,\boldsymbol{\theta}_t} + \gamma\delta_0)$. Define $\tilde{H}_n = \delta_{(y_n,\boldsymbol{x}_n)}$, where $\boldsymbol{x}_n = n\boldsymbol{\theta}_t$ and y_n is uniformly distributed on the interval $[\alpha_t + nt^2 - \frac{1}{n}, \alpha_t + nt^2 + \frac{1}{n}]$. If F_n is the uniform distribution function on $[-\frac{1}{n}, \frac{1}{n}]$, then for any $\boldsymbol{\beta} \in R^p$, $v > 0$ and $\alpha \in R$

$$\begin{aligned} F_{\tilde{H}_n,\alpha,\boldsymbol{\beta}}(v) \;=\;& F_n(v + \alpha - \alpha_t - n(t^2 - \boldsymbol{\beta}'\boldsymbol{\theta}_t)) \\ & - F_n(-v + \alpha - \alpha_t - n(t^2 - \boldsymbol{\beta}'\boldsymbol{\theta}_t)). \end{aligned} \tag{4.6}$$

For any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_{2\lambda}$ let $H_n^\xi(\alpha,\boldsymbol{\theta}) = c\,(H_0 - W_{\alpha,\boldsymbol{\theta}}) + \gamma\tilde{H}_n \in \mathcal{P}_{H_0}(c,\gamma)$. Suppose that $\sup_{n,\alpha,\boldsymbol{\theta}} \|\boldsymbol{T}(H_n^\xi(\alpha,\boldsymbol{\theta}))\| < t$ to find a contradiction. Then, for any $\alpha \in R$ and $\boldsymbol{\theta} \in R^p$ there exists a convergent subsequence, $\{\boldsymbol{T}(H_n^\xi(\alpha,\boldsymbol{\theta}))\}$, such that

$$\lim_{n\to\infty} \boldsymbol{T}(H_n^\xi(\alpha,\boldsymbol{\theta})) = \lim_{n\to\infty} \boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta}) = \tilde{\boldsymbol{\theta}}^\xi(\alpha,\boldsymbol{\theta}), \quad \textit{where} \quad \|\tilde{\boldsymbol{\theta}}^\xi(\alpha,\boldsymbol{\theta})\| = \tilde{t}^\xi(\alpha,\boldsymbol{\theta}) < t.$$

Since $t^2 - \boldsymbol{\theta}_t'\boldsymbol{\theta}_t = 0$, it follows from (4.6) that

$$\lim_{n\to\infty} F_{\tilde{H}_n,\alpha_t,\boldsymbol{\theta}_t}(v) = 1, \quad ^\forall v > 0. \tag{4.7}$$

We show that for any $\alpha \in R$ and $\boldsymbol{\theta} \in R^p$ the subsequence of intercepts corresponding to $\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})$, denoted by $\{T_0(H_n^\xi(\alpha,\boldsymbol{\theta}))\} = \{\alpha_n^\xi(\alpha,\boldsymbol{\theta})\}$ converges to a finite $\hat{\alpha}^\xi(\alpha,\boldsymbol{\theta})$. To do this, assume $\lim_{n\to\infty} |\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*)| = \infty$ for some $\alpha^* \in R$ and $\boldsymbol{\theta}^* \in R^p$. Then, it follows from (4.7) that

$$
\begin{aligned}
(4.8)\qquad & \lim_{n\to\infty} F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*)}(v) \\
&\quad = r \lim_{n\to\infty} F_{\tilde{H}_n,\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*)}(v) \\
&\quad < c\, F_{(H_0-W_{\alpha^*,\boldsymbol{\theta}^*}),\alpha_t,\boldsymbol{\theta}_t}(v) + \gamma\delta_0(v) \\
&\quad = \lim_{n\to\infty} F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_t,\boldsymbol{\theta}_t}(v), \qquad {}^\forall v > 0.
\end{aligned}
$$

