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Game option introduced by Kifer (2000) is studied in this paper. We consider the Black-Scholes model and discuss the value function of a game call option with the exercise price $K > 0$, the penalty $\delta > 0$ and the expiration time T, in connection with exercise regions of the holder and of the writer.

In the case where there is no dividend, we show that the holder's exercise region is empty and the writer's one is just the point $\{K\}$ if $s \le \beta$ and is empty if $s > \beta$, where β is a suitable time less than T determined by the Black-Scholes model and parameters T, K, δ. The value function of the option is represented as that of the corresponding European option minus the writer's premium, which is written using the local time of the price process at the point K.

In the case where the dividend is positive, the holder's exercise region at time s is a non-empty upper half line $[b(s), \infty)$, where $b(s)$ is a nonincreasing function. As for the writer's exercise region, we have a region similar to the case of non-dividend.

Furthermore, the optimal hedging portfolio for the game option will be given.

Key Words: Game option, American call option, Itô-Tanaka-Meyer's formula

1 Introduction

In this paper we study game options introduced by Kifer (2000). We shall be mainly concerned with the game call option; the game put option will be discussed briefly at the final section.

Let B_t be the bank account process defined by $B_t = e^{rt}$, where r is a positive constant called the interest rate. Let S_t be the price process of a stock determined by a SDE

$$dS_t = S_t(\mu dt + \kappa dW(t)), \tag{1.1}$$

where μ, κ are constants such that $\kappa > 0$ and $W(t)$ is a standard Brownian motion. Let P^0 be the risk neutral probability measure. The fair price of the game call option with the exercise price K and the penalty $\delta > 0$ is defined in Kifer (2000) based on the hedging theory. He showed that the fair price can be computed through the formula

$$V_t = \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} E^0[e^{-r(\sigma\wedge\tau - t)}\{((S_\sigma - K)^+ + \delta)I_{\sigma<\tau} + (S_\tau - K)^+ I_{\tau\le\sigma}\}|\mathcal{F}_t],$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times with values in $[t, T]$ and $\{\mathcal{F}_t\}$ is the filtration generated by the price process S_t. "sup" and "inf" are taken in the sense of the essential sup and the essential inf with respect to the measure P. Further, the optimal stopping strategies of the writer of the option and the holder of the option are given by

$$\begin{aligned}\hat{\sigma} &= \inf\{t \in [0, T); \quad V_t \geq (S_t - K)^+ + \delta\} \wedge T, \\ \hat{\tau} &= \inf\{t \in [0, T); \quad V_t \leq (S_t - K)^+\} \wedge T,\end{aligned}$$

respectively. In our Black-Scholes model, the above fair price is written by $V_t = V(S_t, t)$, where $V(x, t)$ is the value function of the form

$$V(x,t) = \inf_{\sigma \in \mathcal{T}_{t,T}} \sup_{\tau \in \mathcal{T}_{t,T}} E^0[e^{-r(\sigma \wedge \tau - t)}\{((S_\sigma - K)^+ + \delta)I_{\sigma < \tau} + (S_\tau - K)^+ I_{\tau \leq \sigma}\}|S_t = x].$$

In this paper, we shall study the value function $V(x, t)$. In the next section, we will redefine the value function in a more definite form together with the writer's cancellation region $\mathcal{E}^A$, the holder's exercise regions $\mathcal{E}^B$ and the continuation region $\mathcal{C}$. Here the holder's exercise region $\mathcal{E}^B$ means the following. The optimal stopping strategy of the holder is that he should stop at the first instant when the price process S_t enter in the region $\mathcal{E}^B$. Regions $\mathcal{E}^A$ and $\mathcal{E}^B$ are quite different between cases where the dividend d of the stock are zero and positive. We will see that the holder's exercise region is empty if the dividend is 0 and that it is nonempty if the dividend is positive. On the other hand, there is writer's exercise region if the penalty δ is small but there is no writer's exercise region if δ is big. In any case, the region is just one point $\{K\}$ (exercise price) or empty. Details of these results are stated in Theorems 2.1 and 2.2 in Section 2. Their proofs will be given through Sections 3-5.

In Section 6, we will decompose the value of the option into those of European options and early exercise premiums of the writer and the holder: The early exercise (cancellation) premium of the writer will be represented by the local time of the price process at the exercise price K, because that the cancellation region is the single point K. In Section 7 we discuss the hedging problem of the game call option. We will define upper hedging price and lower hedging price of the game call and will show that these two coincide with the value function. Then we will determine the optimal hedging portfolio. It will turn out that the optimal cumulative consumptions of the writer and the holder induce early exercise prices of the holder and of the writers, respectively. See Theorems 6.1 and 7.1.

Finally in Section 8, we consider the game put option. Its exercise regions and optimal hedging portfolios will be shown in Theorem 8.1.

During the study of the game options, the authors knew the works by Kyprianou (2004) and Kühn and Kyprianou (2004), where they consider game put options (Israeli puts). Some of their results are close to a part of our Theorem 7.1, but our discussions are quite different from their's.

2 Value functions and exercise regions

The solution of equation (1.1) starting from $x \in \mathbf{R}^+ = \{x; x > 0\}$ at time $s \in [0, T)$ is denoted by $S_{s,t}(x), t \in [s, T)$. It has the flow property $S_{s,u}(x) = S_{t,u}(S_{s,t}(x))$ a.s. for any $s < t < u$ and $x \in \mathbf{R}^+$. It is called the stochastic flow of diffeomorphisms. In our Black-Scholes model, it is represented by

$$S_{s,t}(x) = xH(s,t), \tag{2.1}$$

where

$$H(s,t) = \exp\left\{\kappa(W(t) - W(s)) + \left(\mu - \frac{1}{2}\kappa^2\right)(t-s)\right\}. \tag{2.2}$$

In our market model, the dividend may be paid to the holder of the stock. We assume that its rate d is a nonnegative constant. Then the discounted return process (excess yield process) is given by $R(t) = (\mu + d - r)t + \kappa(W(t) - W(0))$. Define $\theta = \frac{1}{\kappa}(\mu + d - r)$. Then

$$Z_t = \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta(u)^2 du\right\}$$

is a positive martingale with mean 1. We define a risk neutral probability measure P^0 by

$$P^0(A) = E[1_A Z_T]. \tag{2.3}$$

Then, $W^0(t) := W(t) + \theta t$ is a standard Brownian motion with respect to P^0. Further the discounted return process $R(t)$ is a martingale and the discounted price process $S^*_{s,t}(x) = S_{s,t}(x)B_s/B_t$ is a supermartingale with respect to P^0. The latter is a martingale if and only if the dividend d is 0.

Now, we shall consider the game call option with the exercise price $K > 0$ and the penalty $\delta > 0$. The *value function* is defined for $(x, s) \in \mathbf{R}^+ \times [0, T)$ by

$$V(x,s) = V_T(x,s) = \inf_{\sigma \in \mathcal{T}_{s,T}} \sup_{\tau \in \mathcal{T}_{s,T}} J_s^x(\sigma, \tau), \tag{2.4}$$

where

$$J_s^x(\sigma,\tau) = E^0\left[e^{-r(\sigma\wedge\tau - s)}\left\{((S_{s,\sigma}(x) - K)^+ + \delta)1_{\sigma<\tau} + (S_{s,\tau}(x) - K)^+ 1_{\tau\le\sigma}\right\}\right], \tag{2.5}$$

and $\mathcal{T}_{s,T}$ denotes the totality of stopping times with values in the interval $[s, T]$. Then the inequality

$$(x-K)^+ \le V(x,s) \le (x-K)^+ + \delta, \quad \forall (x,s) \in \mathbf{R}^+ \times [0,T)$$

holds.

We will define subsets of $\mathbf{R}^+ \times [0,T)$ concerning the game call by

$$
(2.6) \qquad \begin{array}{rcl} \mathcal{C} &=& \{(x,s) \in \mathbf{R}^+ \times [0,T); (x-K)^+ < V(x,s) < (x-K)^+ + \delta\}, \\ \mathcal{E}^A &=& \{(x,s) \in \mathbf{R}^+ \times [0,T); V(x,s) = (x-K)^+ + \delta\}, \\ \mathcal{E}^B &=& \{(x,s) \in \mathbf{R}^+ \times [0,T); V(x,s) = (x-K)^+\}. \end{array}
$$

These are disjoint subsets of $\mathbf{R}^+ \times [0,T)$ and the union of the three sets is equal to $\mathbf{R}^+ \times [0,T)$. The set $\mathcal{C}$ is called the *continuation region* of the game call option, the set $\mathcal{E}^A$ is called the *cancellation region* of the writer A of the option and the set $\mathcal{E}^B$ is called the *exercise region* of the holder B of the option. Sections of these sets at $s \in (0,T)$ are denoted by $\mathcal{C}_s, \mathcal{E}^A_s, \mathcal{E}^B_s$, respectively.

The infimum and the supremum of J^x_s are attained by the following two stopping times $\hat\sigma^x_s = \tau^x_s(\mathcal{E}^A)$ and $\hat\tau^x_s = \tau^x_s(\mathcal{E}^B)$ of $\mathcal{T}_{s,T}$. Here, $\tau^x_s(\mathcal{D})$ is the hitting time to the Borel subset $\mathcal{D}$ of $\mathbf{R}^+ \times [0,T)$:

$$
(2.7) \qquad \tau^x_s(\mathcal{D}) := \inf\{t \in [s,T); (S_{s,t}(x), t) \in \mathcal{D}\} \wedge T.
$$

We understand that $\tau^x_s(\mathcal{D}) = T$ if $\mathcal{D}$ is an empty set. Namely we have

$$
(2.8) \qquad V(x,s) = J^x_s(\hat\sigma^x_s, \hat\tau^x_s), \quad \forall (x,s) \in \mathbf{R}^+ \times [0,T).
$$

See Kifer (2000). The pair $(\hat\sigma^x_s, \hat\tau^x_s)$ is often called the saddle point of $J^x_s(\sigma,\tau)$.