Hence, by A1(c) we have

$$
J(F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*)}) > J(F_{H_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\alpha_t,\boldsymbol{\theta}_t})
$$

for large enough n. This fact contradicts the definition of $(\alpha_n^\xi(\alpha^*,\boldsymbol{\theta}^*),\boldsymbol{\theta}_n^\xi(\alpha^*,\boldsymbol{\theta}^*))$. Therefore, for any α and $\boldsymbol{\theta}$ we have $\lim_{n\to\infty}|\alpha_n^\xi(\alpha,\boldsymbol{\theta})| = \tilde{\alpha}^\xi(\alpha,\boldsymbol{\theta}) < \infty$. Since $t^2 - |\boldsymbol{\theta}_t'\tilde{\boldsymbol{\theta}}^\xi(\alpha,\boldsymbol{\theta})| = t^2 - t\tilde{t}^\xi(\alpha,\boldsymbol{\theta}) > 0$, it follows from (4.6) that

$$
(4.9)\qquad \lim_{n\to\infty} F_{\tilde{H}_n,\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}(v) = 0, \quad {}^\forall v > 0.
$$

Hence, by (4.9) and $\xi \in \mathcal{F}_{2\lambda}$ we have

$$
\begin{aligned}
(4.10)\qquad \lim_{n\to\infty} F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}(v) &= c\, F^\xi_{\tilde{\alpha}^\xi(\alpha,\boldsymbol{\theta}),\tilde{\boldsymbol{\theta}}^\xi(\alpha,\boldsymbol{\theta})} \\
&\le c\, F^\xi_{0,\mathbf{0}}(v) \\
&= \lim_{n\to\infty}[c\, F^\xi_{0,\mathbf{0}}(v) + \gamma U_n(v)], \quad {}^\forall v > 0.
\end{aligned}
$$

By A1(c) and A1(d) we have

$$
\begin{aligned}
(4.11)\qquad \lim_{n\to\infty} J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}) &\ge \lim_{n\to\infty} J(c\, F^\xi_{0,\mathbf{0}} + \gamma U_n) \\
&= d_\xi = m_\xi(t_1).
\end{aligned}
$$

From (4.7) it follows that

$$
(4.12)\qquad \lim_{n\to\infty} F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_t,\boldsymbol{\theta}_t}(v) = c\, F^\xi_{\alpha_t,\boldsymbol{\theta}_t}(v) + \gamma\delta_0(v)
$$

The equation (4.12) and Lemma 3.3(b) imply

$$
(4.13)\qquad \lim_{n\to\infty} J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_t,\boldsymbol{\theta}_t}) = J(c\, F^\xi_{\alpha_t,\boldsymbol{\theta}_t} + \gamma\delta_0) = m_\xi(t) < m_\xi(t_1).
$$

By (4.11) and (4.13), we have

$$
J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})}) > J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\alpha_t,\boldsymbol{\theta}_t})
$$

for large enough n. This inequality is a contradiction because of $(\alpha_n^\xi(\alpha,\boldsymbol{\theta}),\boldsymbol{\theta}_n^\xi(\alpha,\boldsymbol{\theta})) = \arg\min_{\eta,\boldsymbol{\beta}} J(F_{H_n^\xi(\alpha,\boldsymbol{\theta}),\eta,\boldsymbol{\beta}})$. Thus, for any $t < t_1$ we obtain $\sup_{n,\alpha,\boldsymbol{\theta}} \|\boldsymbol{T}(H_n^\xi(\alpha,\boldsymbol{\theta}))\| \ge t$. This completes the proof. □

Remark 4.1. When $c = 1-\varepsilon$ and $\gamma = \varepsilon$ (i.e., the ε-contamination case), we have $\lambda = 0$ and $\hat{\xi} = \xi^*$. Therefore Theorem 4.1 is reduced to Theorem 1 of Berrendero and Zamar (2001).

5. S- and τ-estimates in the normal distribution case

We consider S-estimates and τ-estimates in the case that H_0 is the multivariate normal distribution $N(\mathbf{0}, \boldsymbol{I}_{p+1})$ with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{I}_{p+1}$. We denote by ϕ the density of the standard normal distribution Φ. For any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_\lambda$ let $\varphi^{\xi}_{\alpha,\boldsymbol{\theta}}$ denote the density of $F^{\xi}_{\alpha,\boldsymbol{\theta}}$. Let $\mathcal{F}_{3\lambda}$ be the set of all $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_{2\lambda}$ such that $\varphi^{\xi}_{0,\boldsymbol{\theta}}$ is expressed in the form of

$$\varphi^{\xi}_{0,\boldsymbol{\theta}}(v) = \frac{1}{\sqrt{1+\|\boldsymbol{\theta}\|^2}}\phi_\xi\left(\frac{v}{\sqrt{1+\|\boldsymbol{\theta}\|^2}}\right), \quad {}^\forall v \geq 0,$$

where ϕ_ξ ($0 \leq \phi_\xi \leq 2\phi$) is some measurable function such that

$$\int_0^\infty \phi_\xi(v)dv = \frac{1-\gamma}{c}.$$

It is easy to see that $\hat{\xi} = \{\hat{W}_{\alpha,\boldsymbol{\theta}}\}$ and $\xi^\circ = \{W^\circ_{\alpha,\boldsymbol{\theta}}\}$ belong to $\mathcal{F}_{3\lambda}$.

Let χ_1 and χ_2 be score functions satisfying the following conditions:

A3. (a) The functions χ_1 and χ_2 are even, bounded, monotone on $[0, \infty)$, continuous at 0 with $0 = \chi_i(0) < \chi_i(\infty) = 1$, $i = 1, 2$ and with at most a finite number of discontinuities.

(b) The function χ_2 is differentiable with $2\chi_2(v) - \chi_2'(v)v \geq 0$.

The S-estimate (Rousseeuw and Yohai,1984) is defined with $J(F) = S(F)$, where

(5.1) $$S(F) \;=\; \inf\left\{s > 0 \;:\; E_F\left[\chi_1\left(\frac{y}{s}\right)\right] \leq b\right\}, \quad 0 < b < 1.$$

For any $\xi = \{W_{\alpha,\boldsymbol{\theta}}\} \in \mathcal{F}_{3\lambda}$ let

$$g_{\xi,i}(s) = E_{F^{\xi}_{0,\mathbf{0}}}\chi_i\left(\frac{y}{s}\right) = \int_0^\infty \chi_i\left(\frac{y}{s}\right)\varphi^{\xi}_{0,\mathbf{0}}(y)dy, \; i = 1, 2.$$

The following theorem gives the lower and upper bounds for the maximum asymptotic bias $B_S(c,\gamma)$ of S-estimates based on χ_1.

Theorem 5.1. *Assume that the nominal distribution H_0 is $N(\mathbf{0}, \boldsymbol{I}_{p+1})$. Then*

$$\underline{B}_S(c,\,\gamma) \leq B_S(c,\,\gamma) \leq \overline{B}_S(c,\,\gamma), \qquad \text{if} \;\; \gamma < \min(b, 1-b),$$

$$B_S(c,\,\gamma) = \infty, \qquad \text{if} \;\; \gamma \geq \min(b, 1-b),$$

where

(5.2) $$\overline{B}_S(c,\,\gamma) \;=\; \sqrt{\left\{g^{-1}_{\xi^*,1}\left(\frac{b-\gamma}{c}\right)\Big/ g^{-1}_{\hat{\xi},1}\left(\frac{b}{c}\right)\right\}^2 - 1}$$

and

$$(5.3) \qquad \underline{B}_S(c,\gamma) \;=\; \sup_{\xi\in\mathcal{F}_{3\lambda}} \sqrt{\left\{ g_{\xi,1}^{-1}\left(\frac{b-\gamma}{c}\right) \Big/ g_{\xi,1}^{-1}\left(\frac{b}{c}\right)\right\}^2 - 1}.$$