Our definition of the value function is different from those in Karatzas and Shreve (1998) and Kühn and Kyprianou (2004). In our model, the terminal time T is assumed to be fixed. However if we were allowed to change T, we have the relation $V_T(x,s) = V_{T-s}(x,0)$. Their value function $v(x,t)$ corresponds to our $V_t(x,0)$. A merit of our definition of V is that formula (2.8) makes clear the relation between the value function and the exercise regions $\mathcal{E}^A, \mathcal{E}^B$.

We will summarize results about exercise and cancellation regions, and then about value functions in two theorems. Proofs will be given in Sections 3-5.

Theorem 2.1. *Let $\mathcal{E}^A$ and $\mathcal{E}^B$ be the writer's cancellation region and the holder's exercise region of the game put option with the exercise price K and penalty δ, respectively.*

1) Let β be the infimum of s satisfying $V(K,s) < \delta$. Then it holds $0 \le \beta < T$. Sections of the writer's exercise region $\mathcal{E}^A$ are characterized by

$$
(2.9) \qquad \mathcal{E}^A_s = \begin{cases} \{K\}, & \text{if } s \le \beta, \\ \phi, & \text{if } s > \beta. \end{cases}
$$

2) a) If $d=0$, the holder's exercise region $\mathcal{E}^B$ is empty.

b) If $d>0$, the holder's exercise region $\mathcal{E}^B$ is nonempty. Its sections $\mathcal{E}^B_s$ are upper half line

$$\mathcal{E}^B_s = \{x; \quad b(s) \le x < \infty\}, \tag{2.10}$$

where $(b(s), s \in [0,T))$ is a nonincreasing function satisfying

$$\max\{K, \frac{r}{d}K\} < b(s) < \infty, \quad \forall s, \tag{2.11}$$

$$\lim_{s\to T} b(s) = \max\{K, \frac{r}{d}K\}. \tag{2.12}$$

Theorem 2.2. *Let $V(x,s), (x,s) \in \mathbf{R}^+ \times [0,T)$ be the value function of the game call option with the exercise price K and the penalty δ defined by (2.4).*

1) The function $V(x,s)$ is positive and locally Lipschitz continuous in (x,s). For any s, $V(x,s)$ is convex and strictly increasing with respect to x. For any x, it is nonincreasing with respect to s.

2) a) Suppose $d=0$. If $s \ge \beta$, we have $V(x,s) = V_E(x,s)$ for any $x \in \mathbf{R}^+$, where $V_E(x,s)$ is the value function of the European call:

$$V_E(x,s) := E^0[e^{-r(T-s)}(S_{s,T}(x) - K)^+]. \tag{2.13}$$

If $s < \beta$, we have $V(x,s) < V_E(x,s)$ for any x.

In particular, if $V_E(K,0) \le \delta$, then $\beta = 0$ and the value function of the game call $V(x,s)$ coincides with that of the European call $V_E(x,s)$ for any (x,s).

b) Suppose $d>0$. If $s \ge \beta$, we have $V(x,s) = V_A(x,s)$ for any x, where $V_A(x,s)$ is the value function of the American call:

$$V_A(x,s) := \sup_{\tau \in \mathcal{T}_{s,T}} E^0[e^{-r(\tau-s)}(S_{s,\tau}(x) - K)^+]. \tag{2.14}$$

If $s < \beta$, we have $V(x,s) < V_A(x,s)$ for any x.

In particular, if $V_A(K,0) \le \delta$, then $\beta = 0$ and the value of the game option and that of the American option coincide each other.

3) $V(x,s)$ is of $C^{2,1}$ class in $\mathcal{C} = (\mathcal{E}^A \cup \mathcal{E}^B)^c$. The first derivative $\frac{\partial V}{\partial x}(x,s)$ is strictly increasing and satisfies $\frac{\partial V}{\partial x}(0+,s) = 0$, $0 < \frac{\partial V}{\partial x}(x,s) < 1$ in $\mathcal{C}_s$. Further at $x=K$, we have

$$0 < \frac{\partial V}{\partial x}(K-,s) \; < \; \frac{\partial V}{\partial x}(K+,s) < 1, \quad \text{if } s < \beta, \tag{2.15}$$

$$0 < \frac{\partial V}{\partial x}(K-,s) \; = \; \frac{\partial V}{\partial x}(K+,s) < 1, \quad \text{if } s \ge \beta. \tag{2.16}$$

Furthermore, we have $\lim_{x\to\infty} \frac{\partial V}{\partial x}(x,s) = 1$ *if* $d = 0$ *and*

$$\frac{\partial V}{\partial x}(b(s)-, s) = \frac{\partial V}{\partial x}(b(s)+, s) = 1, \quad \forall s \in [0, T), \tag{2.17}$$

if $d > 0$.

Remark. If $d = 0$, the time β is characterized as the infimum of s such that $V_E(K, s) < \delta$. By the famous Black-Scholes formula, $V_E(K, s)$ is given by

$$V_E(K,s) = K\Phi\left(\frac{1}{\kappa\sqrt{T-s}}\Big(r + \frac{\kappa^2}{2}\Big)(T-s)\right) - Ke^{-r(T-s)}\Phi\left(\frac{1}{\kappa\sqrt{T-s}}\Big(r - \frac{\kappa^2}{2}\Big)(T-s)\right),$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{z} e^{-x^2/2}dx$. Then the time β is determined by the solution of

$$\frac{\delta}{K} = \Phi\left(\frac{1}{\kappa\sqrt{T-\beta}}\Big(r + \frac{\kappa^2}{2}\Big)(T-\beta)\right) - e^{-r(T-\beta)}\Phi\left(\frac{1}{\kappa\sqrt{T-\beta}}\Big(r - \frac{\kappa^2}{2}\Big)(T-\beta)\right).$$

If $d > 0$, the time β is characterized as the infimum of s such that $V_A(K, s) < \delta$.

In order to understand the previous theorems visually, we present a numerical example of the game call using the simple binomial approach. We assume that the dividend rate is positive. The time interval $[0, T]$ is divided into N time-steps of size $\Delta t = T/N$. Thus the tradable dates are $\{n\Delta t : n = 1, 2, \ldots, N\}$. If the stock price at $n\Delta t$ is S_n then it may take one of only two possible prices λS_n or ρS_n at $(n+1)\Delta t$. We assume $0 < \rho < 1 < e^{r\Delta t} < \lambda$ to avoid an arbitrage opportunity. As usual setting, we have $\lambda = e^{\kappa\sqrt{\Delta t}}$ and $\rho = e^{-\kappa\sqrt{\Delta t}}$. Let $\tilde{p}$ be the risk neutral probability measure. Then $\tilde{p} = (e^{(r-d)\Delta t} - \rho)/(\lambda - \rho)$. Under this probability measure, the stock price moves up to λS_n with probability $\tilde{p}$ and moves down to ρS_n with probability $1 - \tilde{p}$. Since the stock price dynamics is completely determined by the number of upward movements, we can clarify the state of the stock price by a pair (n, i), where i stands for the number of upward movements of the stock price until $n\Delta t$. Given the initial stock price $S_0 > 0$, its price at $n\Delta t$ can be represented by

$$S_n = \lambda^i \rho^{n-i} S_0, \ i = 0, 1, \ldots, n.$$

Let $V_n(i)$ be the price of the game call at $n\Delta t$. Then

$$\begin{aligned} V_n(i) &= \min\{(\lambda^i\rho^{n-i}S_0 - K)^+ + \delta, \ \max\{(\lambda^i\rho^{n-i}S_0 - K)^+, \\ &\quad e^{-r\Delta t}\{\tilde{p}V_{n+1}(i+1) + (1-\tilde{p})V_{n+1}(i)\}\}\}, \ i = 0, \ldots, n, \end{aligned}$$

where the terminal condition is $V_N(i) = (\lambda^i\rho^{N-i}S_0 - K)^+, \ i = 0, \ldots, N$. Applying above recursive equation from the maturity T back to the time 0, we can compute the price of the game call.

We set the option parameters as follows: exercise price $K = 100$, maturity $T = 1$, interest rate $r = 0.1$, dividend rate $d = 0.09$ and volatility $\kappa = 0.3$. We also set the numerical parameter $N = 10000$. Figure 1 shows the behavior of the game call price when the initial stock price S_0 and the penalty δ change. It turns out that the value function V is not smooth at the optimal exercise region of the writer when δ is small (see equations (2.9) and (2.15)). However the value function V is smooth at the optimal exercise boundary of the holder (see equation (2.17)). The optimal exercise regions for both the writer and the holder are represented in Figure 2. It claims that the writer's optimal exercise region is just one point $\{K\}$ or ϕ (see equation (2.9)) and that the holder's one is $[b(s), \infty)$ (see equation (2.10)). Further we see that the function $b(s)$ which represents the optimal exercise boundary of the holder is nonincreasing in time s and satisfies (2.12). Note that the writer's optimal exercise region $\mathcal{E}^A$ is identical to his optimal exercise boundary.

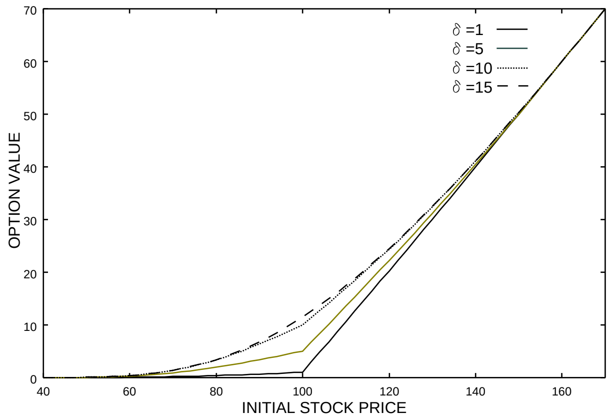

Figure 1: Behavior of the game call price when S_0 and δ change.