Proof. It follows from (5.1) and Lemma 3.4 that

$$d_{\xi^*} = S(c\,F_{0,\mathbf{0}}^{\xi^*} + \gamma\delta_\infty) = g_{\xi^*,1}^{-1}\left(\frac{b-\gamma}{c}\right)$$

and

$$m_{\hat{\xi},S}(\|\boldsymbol{\theta}\|) = S(c\,F_{0,\boldsymbol{\theta}}^{\hat{\xi}} + \gamma\delta_0) = \sqrt{1+\|\boldsymbol{\theta}\|^2}\, g_{\hat{\xi},1}^{-1}\left(\frac{b}{c}\right).$$

Hence, solving $m_{\hat{\xi},S}(\|\boldsymbol{\theta}\|) = d_{\xi^*}$ in $\|\boldsymbol{\theta}\|$, we obtain (5.2). Similarly, we can obtain (5.3). Assume $b \le 0.5$. Then we have $\min(b, 1-b) = b$,

$$\lim_{\gamma\,\uparrow\,b} g_{\xi^*,1}^{-1}\left(\frac{b-\gamma}{c}\right) = \infty \quad\text{and}\quad \lim_{\gamma\,\uparrow\,b} g_{\xi^\circ,1}^{-1}\left(\frac{b-\gamma}{c}\right) = \infty.$$

Therefore, $\lim_{\gamma\,\uparrow\,b} \overline{B}_S(c,\gamma) = \lim_{\gamma\,\uparrow\,b} \underline{B}_S(c,\gamma) = \infty$. This completes the proof. $\square$

The τ-estimate (Yohai and Zamar, 1988) is defined with $J(F) = \tau^2(F)$, where

$$\tau^2(F) = S^2(F) E_F \chi_2\left(\frac{v}{S(F)}\right).$$

As shown in Yohai and Zamar (1988), τ-estimates inherit the breakdown point of the initial S-estimate defined by χ_1 and their efficiencies are mainly determined by χ_2.

The following theorem gives the lower and upper bounds for the maximum asymptotic bias $B_\tau(c,\gamma)$ of τ-estimates which shows how $B_\tau(c,\gamma)$ relates to the maximum asymptotic bias $B_S(c,\gamma)$ of the initial S-estimates based on χ_1.

Theorem 5.2. *Assume that the nominal distribution H_0 is $N(\mathbf{0}, \boldsymbol{I}_{p+1})$. Then*

$$\underline{B}_\tau(c,\gamma) \le B_\tau(c,\gamma) \le \overline{B}_\tau(c,\gamma),$$

where

$$(5.4) \qquad \overline{B}_\tau(c,\gamma) \;=\; \{[1+\overline{B}_S^2(c,\gamma)] H_{\xi^*,\hat{\xi}}(c,\gamma) - 1\}^{1/2},$$

$$(5.5) \qquad \underline{B}_\tau(c,\gamma) \;=\; \sup_{\xi\in\mathcal{F}_{3\lambda}} \left\{ \left[\frac{g_{\xi,1}^{-1}\left(\frac{b-\gamma}{c}\right)}{g_{\xi,1}^{-1}\left(\frac{b}{c}\right)}\right]^2 H_{\xi,\xi}(c,\gamma) - 1\right\}^{1/2},$$

$$H_{\xi_1,\xi_2}(c,\gamma) = \left[\overline{g}_{\xi_1}\left(\frac{b-\gamma}{c}\right) + \frac{\gamma}{c}\right] \Big/ \overline{g}_{\xi_2}\left(\frac{b}{c}\right) \quad \textit{and} \quad \overline{g}_\xi(s) = g_{\xi,2}[g_{\xi,1}^{-1}(s)].$$

Proof. We note that

$$\begin{aligned} d_{\xi^*} &= \tau^2(c\,F_{0,\mathbf{0}}^{\xi^*} + \gamma\delta_\infty) \\ &= \left[g_{\xi^*,1}^{-1}\left(\frac{b-\gamma}{c}\right)\right]^2 \left[c\,\overline{g}_{\xi^*}\left(\frac{b-\gamma}{c}\right) + \gamma\right] \end{aligned}$$