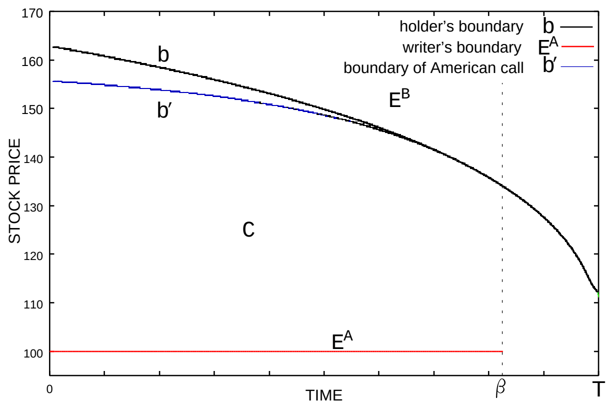

Figure 2: Optimal exercise boundaries for the writer and the holder when $\delta = 5$.

3 Lipschitz continuity and monotonicity of the value function

We will first discuss the (weak) derivative of the value function $V(x, s)$ with respect to x.

Lemma 3.1. *The function $V(x, s)$ is positive for any $(x, s) \in \mathbf{R}^+ \times [0, T)$ and is nondecreasing with respect to x for any s. Further, $V(x, s)$ is Lipschitz continuous with respect to x for any s and its Radon-Nikodym derivative satisfies*

$$0 \le \frac{\partial V}{\partial x}(x, s) \le 1, \quad a.e. \;\; x \tag{3.1}$$

for any s.

Proof. The positivity of $V(x,s)$ is obvious since $V(x,s)$ is represented by (2.8). We will prove the second assertion. We have $S_{s,t}(x) < S_{s,t}(y)$ for all $t \geq s$ a.s. P^0 if $x < y$, because of (2.1). Since $g(x) = (x-K)^+$ is a nondecreasing function, we have $J_s^x(\sigma,\tau) \leq J_s^y(\sigma,\tau)$ for any $\sigma, \tau \in \mathcal{T}_{s,T}$. This implies $V(x,s) \leq V(y,s)$ if $x \leq y$, for any $0 \leq s < T$.

We will prove the Lipschitz continuity of $V(x,s)$ with respect to x for any fixed s. Observe two inequalities

$$V(y,s) \leq J_s^y(\hat{\sigma}_s^x, \hat{\tau}_s^y), \quad V(x,s) \geq J_s^x(\hat{\sigma}_s^x, \hat{\tau}_s^y).$$

The first inequality follows from replacing the optimal stopping time $\hat{\sigma}_s^y$ by the nonoptimal one $\hat{\sigma}_s^x$. The second one is obtained similarly. Note $z_1^+ - z_2^+ \leq (z_1 - z_2)^+$. Then we have for any $y > x$,

$$\begin{aligned} 0 &\leq V(y,s) - V(x,s) \\ &\leq E^0[e^{-r(\hat{\sigma}_s^x \wedge \hat{\tau}_s^y - s)}(S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^y}(y) - S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^y}(x))^+] \\ &\leq (y-x)E^0[e^{-r(\hat{\sigma}_s^x \wedge \hat{\tau}_s^y - s)} H(s, \hat{\sigma}_s^x \wedge \hat{\tau}_s^y)]. \end{aligned}$$

The last expectation is less than or equal to 1, since $e^{-r(t-s)}H(s,t), t \geq s$ is a supermartingale with value 1 at $t = s$. Therefore we have

$$0 \leq V(y,s) - V(x,s) \leq (y-x), \quad \text{if } \ y > x. \tag{3.2}$$

This proves that $V(x,s)$ is Lipschitz continuous in x and its Radon-Nikodym derivative satisfies (3.1). □

Before we proceed the study of the monotonicity of V with respect to time s, we study the value function near $x = K$.

Lemma 3.2. *Let β be the infimum of s such that $V(K,s) < \delta$. Then $0 \leq \beta < T$. It coincides with the infimum of s such that $V_A(K,s) < \delta$, where $V_A(x,s)$ is the value function of the American call. Further, $V(x,s) = V_A(x,s)$ holds for any $x \in \mathbf{R}^+$ if $s > \beta$.*

Proof. Let us compare $V(x,s)$ and $V_A(x,s)$. The inequality $V(x,s) \leq V_A(x,s)$ holds for any x, s. Since $V_A(x,s)$ converges to $(x-K)^+$ as $s \to T$, $V(K,s) \to 0$ as $s \to T$. Therefore we have $\beta < T$.

Now for any $\epsilon > 0$, there exists $s \in (\beta, \beta+\epsilon)$ such that $V(K,s) < \delta$. Since $0 \leq \frac{\partial V}{\partial x}(x,s) \leq 1$ holds a.e., the inequality $V(x,s) < (x-K)^+ + \delta$ holds for all x. Therefore the set $\mathcal{E}_s^A$ is empty. Then the optimal stopping time $\hat{\sigma}_s^x$ is equal to T a.s. Consequently we have

$$V(x,s) = \sup_{\tau \in \mathcal{T}_{s,T}} J_s^x(T,\tau) = V_A(x,s), \quad \forall x.$$

Furthermore, since $V_A(K,s)$ is nonincreasing with respect to s, we have $V(K,s) \leq V_A(K,s) < \delta$ for any $s' \in (s, s+T)$. Therefore we have $\mathcal{E}^A_{s'} = \phi$ and $V(x,s') = V_A(x,s')$ holds for any x. We have thus shown that $V(x,s) = V_A(x,s)$ holds for any $s > \beta$.

Let α be the infimum of s such that $V_A(K,s) < \delta$. We want to prove $\alpha = \beta$. If $s > \beta$, we have $V(x,s) = V_A(x,s)$ for any x, proving $s \geq \alpha$. This implies $\beta \geq \alpha$. Suppose next $s < \beta$. Then it holds $V(K,s) \geq \delta$. Therefore we have $V_A(K,s) \geq V(K,s) \geq \delta$ holds. This yields $s \leq \alpha$, showing $\beta \leq \alpha$. We have thus proved $\beta = \alpha$. □

We will now study the monotonicity of the value function with respect to the time.

Lemma 3.3. *The value function $V(x,s)$ of the game call option is nonincreasing with respect to s for any x.*

Proof. The fact is obvious for $s > \beta$ since $V(x,s)$ coincides with $V_A(x,s)$ and the latter is continuous and nonincreasing with respect to time s. We shall consider the case where $s < \beta$. Let $\epsilon \geq 0$. Let $\hat{\sigma}^x_{s-\epsilon}$ and $\hat{\tau}^x_{s-\epsilon}$ be the optimal stopping times for the writer and the holder at $(x, s-\epsilon)$, respectively. We consider the following stopping times

$$
\begin{aligned}
\tau_* &= \begin{cases} \inf\{u \in (s-\epsilon, T-\epsilon); (S_{s-\epsilon,u}(x), u+\epsilon) \in \mathcal{E}^B\} & \text{if } \{\ \ \} \neq \phi, \\ \hat{\tau}^y_{T-\epsilon}, & \text{if } \{\ \ \} = \phi, \end{cases} \\
\sigma_* &= \inf\{u \in (s,T); (S_{s,u}(x), u-\epsilon) \in \mathcal{E}^A\} \wedge T,
\end{aligned}
$$

where $y = S_{s-\epsilon,T-\epsilon}(x)$. Then, we have $V(x,s-\epsilon) \geq J^x_{s-\epsilon}(\hat{\sigma}^x_{s-\epsilon}, \tau_*)$ and $J^x_s(\sigma_*, \tau^x_s) \geq V(x,s)$. We want to prove $J^x_{s-\epsilon}(\hat{\sigma}^x_{s-\epsilon}, \tau_*) \geq J^x_s(\sigma_*, \hat{\tau}^x_s)$. If the inequality is verified, then the inequality $V(x,s-\epsilon) \geq V(x,s)$ will follow. Setting

$$
R^x_s(u,v) = ((S_{s,u}(x) - K)^+ + \delta)1_{u<v} + (S_{s,v}(x) - K)^+ 1_{v \leq u}, \tag{3.3}
$$

we have

$$
\begin{aligned}
& J^x_{s-\epsilon}(\hat{\sigma}^x_{s-\epsilon}, \tau_*) - J^x_s(\sigma_*, \hat{\tau}^x_s) \\
&= \Big\{ E^0[e^{-r\{\hat{\sigma}^x_{s-\epsilon} \wedge \tau_* - (s-\epsilon)\}} R^x_{s-\epsilon}(\hat{\sigma}^x_{s-\epsilon}, \tau_*) 1_{\{\hat{\sigma}^x_{s-\epsilon} < T-\epsilon\} \cup \{\tau_* < T-\epsilon\}})] \\
&\quad - E^0[e^{-r\{\sigma_* \wedge \hat{\tau}^x_s - s\}} R^x_s(\sigma_*, \hat{\tau}^x_s) 1_{\{\sigma_* < T \cup \hat{\tau}^x_s < T\}}] \Big\} \\
&\quad + \Big\{ E^0[e^{-r\{\hat{\sigma}^x_{s-\epsilon} \wedge \tau_* - (s-\epsilon)\}} R^x_{s-\epsilon}(\hat{\sigma}^x_{s-\epsilon}, \tau_*) 1_{\hat{\sigma}^x_{s-\epsilon} \geq T-\epsilon, \tau_* \geq T-\epsilon}] \\
&\quad - E^0[e^{-r\{\sigma_* \wedge \hat{\tau}^x_s\}} R^x_s(\sigma_*, \hat{\tau}^x_s) 1_{\sigma_* \geq T, \hat{\tau}^x_s \geq T}] \Big\} \\
&= \{I_1 - J_1\} + \{I_2 - J_2\}.
\end{aligned}
$$

We shall consider I_1. Note that the law of $S_{s,t}$ is stationary. Then the law of the triple $\{S_{s-\epsilon,t-\epsilon}(x), \hat{\sigma}^x_{s-\epsilon} - (s-\epsilon), \tau_* - (s-\epsilon)\}$ coincides with that of the triple $\{S_{s,t}(x), \sigma_* -$

$s, \hat{\tau}_s^x - s\}$, as far as $\hat{\sigma}_s^x \le \beta$. Then we can replace the notations $S_{s-\epsilon,t-\epsilon}(x)$ etc by $S_{s,t}(x)$ etc. in I_1. Then we obtain

$$I_1 = E^0[e^{-r\{\sigma_* \wedge \hat{\tau}_s^x - s\}} R_s^x(\sigma_*, \hat{\tau}_s^x) 1_{\{\sigma_* < T\} \cup \{\hat{\tau}_s^x < T\}}],$$

which coincides with J_1. Therefore we get $I_1 - J_1 = 0$.

We shall next consider I_2. On the set $\{\hat{\sigma}_{s-\epsilon}^x \ge T - \epsilon, \tau_* \ge T - \epsilon\}$, we have $\hat{\sigma}_{s-\epsilon}^x = \hat{\sigma}_{T-\epsilon}^y$ and $\tau^* = \hat{\tau}_{T-\epsilon}^y$, where $y = S_{s-\epsilon,T-\epsilon}(x)$. Therefore using the flow property $S_{s-\epsilon,\hat{\sigma}_{s-\epsilon}^x} = S_{T-\epsilon,\hat{\sigma}_{T-\epsilon}^y}(S_{s-\epsilon,T-\epsilon}(x))$, we get

$$\begin{aligned} \tilde{I}_2(\omega) &:= e^{-r\{\hat{\sigma}_{s-\epsilon}^x \wedge \tau_* - (s-\epsilon)\}} R_{s-\epsilon}^x(\hat{\sigma}_{s-\epsilon}^x, \tau_*) 1_{\hat{\sigma}_{s-\epsilon}^x \ge T-\epsilon, \tau_* \ge T-\epsilon} \\ &= e^{-r(T-s)} e^{-r(\hat{\sigma}_{T-\epsilon}^y \wedge \hat{\tau}_{T-\epsilon}^y - (T-\epsilon))} R_{T-\epsilon}^y(\hat{\sigma}_{T-\epsilon}^y, \hat{\tau}_{T-\epsilon}^y) 1_{\hat{\sigma}_{s-\epsilon}^x \ge T-\epsilon, \tau_* \ge T-\epsilon}. \end{aligned}$$

Then its conditional expectation is computed as

$$\begin{aligned} E[\tilde{I}_2 | \mathcal{F}_{T-\epsilon}] &= e^{-r(T-s)} J_{T-\epsilon}^y(\hat{\sigma}_{T-\epsilon}^y, \hat{\tau}_{T-\epsilon}^y) 1_{\hat{\sigma}_{s-\epsilon}^x \ge T-\epsilon, \tau_* \ge T-\epsilon} \\ &= e^{-r(T-s)} V(y, T-\epsilon) 1_{\hat{\sigma}_{s-\epsilon}^x \ge T-\epsilon, \tau_* \ge T-\epsilon}. \end{aligned}$$

Therefore,

$$I_2 = E[\tilde{I}_2] = E^0[e^{-r(T-s)} V(S_{s,T}(x), T-\epsilon) 1_{\sigma_* \ge T, \hat{\tau}_s^x \ge T}].$$

Note that $V(x, T-\epsilon) \ge V(x,T)$ holds for any x if $T - \epsilon > \beta$. Then we get

$$I_2 \ge E^0[e^{-r(T-s)} V(S_{s,T}(x), T) 1_{\sigma_* \ge T, \hat{\tau}_s^x \ge T}].$$

We can show by a similar method that J_2 is equal to the right hand side of the above. Therefore we get $I_2 - J_2 \ge 0$. $\square$

Lemma 3.4. *The value function $V(x,s)$ is locally Lipschitz continuous in (x,s).*

Proof. We showed in Lemma 3.1 that $V(x,s)$ is Lipschitz continuous with respect to x for each s. We will prove the local Lipschitz continuity with respect to s. Let σ^* and τ^* be stopping times defined by

$$\begin{aligned} \sigma^* &= \begin{cases} \inf\{u \in (s-\epsilon, T-\epsilon); (S_{s-\epsilon,u}(x), u+\epsilon) \in \mathcal{E}^A\} & \text{if } \{\ \} \ne \phi, \\ \hat{\tau}_{T-\epsilon}^y, & \text{if } \{\ \} = \phi, \end{cases} \\ \tau^* &= \inf\{u \in (s,T); (S_{s,u}(x), u-\epsilon) \in \mathcal{E}^B\} \wedge T, \end{aligned}$$

where $y = S_{s-\epsilon,T-\epsilon}(x)$. Then we have

$$V(x, s-\epsilon) - V(x,s) \le J_{s-\epsilon}^x(\sigma^*, \hat{\tau}_{s-\epsilon}^x) - J_s^x(\hat{\sigma}_s^x, \tau^*).$$

Using the flow property, we have similarly as in the proof of Lemma 3.3,

$$\begin{aligned}
&J^x_{s-\epsilon}(\sigma^*, \hat{\tau}^x_{s-\epsilon}) - J^x_s(\hat{\sigma}^x_s, \tau^*) \\
&= \Big\{E^0[e^{-r\{\sigma^*\wedge\hat{\tau}^x_{s-\epsilon}-(s-\epsilon)\}}R^x_{s-\epsilon}(\sigma^*, \hat{\tau}^x_{s-\epsilon})1_{\{\sigma^*<T-\epsilon\}\cup\{\hat{\tau}^x_{s-\epsilon}<T-\epsilon\}}] \\
&\quad -E^0[e^{-r\{\hat{\sigma}^x_s\wedge\tau^*-s\}}R^x_s(\hat{\sigma}^x_s, \tau^*)1_{\{\hat{\sigma}^x_s<T\}\cup\{\tau^*<T\}}]\Big\} \\
&\quad +\Big\{E^0[e^{-r(T-s)}J^y_{T-\epsilon}(\sigma^*, \hat{\tau}^y_{T-\epsilon})|_{y=S_{s-\epsilon,T-\epsilon}(x)}1_{\{\sigma^*\geq T-\epsilon, \hat{\tau}^x_{s-\epsilon}\geq T-\epsilon\}}] \\
&\qquad -E^0[e^{-r(T-s)}J^{y'}_T(T,T)|_{y'=S_{s,T}(x)}1_{\{\sigma^*\geq T,\tau^*\geq T\}}]\Big\} \\
&= K_1 + K_2.
\end{aligned}$$

The law of $\{S_{s-\epsilon,t-\epsilon}(x), \sigma^* - (s-\epsilon), \hat{\tau}^x_{s-\epsilon} - (s-\epsilon)\}$ coincides with that of $\{S_{s,t}(x), \hat{\sigma}^x_s - s, \tau^* - s\}$ on $\{\tau^* < T\}$. Then we get $K_1 = 0$. Further, we have

$$K_2 \;=\; E^0[e^{-r(T-s)}\Big\{V(S_{s,T}(x), T-\epsilon) - (S_{s,T}(x) - K)^+\Big\}1_{\{\hat{\sigma}^x_s\geq T,\tau^*\geq T\}}].$$

If $T - \epsilon \geq \beta$, then $V(x, T-\epsilon)$ is equal to $V_E(x, T-\epsilon)$ or to $V_A(x, T-\epsilon)$. We know that the value function $V_E(x,s)$ or $V_A(x,s)$ is locally Lipschitz continuous with respect to s. Further the function $\Phi(x,s) = V_E(x,s) - (x-K)^+$ or $\Phi(x) = V_A(x,s) - (x-K)^+$ is bounded (See the proof of Lemma 5.2). Then $|K_2|$ is bounded by $L\epsilon$ where L is a positive constant. Therefore $|V(x, s-\epsilon) - V(x,s)|$ is also bounded by the same quantity, proving the local Lipschitz continuity. □

4 Exercise regions for the writer and the holder.

The exercise regions $\mathcal{E}^B$ and $\mathcal{E}^A$ are closed subsets and the continuation region $\mathcal{C}$ is an open subset of $\mathbf{R}^+ \times [0,T)$, since $V(x,s)$ is a continuous function.

We shall first study the holder's exercise region.

Lemma 4.1. *1) Suppose that dividend d is 0. Then $\mathcal{E}^B$ is empty.*
2) Suppose that dividend d is positive. Then $\mathcal{E}^B$ is nonempty. Further, each section $\mathcal{E}^B_s$ is an upper half line: there exists a nonincreasing function $b(s), s \in [0,T)$ such that $K < b(s) < \infty$ and $\mathcal{E}^B_s = [b(s), \infty)$ holds for any s.

Proof. We first assume that $d = 0$. Observe that $J^x_s(\sigma,\tau)$ of (2.5) is written by

$$J^x_s(\sigma,\tau) = E^0[e^{-r(\sigma\wedge\tau-s)}\{(S_{s,\sigma\wedge\tau}(x) - K)^+ + \delta 1_{\sigma<\tau}\}]. \tag{4.1}$$

For a given s and $\sigma \in \mathcal{T}_{s,T}$, we consider a stochastic process

$$Y^\sigma_t = e^{-r(t\wedge\sigma-s)}\Big\{(S_{s,t\wedge\sigma}(x) - K)^+ + \delta e^{-r(t\wedge\sigma-s)}1_{\sigma<t}\Big\}, \quad t \in [s,T]. \tag{4.2}$$

We will show that Y_t^σ is a submartingale. Since the discounted price process $e^{-r(t-s)}S_{s,t}(x)$ is a martingale with respect to P^0, $e^{-r(t\wedge\sigma-s)}S_{s,t\wedge\sigma}(x)$ is a martingale by Doob's optional sampling theorem. Then the stopped process $e^{-r(t\wedge\sigma-s)}(S_{s,t\wedge\sigma}(x)-K)$ is a submartingale. Therefore its positive part is again a submartingale. Further, $\delta e^{-r(\sigma-s)}1_{\sigma\le t}, t\in[s,T]$ is an increasing process and hence it is a submartingale. We have thus seen that Y_t^σ is a submartingale.