and that

$$\begin{aligned} m_{\hat{\xi},\tau}(\|\boldsymbol{\theta}\|) &= \tau^2(c\,F_{0,\boldsymbol{\theta}}^{\hat{\xi}} + \gamma\delta_0) \\ &= m_{\hat{\xi},S}^2(\|\boldsymbol{\theta}\|)\cdot c\,E_{F_{0,\boldsymbol{\theta}}^{\hat{\xi}}}\chi_2\left(\frac{y-\boldsymbol{\theta}'\boldsymbol{x}}{m_{\hat{\xi},S}(\|\boldsymbol{\theta}\|)}\right) \qquad (5.6) \\ &= (1+\|\boldsymbol{\theta}\|^2)\left[g_{\hat{\xi},1}^{-1}\left(\frac{b}{c}\right)\right]^2 c\,\overline{g}_{\hat{\xi}}\left(\frac{b}{c}\right). \end{aligned}$$

Solving $m_{\hat{\xi},\tau}(\|\boldsymbol{\theta}\|) = d_{\xi^*}$, we obtain.

$$\|\boldsymbol{\theta}\| = m_{\hat{\xi},\tau}^{-1}(d_{\xi^*}) = \{(1+\overline{B}_S(c,\gamma)^2)H_{\xi^*,\hat{\xi}}(c,\gamma) - 1\}^{1/2}.$$

which implies (5.4). Similarly, we can obtain (5.5). □

Remark 5.1. We can easily check that S (based on χ_1) and τ^2 (based on χ_1 and χ_2) satisfy A1(a) under A3. As pointed out in Berrendero and Zamar (2001), we can see that S and τ^2 also satisfy A1(b), A1(c) and A1(d) under A3.

Remark 5.2. The upper bound $\overline{B}_S(c,\gamma)$ in (5.2) is the same as (4.7) in Ando and Kimura (2004). Note that $h_\xi(\tau)$ in (4.7) satisfies the relation $h_\xi(\tau) = g_{\xi,1}(\frac{1}{\tau})$. We should notice that when χ_1 is a jump function, $\overline{B}_S(c,\gamma) = B_S(c,\gamma)$ holds for $c \le 1$ (see Theorem 4.1 of Ando and Kimura, 2004).

Remark 5.3. Regarding the intercept estimates, we can see the arguments in Section 7 of Berrendero and Zamar (2001). Here, we should point out that a (c,γ)-neighborhood version of their Theorem 6 is also obtained.

We give some tables of the upper bounds $\overline{B}_\tau(c,\gamma)$ for the asymptotic bias of τ-estimates based on the following three score functions χ_1 and χ_2:

(a) Huber score function:

$$\chi_H(y) = \min\{(y/c_H)^2, 1\},$$

$$\chi_1 = \chi_H \text{ with } c_H = 1.041 \quad \text{and} \quad \chi_2 = \chi_H \text{ with } c_H = 2.832.$$

(b) Tukey score function:

$$\chi_T(y) = \min\{3(y/c_T)^2 - 3(y/c_T)^4 + (y/c_T)^6, 1\},$$

$$\chi_1 = \chi_T \text{ with } c_T = 1.548 \quad \text{and} \quad \chi_2 = \chi_T \text{ with } c_T = 6.039.$$

(c) Dennis-Welsch score function:

$$\chi_{DW}(y) = 1 - \exp\{-(y/c_{DW})^2\},$$

$$\chi_1 = \chi_{DW} \text{ with } c_{DW} = 0.816 \quad \text{and} \quad \chi_2 = \chi_{DW} \text{ with } c_{DW} = 4.043.$$

For comparison we also consider the following score function.