It holds $J_s^x(\hat\sigma_s^x,\tau) = E^0[Y_\tau^{\hat\sigma_s^x}]$. Then the maximum of $J_s^x(\hat\sigma_s^x,\tau)$ is attained by the constant time $\tau = T$ by Doob's optional sampling theorem. This proves that $\hat\tau_s^x = T$ a.s. and $\mathcal{E}^B$ is an empty set.

We will next consider the case where $d > 0$. Let $V_A(x,s)$ be the value function of the American call with the exercise price K: Let $\mathcal{E}_s = \{x; V_A(x,s) = (x-K)^+\}$ be the exercise region of the American call. It is known that $\mathcal{E}_s$ is a nonempty upper half line $[b'(s),\infty)$ for any s. Since $V(x,s)\le V_A(x,s)$ holds for any (x,s), $\mathcal{E}_s^B \supset \mathcal{E}_s \ne \phi$. We will show that the set $\mathcal{E}_s^B$ is also an upper half line. Let $b(s)$ be the smallest b such that the interval $[b,\infty)$ is in $\mathcal{E}_s^B$. Then $b(s)\le b'(s)$. For any $\epsilon>0$, the set $\mathcal{C}_s\cap(b(s)-\epsilon, b(s))$ is nonempty. Take x_0 from the set. Since $V(x,s)$ satisfies $\frac{\partial}{\partial x}V(x_0,s)\le 1$, we have $V(x,s) > (x-K)^+$ for any $x < x_0$. Therefore we obtain $\mathcal{E}_s^B = [b(s),\infty)$. It holds $b(s) > K$, since $V(x,s) > 0$ for any x.

We have the relation $\mathcal{E}_{s'}^B \subset \mathcal{E}_s^B$ if $s' < s$ and $\mathcal{E}_{s'}^A \supset \mathcal{E}_s^A$ if $s' < s$, since $V(x,s)$ is nonincreasing with respect to s. Then the function $b(s)$ is nonincreasing since the sets $\mathcal{E}_s^B$ are nondecreasing with respect to s. □

Remark. Let us consider the case $s > \beta$. If the dividend d is 0, the value function $V(x,s)$ coincides with that of the European call $V_E(x,s)$ since $\mathcal{E}^B = \phi$.

Lemma 4.2. *If $0\le s\le\beta$, we have $\mathcal{E}_s^A = \{K\}$, and if $s>\beta$ we have $\mathcal{E}_s^A = \phi$.*

Proof. The writer's cancellation region $\mathcal{E}^A$ is an interval $[a_1(s), a_2(s)]$ including the point $\{K\}$, which can be verified similarly as in the case of holder's exercise region $\mathcal{E}^B$. We want to prove $\mathcal{E}_s^A = \{K\}$ if $s\le\beta$.

We first consider the case $d = 0$. We consider the function $U(x,s) = V_E(x,s) - V(x,s)$. Since $\hat\tau_s^x = T$ holds a.s., we have

$$\begin{aligned} U(x,s) &= \sup_{\sigma\in\mathcal{T}_{s,T}} \{V_E(x,s) - J_s^x(\sigma,T)\} \\ &= \sup_{\sigma\in\mathcal{T}_{s,T}} \Big\{E^0[e^{-r(T-s)}(S_{s,T}(x)-K)^+1_{\sigma<T}] \\ &\qquad -E^0[e^{-r(\sigma-s)}((S_{s,\sigma}(x)-K)^+ + \delta)1_{\sigma<T}]\Big\}. \end{aligned}$$

Using the flow property $S_{s,T}(x) = S_{\sigma,T}(S_{s,\sigma}(x))$ and the independence of $S_{\sigma,T}$ and $\mathcal{F}_\sigma$, we have

$$
\begin{aligned}
& E^0[e^{-r(T-s)}(S_{s,T}(x)-K)^+|\mathcal{F}_\sigma] \\
(4.3)\qquad =\ & e^{-r(\sigma-s)}E^0[e^{-r(T-\sigma)}(S_{\sigma,T}(S_{s,\sigma}(x))-K)^+|\mathcal{F}_\sigma] \\
=\ & e^{-r(\sigma-s)}E^0[e^{-r(T-u)}(S_{u,T}(y)-K)^+]|_{u=\sigma,y=S_{s,\sigma}(x)} \\
=\ & e^{-r(\sigma-s)}V_E(S_{s,\sigma}(x),\sigma).
\end{aligned}
$$

Therefore,

$$
(4.4)\qquad U(x,s) = \sup_{\sigma\in\mathcal{T}_{s,T}} E^0[e^{-r(\sigma-s)}\{V_E(S_{s,\sigma}(x),\sigma) - (S_{s,\sigma}(x)-K)^+ - \delta\}1_{\sigma<T}].
$$

This is an optimal stopping problem with respect to the function $\Phi(x,s) = V_E(x,s) - (x - K)^+ - \delta$. The supremum is attained by $\hat{\sigma}_s^x = \tau_s^x(\mathcal{E}^A)$ and the equality $U(x,s) = \Phi(x,s)$ holds for $x \in \mathcal{E}^A$, since $V(x,s) = (x-K)^+ + \delta$ holds on $\mathcal{E}^A$.

The function $\Phi(x,s)$ satisfies $\frac{\partial\Phi}{\partial x}(x,s) > 0$ if $x < K$ and $\frac{\partial\Phi}{\partial x}(x,s) < 0$ if $x > K$. Therefore it takes the maximum value $V_E(K,s) - \delta$ at $x = K$. Then it is plausible that the maximum of (4.4) will be attained by the stopping time $\hat{\sigma}_s^x = \tau_s^x(\mathcal{E}^K)$ where $\mathcal{E}^K = \{(K,s); s \le \beta\}$. In order to prove the fact, we have to modify the above optimal stopping problem.

Consider optimization problems

$$
(4.5)\qquad \tilde{U}(x,s) = \sup_{\sigma\in\mathcal{T}_{s,T}} E^0[e^{-r(\sigma-s)}\Phi(S_{s,\sigma}(x),\sigma)],
$$

and for $\epsilon > 0$,

$$
\tilde{U}_\epsilon(x,s) = \sup_{\sigma\in\mathcal{T}_{s,T-\epsilon}} E^0[e^{-r(\sigma-s)}\Phi(S_{s,\sigma}(x),\sigma)].
$$

Then $\tilde{U}_\epsilon \le \tilde{U}$ and $\lim_{\epsilon\to 0}\tilde{U}_\epsilon(x,s) = \tilde{U}$ holds. Since $\tilde{U}_\epsilon \le U$ holds for any $\epsilon > 0$, we have $\tilde{U} \le U$.

We will prove that the converse inequality $U \le \tilde{U}$ holds on $\mathcal{E}^+ := \{(x,s); \Phi(x,s) \ge 0\}$ (Note that $\mathcal{E}^+ \supset \mathcal{E}^A$, since $V_E \ge V$). Let $\hat{\sigma} = \tau_s^x((\mathcal{E}^+)^c)$. Then for any $\sigma \in \mathcal{T}_{s,T}$ and $(x,s) \in \mathcal{E}^+$, we have

$$
\begin{aligned}
E^0[e^{-r(\sigma-s)}\Phi(S_{s,\sigma}(x),\sigma)1_{\sigma<T}] &\le E^0[e^{-r(\sigma\wedge\hat{\sigma}-s)}\Phi(S_{s,\sigma\wedge\hat{\sigma}}(x),\sigma\wedge\hat{\sigma})1_{\sigma\wedge\hat{\sigma}<T}] \\
&\le E^0[e^{-r(\sigma\wedge\hat{\sigma}-s)}\Phi(S_{s,\sigma\wedge\hat{\sigma}}(x),\sigma\wedge\hat{\sigma})].
\end{aligned}
$$

The last inequality follows since $\Phi(S_{s,\sigma\wedge\hat{\sigma}}(x),\sigma\wedge\hat{\sigma})1_{\sigma\wedge\hat{\sigma}=T} \ge 0$ holds a.s. if $(x,s) \in \mathcal{E}^+$. The last term is dominated by $\tilde{U}(x,s)$. Then we get $U \le \tilde{U}$ on $\mathcal{E}^+$. We have thus shown the equality $U = \tilde{U}$ on $\mathcal{E}^+$.

We want to show that the stopping time $\hat{\sigma}_s^x = \tau_s^x(\mathcal{E}^K)$ attains the maximum at (4.5). Take $\hat{\sigma}_s^x$ at (4.5). Then we get

$$\begin{aligned}\tilde{U}(x,s) &\geq E^0[e^{-r(\hat{\sigma}_s^x - s)}V_E(S_{s,\hat{\sigma}_s^x}(x), \hat{\sigma}_s^x)] - \delta E^0[e^{-r(\hat{\sigma}_s^x - s)}] \\ &= V_E(x,s) - \delta E^0[e^{-r(\hat{\sigma}_s^x - s)}],\end{aligned}$$

where the last equality follows from (4.3). Now take any (x,s) from $\mathcal{E}^A$. If $x \neq K$, we have

$$U(x,s) = \tilde{U}(x,s) > \Phi(x,s)$$

because $0 \leq E^0[e^{-r(\hat{\sigma}_s^x - s)}1_{\hat{\sigma}_s^x < T}] < 1$, which contradicts to the fact $(x,s) \in \mathcal{E}^A$. Therefore we have $x = K$. We have thus shown $\mathcal{E}^A = \mathcal{E}^K$.

We will next study the case where $d > 0$. We consider a new market where the price process is given by $\tilde{S}_{s,t}(x) = e^{d(t-s)}S_{s,t}(x)$, which has no longer dividend. The price process is a martingale with respect to P^0. Let $\tilde{V}(x,s)$ be the value function of the game call with respect to $\tilde{S}_{s,t}(x)$ with the exercise price K and the penalty δ. Then, it holds $\tilde{V}(x,s) \geq V(x.s)$ for any (x,s). Let $\tilde{\mathcal{E}}^A$ be the cancellation region of the writer. Then we have $\mathcal{E}^A \subset \tilde{\mathcal{E}}^A$. Since $\tilde{S}_{s,t}$ has no dividend, $\tilde{\mathcal{E}}_s^A$ is equal to $\tilde{\mathcal{E}}^K = \{(K,s); s < \tilde{\beta}\}$. Consequently $\mathcal{E}^A$ should be of the form $\mathcal{E}^K$. $\square$

5 Derivatives of the value function

It is known (e.g. Karatzas and Shreve (1998)) that the value function $V(x,s)$ is of $C^{2,1}$ class in the domain $\mathcal{C}$, i.e., it is twice continuously differentiable with respect to x and once continuously differentiable with respect to s. Define the partial differential operator $\mathcal{L}$ by

$$\mathcal{L}U = \frac{\partial U}{\partial s} + \frac{1}{2}\kappa^2 x^2 \frac{\partial^2 U}{\partial x^2} + (r-d)x\frac{\partial U}{\partial x} - rU. \tag{5.1}$$

Then V satisfies

$$\mathcal{L}V(x,s) = 0, \quad \text{in } \mathcal{C}. \tag{5.2}$$

The following lemma will be useful in later discussions.

Lemma 5.1. *Let $\mathcal{D}$ be a convex domain in $\mathbf{R}^+ \times [0,T)$. Let $U(x,s)$ be of $\mathcal{C}^{2,1}$ class function on $\mathcal{D}$ satisfying $\mathcal{L}U(x,s) = 0$ for any $(x,s) \in \mathcal{D}$, where $\mathcal{L}$ is the differential operator of (5.1).*

Suppose that $U(x,s)$ is positive, nonincreasing with respect to s. Suppose further that it is either nonincreasing or nondecreasing with respect to x. Then for each s, $U(x,s)$ is strictly convex with respect to x in $\mathcal{D}_s = \{x; (x,s) \in \mathcal{D}\}$, i.e., $\frac{\partial^2}{\partial x^2}U(x,s) > 0$ holds for *any $x \in \mathcal{D}_s$ so that $\frac{\partial U}{\partial x}$ is strictly increasing. In particular, $U(x,s)$ is strictly increasing or strictly decreasing for $x \in \mathcal{D}_s$, according as $U(x,s)$ is nondecreasing or nonincreasing, respectively.*

Proof. Note that U satisfies the equality

$$\frac{1}{2}\kappa^2 x^2 \frac{\partial^2 U}{\partial x^2} = -\frac{\partial U}{\partial s} - (r-d)x\frac{\partial U}{\partial x} + rU, \quad \forall (x,s) \in \mathcal{D}$$

by (5.1). Suppose first that $r \le d$ and $\frac{\partial U}{\partial x} \ge 0$. Then we have $-\frac{\partial U}{\partial s} \ge 0$, $-(r-d)x\frac{\partial U}{\partial x} \ge 0$ and $rU > 0$. Therefore $\frac{\partial^2 U}{\partial x^2}(x,s) > 0$. Then $U(x,s)$ is strictly convex.

Suppose next that $r > d$ and $\frac{\partial U}{\partial x} \ge 0$. Set $\tilde{U}(x,s) = U(-x,s)$ for $x < 0$. Then,

$$\begin{aligned}
&\frac{1}{2}\kappa^2 x^2 \frac{\partial^2 \tilde{U}}{\partial x^2} + \frac{\partial \tilde{U}}{\partial s} - (r-d)x\frac{\partial \tilde{U}}{\partial x} - r\tilde{U} \\
&\quad = \frac{1}{2}\kappa^2 x^2 \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial s} + (r-d)x\frac{\partial U}{\partial x} - rU = 0.
\end{aligned}$$

We can repeat the similar argument as the above to the function $\tilde{U}$, and we find that the function $\tilde{U}$ is a strictly convex function. Then U is also a strictly convex function.

We can prove the assertion of the lemma in the case where $\frac{\partial U}{\partial x} \le 0$, similarly. □

In the remainder of this section, we will complete the proof of Theorem 2.2. If $s \ge \beta$, $V(x,s)$ coincides with the value function of the European call or American call. Then assertion (3) is known. So we will restrict our attention to the case $s < \beta$.

Lemma 5.2. *Suppose $d = 0$. Let $s < \beta$. a) The first derivative $\frac{\partial V}{\partial x}(x,s)$ is continuous in x except $x = K$. It is strictly increasing, satisfies $\frac{\partial V}{\partial x}(0+,s) = 0$, $0 < \frac{\partial V}{\partial x}(x,s) < 1$ in $(0,K) \cup (K,\infty)$ and $\lim_{x\to\infty} \frac{\partial V}{\partial x}(x,s) = 1$. b) At $x = K$ we have (2.15). c) It holds $V(x,s) < V_E(x,s)$ for any x. d) $V(x,s)$ is a convex function of $x \in \mathbf{R}^+$ for any s.*

Proof. We shall consider the function $U(x,s) = V_E(x,s) - V(x,s)$. It satisfies (4.3), where the supremum is attained by the stopping time $\hat{\sigma}_s^x = \tau_s^x(\mathcal{E}^A)$. Therefore

$$(5.3) \qquad \begin{aligned} U(x,s) &= E^0[e^{-r(\hat{\sigma}_s^x - s)}\{V_E(S_{s,\hat{\sigma}_s^x}(x), \hat{\sigma}_s^x) - (S_{s,\hat{\sigma}_s^x}(x) - K)^+ - \delta\}1_{\hat{\sigma}_s^x < T}] \\ &= E^0[e^{-r(\hat{\sigma}_s^x - s)}(V_E(K, \hat{\sigma}_s^x) - \delta)1_{\hat{\sigma}_s^x < T}]. \end{aligned}$$

It is a bounded positive function since $V_E(K,\hat{\sigma}_s^x) - \delta > 0$ holds if $\hat{\sigma}_s^x < T$. It is nondecreasing with respect to x for $x < K$. Indeed, noting that $\hat{\sigma}_s^x \downarrow$ as $x \uparrow$ for $x < K$ and $\hat{\sigma}_s^x \uparrow$ as $x \uparrow$ for $x > K$, then the family of random variables

$$R^x = e^{-r(\hat{\sigma}_s^x - s)}(V_E(K,\hat{\sigma}_s^x) - \delta)1_{\hat{\sigma}_s^x < T}$$

are nondecreasing with respect to x if $x < K$ and is nonincreasing with respect to x if $x > K$. Then the expected value $U(x,s) = E^0[R^x]$ is nondecreasing if $x < K$ and is nonincreasing if $x < K$. Consequently we have $\frac{\partial U}{\partial x} \geq 0$ if $x < K$ and $\frac{\partial U}{\partial x} \leq 0$ if $x > K$.

We will prove (a). The strict increasing property of $\frac{\partial V}{\partial x}$ follows from Lemma 5.1. Observe the equality

$$\frac{\partial U}{\partial x}(0+,s) = \frac{\partial V_E}{\partial x}(0+,s) - \frac{\partial V}{\partial x}(0+,s).$$

Each term is nonnegative. We know $\frac{\partial V_E}{\partial x}(0+,s) = 0$. Therefore we have $\frac{\partial V}{\partial x}(0+,s) = 0$. We will next show that $c(s) = \lim_{x\to\infty} \frac{\partial V}{\partial x}(x,s)$ is equal to 1. If it is not the case, we have $\lim_{x\to\infty} \frac{\partial V}{\partial x}(x,s) = 1 - c(s) > 0$. Then, since $\lim_{x\to\infty} \frac{\partial V_E}{\partial x}(x,s) = 1$, we have $\lim_{x\to\infty} U(x,s) = \infty$ holds. But this is a contradiction since $U(x,s)$ is a bounded function.

We will next prove (b). Note that $U(x,s)$ is nonincreasing with respect to s for any x in view of (4.3), since $\mathcal{T}_{s,T} \subset \mathcal{T}_{s',T}$ holds if $s' < s$. Furthermore it satisfies $\mathcal{L}U = 0$ in two convex domains $(0,K) \times [0,T)$ and $(K,\infty) \times [0,T)$. We can apply Lemma 5.1 to each domain and we can conclude that $U(x,s)$ is strictly convex with respect to x both in the intervals $(0,K)$ and (K,∞). Consequently, $\frac{\partial U}{\partial x}(x,s)$ is positive and strictly increasing with respect to $x \in (0,K)$, which implies $\frac{\partial U}{\partial x}(K-,s) > 0$. Also if $x \in (K,\infty)$ $\frac{\partial U}{\partial x}$ is strictly increasing. It converges to 0 as $x \to \infty$, since both of $\frac{\partial V_E}{\partial x}(x,s)$ and $\frac{\partial V}{\partial x}(x,s)$ converges to 1 as $x \to \infty$. Therefore we get $\frac{\partial U}{\partial x}(K+,s) < 0$. Since $V_E(x,s)$ is continuously differentiable at $x = K$, we obtain at $x = K$,

$$\frac{\partial V}{\partial x}(K-,s) < \frac{\partial V_E}{\partial x}(K-,s) = \frac{\partial V_E}{\partial x}(K+,s) < \frac{\partial V}{\partial x}(K+,s).$$

We have thus proved (2.15) in the case $s < \beta$.