(d) Jump score function:

$$\chi_s(y) = \begin{cases} 0, & y \le c_s, \\ 1, & y > c_s, \end{cases}$$
$$\chi_1 = \chi_2 = \chi_s \text{ with } c_s = 0.67$$

The constants c_H, c_T, c_{DW} and c_s are chosen so that the corresponding τ-estimates have 95% efficiency and 0.5 breakdown point. Note that τ-estimates are reduced to S-estimates in the case of $\chi_1 = \chi_2$. Tables 1,2 and 3 exhibit $\overline{B}_\tau(c,\gamma)$ for τ-estimates based on Huber, Tukey and Dennis-Welsch score functions, respectively. Table 4 presents $\overline{B}_S(c,\gamma)$ for the S-estimate based on the jump score function. As pointed out in Remark 5.2, we have $\overline{B}_S(c,\gamma) = B_S(c,\gamma)$ for $c \le 1$ (taking $c = 1-\varepsilon$ and $\gamma = \varepsilon + \delta$, we have Rieder's (ε,δ)-neighborhood case). In all the tables, $\overline{B}_\tau(c,\gamma)$ and $\overline{B}_S(c,\gamma)$ on the diagonal lines correspond to γ-contamination neighborhoods $\mathcal{P}_{H_0}(1-\gamma,\gamma)$ and are equal to $B_\tau(1-\gamma,\gamma)$ and $B_S(1-\gamma,\gamma)$, respectively. We can see that Huber type τ-estimate gives the smallest $\overline{B}_\tau(c,\gamma)$ among the three types of τ-estimates and Tukey type τ-estimate does the second smallest one. Although the S-estimate based on the jump score function, which has minimax bias in the class of M-estimates with general scale for the γ-contamination case, its efficiency is not high (at most 33 %, see Hössjer, 1992).

Acknowledgments. The research was supported in part by Grant-in-Aid for Scientific Research (KAKENHI(16.12163)) and Nanzan Pache Research Subsidy I-A-2.

References

[1] Adrover,J.G.(1998). Minimax bias robust estimation of the dispersion matrix of a multivariate distribution, *Ann. Statist.* **26**, 2301-2320.

[2] Ando,M. and Kimura,M.(2003). A characterization of the neighborhoods defined by certain special capacities and their applications to bias-robustness of estimates, *J. Statist. Plann. Inference.* **116**, 61-90.

[3] Ando,M. and Kimura,M.(2004). The maximum asymptotic bias of S-estimates for regression over the neighborhoods defined by certain special capacities, *J. Multivariate Anal.* **90**, 407-425.

[4] Bednarski, T.(1981). On solutions of minimax test problems for special capacities. *Z. Wakreck. verw. Gebiete.* **10**, 269-278.

[5] Berrendero,J.R. and Zamar,R.H.(2001). Maximum bias curves for robust regression with non-elliptical regressors, *Ann. Statist.* **29**, 224-251.

[6] He,X. and Simpson,D.G.(1993). Lower bounds for contamination bias: globally minimax versus locally linear estimation, *Ann. Statist.* **21**, 314-337.

[7] Hössjer,O.(1992). On the optimality of S-estimators, *Statist. Probab. Lett.* **14**, 413-419.

[8] Huber,P.J.(1964). Robust estimation of a location parameter, *Ann. Math. Statist.* **35**, 73-101.

[9] Martin,R.D., Yohai,V.J. and Zamar,R.H.(1989). Min-max bias robust regression, *Ann. Statist.* **17**, 1608-1630.

[10] Martin,R.D. and Zamar,R.H.(1989). Asymptotically min-max robust M-estimates of scale for positive random variables, *J. Amer. Statist. Assoc.* **84**, 494-501.

[11] Martin,R.D. and Zamar,R.H.(1993). Bias robust estimation of scale, *Ann. Statist.* **21**, 991-1017.

[12] Rieder,H.(1977). Least favorable pairs for special capacities. *Ann. Statist.* **6**, 1080-1094.

[13] Rousseeuw, P.J.(1984). Least median of squares regression, *J. Amer. Statist. Assoc.* **79**, 871-880.

[14] Rousseeuw, P.J. and Yohai, V.(1984). Robust regression by means of S-estimators, *Robust and Nonlinear Time Series Analysis. Lecture Notes in Statist*, **26**, Springer, New York, 256-272.

[15] Yohai,V.J. and Zamar,R.H.(1988). High breakdown-point estimates of regression by means of the minimization of an efficient scale. *J. Amer. Statist. Assoc.* **83**, 406-413.