We will next show (c). The function $U(x,s) = V_E(x,s) - V(x,s)$ is strictly increasing in $(0,K)$. Since $V_E(0+,s) = V(0+,s) = 0$, we have $V_E(x,s) > V(x,s)$ for any $x \in (0,K)$. For $x \in (K,\infty)$, $\frac{\partial U}{\partial x}$ is strictly increasing and $\frac{\partial U}{\partial x}(+\infty,s) = 0$. Then $\frac{\partial U}{\partial x}(x,s) < 0$ for any $x \in (K,\infty)$. Since $U(+\infty,s) = 0$ holds, we get $U(x,s) > 0$ for any $x \in (K,\infty)$, proving $V_E(x,s) > V(x,s)$ for $x \in (K,\infty)$.

So far we have shown that $\frac{\partial V}{\partial x}(x,s)$ is increasing with respect to $x \in \mathbf{R}^+$. Therefore the function $V(x,s)$ is a convex function of $x \in \mathbf{R}^+$ for any $s < \beta$, proving (d). □

We will next consider the case where $d > 0$.

Lemma 5.3. *Assume $d > 0$. Then $\frac{\partial V}{\partial x}$ is continuous at $x = b(s)$. We have (2.17).*

Proof. We have clearly $\frac{\partial V}{\partial x}(b(s)+,s) = 1$. Further it holds $\frac{\partial V}{\partial x}(b(s)-,s) \leq 1$, by Lemma 3.1. We want to show that the equality holds. The following argument is consulted with

Karatzas and Shreve (1998). For a given (x,s) and $\epsilon \geq 0$, set $\hat{\tau}_s^{x-\epsilon} = \tau_s^{x-\epsilon}(\mathcal{E}^B) := \inf\{t \in [s,T); S_{s,t}(x-\epsilon) \geq b(t)\} \wedge T$. In the following we consider the case $x = b(s)$ $(> K)$. Let $\epsilon > 0$ be such that $x - \epsilon > K$. Then $\hat{\tau}_s^{x-\epsilon}$ is nondecreasing in ϵ and $\hat{\tau}_s^x = s$ a.s. Further $\hat{\tau}_s^{x-\epsilon} \downarrow$ as $\epsilon \downarrow 0$ a.s. Observe the inequality

$$V(x-\epsilon,s) - V(x,s) \quad \leq \quad J_s^{x-\epsilon}(\hat{\sigma}_s^x, \hat{\tau}_s^{x-\epsilon}) - J_s^x(\hat{\sigma}_s^x, \hat{\tau}_s^{x-\epsilon}),$$

where $\hat{\sigma}_s^x := \tau_s^x(\mathcal{E}^A)$. It holds by (2.5),

$$\begin{aligned} J_s^{x-\epsilon}(\hat{\sigma}_s^x, \hat{\tau}_s^{x-\epsilon}) &= E^0\left[e^{-r(\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon} - s)}\{(S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x-\epsilon) - K)^+ + \delta 1_{\hat{\sigma}_s^x < \hat{\tau}_s^{x-\epsilon}}\}\right], \\ J_s^x(\hat{\sigma}_s^x, \hat{\tau}_s^{x-\epsilon}) &= E^0\left[e^{-r(\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon} - s)}\{(S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x) - K)^+ + \delta 1_{\hat{\sigma}_s^x < \hat{\tau}_s^{x-\epsilon}}\}\right]. \end{aligned}$$

Since $x - \epsilon > K$, we have $S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x-\epsilon) \geq K$ and $S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x) \geq K$ if $\hat{\tau}_s^{x-\epsilon} < \hat{\sigma}_s^x$. Then,

$$(S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x-\epsilon) - K)^+ - (S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x) - K)^+ = -\epsilon H(s, \hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}),$$

if $\hat{\tau}_s^{x-\epsilon} < \hat{\sigma}_s^x$. Further,

$$(S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x-\epsilon) - K)^+ - (S_{s,\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}}(x) - K)^+ \leq \epsilon H(s,T)$$

on the set $\hat{\tau}_s^{x-\epsilon} \geq \hat{\sigma}_s^x$. Therefore we have

$$\begin{aligned} V(x-\epsilon,s) - V(x,s) \quad \leq \quad & -\epsilon E^0[e^{-r(\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon} - s)} H(s, \hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}) 1_{\hat{\tau}_s^{x-\epsilon} < \hat{\sigma}_s^x}] \\ & + \epsilon E^0[e^{-r(T-s)} H(s,T) 1_{\hat{\tau}_s^{x-\epsilon} \geq \hat{\sigma}_s^x}]. \end{aligned}$$

Divide the above inequality by $-\epsilon < 0$ and let ϵ tend to 0. Then we obtain

$$\begin{aligned} & \liminf_{\epsilon \downarrow 0} \frac{V(x-\epsilon,s) - V(x,s)}{-\epsilon} \\ & \geq \lim_{\epsilon \downarrow 0} E^0[e^{-r(\hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon} - s)} H(s, \hat{\sigma}_s^x \wedge \hat{\tau}_s^{x-\epsilon}) 1_{\hat{\tau}_s^{x-\epsilon} < \hat{\sigma}_s^x}] \\ & \quad - \lim_{\epsilon \downarrow 0} E^0[e^{-r(T-s)} H(s, \hat{\sigma}_s^x) 1_{\hat{\tau}_s^{x-\epsilon} \geq \hat{\sigma}_s^x}]. \end{aligned}$$

The first term of the right hand side is equal to 1, since $\hat{\tau}_s^{x-\epsilon} \to s$ a.s. P^0 as $\epsilon \downarrow 0$. The last term of the above is 0, since $P^0(\hat{\tau}_s^{x-\epsilon} \geq \hat{\sigma}_s^x)$ tends to 0 as $\epsilon \downarrow 0$. The above lim inf is greater than or equal to 1. Therefore we get $\frac{\partial V}{\partial x}(b(s)-,s) \geq 1$. □

As an application of Lemmas 5.1 and 5.3, we show the following.

Lemma 5.4. *Suppose $d > 0$. Let $\mathcal{E}_s^B = [b(s), \infty)$ be the exercise region of the holder. Then, $b(s)$ satisfies (2.11) and (2.12).*

Proof. We saw the inequality $b(s) > K$ already. We will prove $dx - rK > 0$ if $x \geq b(s)$. We will first prove

$$\mathcal{L}V(b(s)+, s) < 0, \quad \forall s.$$

We have $\mathcal{L}V(b(s)-, s) = 0$ since $\mathcal{L}V = 0$ holds in $\mathcal{C}$. Since V is strictly convex with respect to $x \in \mathcal{C}_s$ by Lemma 5.1, we have $\frac{\partial^2 V}{\partial x^2}(b(s)-, s) > 0$. On the other hand we have $\frac{\partial^2 V}{\partial x^2}(b(s)+, s) = 0$, because $V(x, s) = x - K$ for $x \geq b(s)$. Therefore,

$$\frac{\partial^2 V}{\partial x^2}(b(s)+, s) < \frac{\partial^2 V}{\partial x^2}(b(s)-, s).$$

Since $\frac{\partial V}{\partial x}$ and V are continuous at $(b(s), s)$, we get $\mathcal{L}V(b(s)+, s) < \mathcal{L}V(b(s)-, s) = 0$.

Now, since $V(x, s) = x - K$ holds for $x > b(s)$, we have

$$\mathcal{L}V(x, s) = -(dx - rK), \quad x > b(s) \tag{5.4}$$

by a direct computation. Therefore we have $\mathcal{L}V(b(s)+, s) = -(db(s) - rK) < 0$. We have thus proved the inequality of (2.11).

Now if $s \geq \beta$, $V(x, s) = V_A(x, s)$ holds. Then $b(s), s \geq \beta$ coincides with the boundary of the exercise region of the American option. Therefore we have (2.12). See e.g. Karatzas and Shreve (1998). □

Lemma 5.5. *Suppose $d > 0$. Let $s < \beta$. a) The first derivative $\frac{\partial V}{\partial x}(x, s)$ is continuous in x except $x = K$. It is strictly increasing, satisfies $\frac{\partial V}{\partial x}(0+, s) = 0$, $0 < \frac{\partial V}{\partial x}(x, s) < 1$ in $(0, K) \cup (K, b(s))$ and $\frac{\partial V}{\partial x}(x, s) = 1$ in $(b(s), \infty)$. b) At $x = K$ we have (2.15). c) $V(x, s) < V_A(x, s)$ holds valid for any x. d) $V(x, s)$ is a convex function of x for any s.*

Proof. Define

$$\tilde{V}_A(x, s) = E^0[e^{-r(\hat{\tau}_s^x - s)}(S_{s, \hat{\tau}_s^x}(x) - K)^+]. \tag{5.5}$$

We consider the function $\tilde{U}(x, s) = \tilde{V}_A(x, s) - V(x, s)$. It enjoys properties similar to those of $U(x, s)$ in the previous lemma. Then we can show (a)-(d) similarly as in lemma 5.2. □

6 Early exercise premiums

The value function $V(x, s)$ is locally Lipschitz continuous in $\mathbf{R}^+ \times [0, T)$ and of $\mathcal{C}^{2,1}$-class in the domain $\mathcal{C} = (\mathcal{E}^A \cup \mathcal{E}^B)^c$. At $(x, s) = (K, s) \in \mathcal{E}^A$, V is differentiable with respect to s except the point $s = \beta$ and the derivative is bounded for $s \in (0, \beta) \cup (\beta, T)$. On the other hand, $V(x, s)$ is not differentiable with respect to x at $x = K$ if $s < \beta$, since the right derivative and the left derivative do not coincide. However the function $V(x, s)$ is convex with respect to x for any s. Then Itô-Tanaka-Meyer's formula for convex function can be extended to time depending convex function $V(x, t)$. The formula can be written using the local time of the price process. Let $L(t) = L^{(s,x)}(t)$ be the local time of the semimartingale $\{S_{s,t}(x), t \geq s\}$ at the point K. Then we have

$$
\begin{aligned}
& e^{-r(t-s)}V(S_{s,t}(x), t) - V(x, s) \\
= & \int_s^t e^{-r(u-s)} \kappa S_{s,u}(x) \frac{\partial V}{\partial x}(S_{s,u}(x), u) dW^0(u) \\
& + \int_s^t e^{-r(u-s)} \mathcal{L}V(S_{s,u}(x), u) du \\
& + \int_s^t e^{-r(u-s)} \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u).
\end{aligned} \tag{6.1}
$$

(e.g. Karatzas and Shreve (1991)).