[16] Yohai,V.J. and Zamar,R.H.(1993). A minimax-bias property of the least α-quantile estimates, *Ann. Statist.* **21**, 1824-1842.

Table 1: $\overline{B}_\tau(c,\ \gamma)$ (Huber score function)

$c \setminus \gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	20.22
0.65	—	—	—	—	—	—	—	—	—	5.05	26.03
0.75	—	—	—	—	—	—	—	—	2.63	6.54	32.16
0.80	—	—	—	—	—	—	—	2.00	3.11	7.31	35.37
0.85	—	—	—	—	—	—	1.52	2.44	3.62	8.10	38.58
0.90	—	—	—	—	—	1.10	1.92	2.87	4.13	8.90	41.94
0.95	—	—	—	—	0.71	1.46	2.27	3.29	4.64	9.70	45.14
0.97	—	—	—	0.53	0.86	1.59	2.40	3.45	4.83	10.03	46.59
0.98	—	—	0.42	0.62	0.93	1.64	2.47	3.53	4.94	10.19	47.25
0.99	—	0.29	0.53	0.70	0.99	1.70	2.54	3.61	5.04	10.36	47.87
1.00	0.00	0.42	0.61	0.76	1.05	1.76	2.60	3.69	5.14	10.52	48.68
1.10	0.84	0.97	1.11	1.24	1.52	2.29	3.25	4.48	6.14	12.19	55.40
1.20	1.20	1.34	1.48	1.63	1.93	2.80	3.88	5.28	7.15	13.90	62.45
1.50	2.08	2.28	2.47	2.68	3.10	4.30	5.79	7.70	10.22	19.15	84.29
2.00	3.47	3.77	4.07	4.38	5.02	6.84	9.06	11.87	15.56	28.42	123.01
3.00	6.29	6.83	7.37	7.91	9.05	12.18	15.95	20.68	26.90	48.12	205.18
5.00	12.32	13.39	14.44	15.48	17.67	23.70	30.79	39.68	51.21	90.43	379.09
10.00	28.94	31.37	33.82	36.35	41.38	55.18	71.56	91.73	117.22	208.27	863.63

Table 2: $\overline{B}_\tau(c,\ \gamma)$ (Tukey score function)

$c \setminus \gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	24.40
0.65	—	—	—	—	—	—	—	—	—	6.42	31.66
0.75	—	—	—	—	—	—	—	—	3.42	8.72	39.46
0.80	—	—	—	—	—	—	—	2.61	4.14	9.87	43.34
0.85	—	—	—	—	—	—	1.97	3.19	4.81	11.04	47.50
0.90	—	—	—	—	—	1.43	2.45	3.72	5.46	12.22	51.62
0.95	—	—	—	—	0.91	1.82	2.86	4.23	6.11	13.42	55.80
0.97	—	—	—	0.68	1.06	1.95	3.02	4.43	6.37	13.88	57.60
0.98	—	—	0.54	0.76	1.13	2.02	3.10	4.53	6.51	14.13	58.42
0.99	—	0.38	0.63	0.83	1.18	2.08	3.18	4.63	6.64	14.37	59.25
1.00	0.00	0.49	0.71	0.89	1.24	2.15	3.26	4.73	6.77	14.61	60.20
1.10	0.82	1.02	1.20	1.38	1.75	2.77	4.05	5.74	8.08	17.05	68.77
1.20	1.19	1.39	1.59	1.79	2.20	3.37	4.83	6.75	9.40	19.52	77.62
1.50	2.08	2.37	2.65	2.93	3.52	5.16	7.20	9.87	13.5	27.23	105.26
2.00	3.48	3.93	4.37	4.81	5.72	8.24	11.31	15.28	20.63	40.62	153.33
3.00	6.33	7.14	7.94	8.73	10.34	14.74	20.03	26.79	35.79	69.45	256.06
5.00	12.43	14.03	15.59	17.16	20.32	28.83	38.86	51.54	68.56	130.78	476.23
10.00	29.11	32.93	36.65	40.38	47.59	67.51	90.73	119.83	158.96	301.82	1076.98

Table 3: $\overline{B}_{\tau}(c,\ \gamma)$ (Dennis-Welsch score function)

$c \setminus \gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	28.25
0.65	—	—	—	—	—	—	—	—	—	7.66	37.04
0.75	—	—	—	—	—	—	—	—	4.05	10.57	46.35
0.80	—	—	—	—	—	—	—	3.07	4.92	12.01	51.17
0.85	—	—	—	—	—	—	2.31	3.76	5.72	13.46	56.06
0.90	—	—	—	—	—	1.67	2.85	4.37	6.50	14.92	61.04
0.95	—	—	—	—	1.06	2.10	3.33	4.97	7.29	16.40	66.09
0.97	—	—	—	0.79	1.21	2.25	3.52	5.21	7.60	16.99	68.12
0.98	—	—	0.63	0.87	1.28	2.32	3.61	5.33	7.76	17.29	69.15
0.99	—	0.44	0.72	0.94	1.34	2.39	3.70	5.45	7.92	17.59	70.17
1.00	0.00	0.54	0.79	1.00	1.40	2.47	3.79	5.57	8.07	17.89	71.20
1.10	0.83	1.07	1.29	1.51	1.95	3.17	4.71	6.77	9.66	20.92	81.61
1.20	1.19	1.45	1.70	1.94	2.45	3.85	5.63	7.99	11.27	24.01	92.24
1.50	2.09	2.46	2.82	3.18	3.90	5.92	8.42	11.71	16.23	33.55	125.23
2.00	3.48	4.08	4.65	5.22	6.36	9.48	13.28	18.19	24.90	50.25	183.11
3.00	6.34	7.43	8.47	9.49	11.54	17.03	23.60	32.00	43.36	85.84	306.47
5.00	12.45	14.62	16.69	18.71	22.73	33.36	45.95	61.90	83.29	162.66	572.37
10.00	29.19	34.37	39.29	44.07	53.52	78.32	107.40	144.00	192.81	372.75	1297.44

Table 4: $\overline{B}_{S}(c,\ \gamma)$ (Jump score function)

$c \setminus \gamma$	0.00	0.01	0.02	0.03	0.05	0.10	0.15	0.20	0.25	0.35	0.45
0.55	—	—	—	—	—	—	—	—	—	—	14.77
0.65	—	—	—	—	—	—	—	—	—	3.96	18.29
0.75	—	—	—	—	—	—	—	—	2.01	4.95	21.90
0.80	—	—	—	—	—	—	—	1.51	2.30	5.46	23.73
0.85	—	—	—	—	—	—	1.14	1.76	2.59	5.97	25.58
0.90	—	—	—	—	—	0.82	1.36	2.01	2.88	6.49	27.45
0.95	—	—	—	—	0.52	1.05	1.58	2.25	3.17	7.01	29.34
0.97	—	—	—	0.39	0.63	1.13	1.67	2.35	3.29	7.22	30.09
0.98	—	—	0.31	0.45	0.68	1.17	1.71	2.40	3.35	7.33	30.47
0.99	—	0.22	0.39	0.51	0.72	1.21	1.75	2.44	3.41	7.44	30.85
1.00	0.00	0.31	0.45	0.56	0.77	1.25	1.80	2.49	3.46	7.54	31.24
1.1	0.73	0.82	0.90	0.99	1.17	1.64	2.22	2.98	4.06	8.61	35.08
1.2	1.09	1.17	1.26	1.34	1.51	2.01	2.64	3.47	4.66	9.70	38.97
1.5	2.01	2.10	2.19	2.29	2.50	3.10	3.89	4.96	6.50	13.05	50.91
2.0	3.47	3.59	3.71	3.84	4.13	4.96	6.05	7.55	9.70	18.89	71.52
3.0	6.50	6.69	6.89	7.10	7.55	8.88	10.64	13.05	16.52	31.24	114.59
5.0	3.05	13.40	13.76	14.15	14.97	17.40	20.60	24.97	31.24	57.69	205.51
10.0	31.24	32.02	32.83	33.69	35.51	40.94	48.04	57.69	71.52	129.37	447.93