Theorem 6.1. *The value function $V(x, s)$ is represented by*

$$
\begin{aligned}
V(x, s) = & V_E(x, s) \\
& + E^0 \left[\int_s^T e^{-r(u-s)} (dS_{s,u}(x) - rK) I_{S_{s,u}(x) > b(u)} du \right] \\
& - E^0 \left[\int_s^\beta e^{-r(u-s)} \left(\frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right) dL(u) \right]
\end{aligned} \tag{6.2}
$$

if $d > 0$, and by

$$
V(x, s) = V_E(x, s) - E^0 \left[\int_{s\wedge\beta}^\beta e^{-r(u-s)} \left(\frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right) dL(u) \right] \tag{6.3}
$$

if $d = 0$.

Proof. Suppose first $d > 0$. If $(S_{s,u}(x), u) \in \mathcal{C}$ or equivalently if $S_{s,u}(x) < b(u)$, we have $\mathcal{L}V(S_{s,u}(x), u) = 0$, and if $S_{s,u}(x) > b(u)$, we have $\mathcal{L}V(S_{s,u}(x), u) = -(dS_{s,u}(x) - rK) < 0$ by Lemma 5.4. Since $\frac{\partial V}{\partial x}$ is continuous at $x = K$ if $u > \beta$. Then the last term of (6.1) is equal to

$$
\int_{s\wedge\beta}^{t\wedge\beta} e^{-r(u-s)} \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u).
$$

Consequently we obtain

$$
\begin{aligned}
e^{-r(t-s)}V(S_{s,t}(x), t) = & V(x, s) \\
& + \int_s^t e^{-r(u-s)} \kappa S_{s,u}(x) \frac{\partial V}{\partial x}(S_{s,u}(x), u) dW^0(u) \\
& - \int_s^t e^{-r(u-s)} (dS_{s,u}(x) - rK) I_{S_{s,u}(x) > b(u)} du \\
& + \int_{s\wedge\beta}^{t\wedge\beta} e^{-r(u-s)} \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u).
\end{aligned} \tag{6.4}
$$

Now set $t = T$ in (6.4) and take the expectation. Since $V(x, T) = (x - K)^+$, we obtain

$$
\begin{aligned}
&E^0[e^{-r(T-s)}(S_{s,T}(x) - K)^+] = \\
&\quad V(x, s) - E^0 \left[\int_s^T e^{-r(u-s)} (dS_{s,u}(x) - rK) I_{S_{s,u}(x) > b(u)} du \right] \\
&\quad + E^0 \left[\int_{s \wedge \beta}^{\beta} e^{-r(u-s)} \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u) \right].
\end{aligned}
$$

This proves (6.2).

Suppose next that $d = 0$. Then there is no holder's exercise region. Then $\mathcal{L}V(x, s) = 0$ holds for any $(x, s) \in \mathcal{C}$. Then the second term of the right hand side of (6.1) is equal to 0 a.s. Then the corresponding term disappear in (6.2) and we obtain (6.3). $\square$

Remark. The last term of (6.2) is the writer's early exercise premium. Let $u < \beta$. Let $h_u(x, dt)$ be the hitting distribution of $\hat{\sigma}_u^x = \tau_u^x(\mathcal{E}^A)$, i.e., $h_u(x, dt) = P^0(\hat{\sigma}_u^x \in dt)$. Then in view of (5.3),

$$
U(x, u) = \int_u^{\beta} e^{-r(t-u)} (V_E(K, u) - \delta) h_u(x, dt).
$$

We can define a positive measure $m_u(dt)$ on $[u, \beta]$ by

$$
m_u(dt) = \frac{\partial}{\partial x} h_u(x, dt) \bigg|_{x=K-} - \frac{\partial}{\partial x} h_u(x, dt) \bigg|_{x=K+}.
$$

Then,

$$
\begin{aligned}
\frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) &= -\frac{\partial U}{\partial x}(K+, u) + \frac{\partial U}{\partial x}(K-, u) \\
&= \int_u^{\beta} e^{-r(t-u)} (V_E(K, t) - \delta) m_u(dt).
\end{aligned}
$$

Therefore the writer's early exercise premium is written by

$$
E^0 \left[\int_{s \wedge \beta}^{\beta} \left(\int_u^{\beta} e^{-r(t-u)} (V_E(K, t) - \delta) m_u(dt) \right) dL(u) \right], \tag{6.5}
$$

Here we note that the measure $m_u(dt)$ and the local time L_t are computed from the price process S_t.

In the case where $d > 0$, the writer's early exercise premium is written by

$$
E^0 \left[\int_s^{\beta} \left(\int_u^{\beta} e^{-r(t-u)} (\tilde{V}_A(K, t) - \delta) m_u(dt) \right) dL(u) \right].
$$

where $\tilde{V}_A$ is defined by (5.5).

The second term of the right hand side of (6.2) is positive if $d > 0$. It is called the *holder's early exercise premium*. It increases when the price process $S_{s,t}(x)$ is in the exercise region $[b(t), \infty)$, since $dS_{s,u}(x) - rK > 0$.

7 Hedgings of game call options

In this section we shall discuss the hedging problem of the game call option. Let $S_t, t \in [0,T)$ be a price process starting from S_0 at time 0. Let $z > 0$, let $\pi = (\beta(t), \gamma(t)), t \in [0,T)$ be a predictable process and let $C(t), t \in [0,T)$ be a continuous predictable increasing process. These are called the *initial wealth, the self-financing portfolio and the cumulative consumption process*, respectively. The *wealth process* is defined by

$$Z^{z,\pi,C}(t) = z + \int_0^t \beta(s)dB_s + \int_0^t \gamma(s)dS_s - C(t).$$

Let $R(u,v)$ be the pay-off function of the game call:

$$R(u,v) = ((S_u - K)^+ + \delta)1_{u<v} + (S_v - K)^+ 1_{v\le u}. \tag{7.1}$$

Let σ be a stopping time. A triple (σ, π, C) is called the *writer's upper hedging portfolio* of the pay-off function $R(u,v)$ if

$$R(\sigma, t) \le Z^{z,\pi,C}(t \wedge \sigma), \quad \forall t \;\; a.s. \tag{7.2}$$

is satisfied. The *upper hedging price* of the game call is defined by

$$h_{up} := \inf\{z;\ \exists(\sigma,\pi,C) \text{ such that } R(\sigma,t) \le Z^{z,\pi,C}(t\wedge\sigma) \;\; \forall t \; a.s.\}. \tag{7.3}$$

We will call the triple (σ^*, π^*, C^*) the *writer's optimal upper hedging portfolio* if it attains h_{up}.

Now let τ be a stopping time. A triple (τ, π', C') is called the *holder's hedging portfolio* if it satisfies

$$R(t,\tau) \ge -Z^{-z,\pi',C'}(t\wedge\tau), \quad \forall t \;\; a.s. \tag{7.4}$$

The *lower hedging price* is defined by

$$h_{low} = \sup\{z;\ \exists(\tau,\pi',C') \text{ such that } R(t,\tau) \ge -X^{-z,\pi',C'}(t\wedge\tau) \;\; \forall t \; a.s.\}. \tag{7.5}$$

We will call (τ_*, π_*, C_*) the *holder's optimal hedging portfolio* if it attains h_{low}.

Kifer (2000) defined the hedging price h by the infimum of z such that there exists $(\sigma, \pi, 0)$ satisfying $R(\sigma, t) \le Z^{z,\pi,0}_{t\wedge\sigma}$ for any t. Then we have

$$h = h_{up} \ge h_{low}. \tag{7.6}$$

He shows $h = V(S_0, 0)$.

Theorem 7.1. *Both h_{up} and h_{low} coincide with the value function $V(S_0, 0)$. Set*

$$
\begin{aligned}
\sigma^* &= \tau(\mathcal{E}^A), \\
\beta^*(t) &= \begin{cases} e^{-rt}\{V(S_t,t) - S_t \frac{\partial V}{\partial x}(S_t,t)\}, & \text{if } t < \tau(\mathcal{E}^A), \\ 0, & \text{if } t \geq \tau(\mathcal{E}^A), \end{cases} \\
\gamma^*(t) &= \begin{cases} \frac{\partial V}{\partial x}(S_t,t), & \text{if } t < \tau(\mathcal{E}^A), \\ 0, & \text{if } t \geq \tau(\mathcal{E}^A), \end{cases} \\
C^*(t) &= \begin{cases} 0, & \text{if } d = 0, \\ \int_0^{t\wedge\sigma^*}(dS_u - rK)I_{S_u \geq b(u)}du, & \text{if } d > 0, \end{cases}
\end{aligned}
\tag{7.7}
$$

where $\tau(\mathcal{E}^A)$ is the hitting time of the process S_t to the set $\mathcal{E}^A$. Then $(\sigma^*, \pi^* = (\beta^*, \gamma^*), C^*)$ is an optimal portfolio of the writer. It holds $R(\sigma^*, t) = Z_{t\wedge\sigma^*}^{z^*,\pi^*,C^*}$ for any t, where $z^* = V(S_0, 0)$.

Set

$$
\begin{aligned}
\tau_* &= \tau(\mathcal{E}^B), \\
\beta_*(t) &= \begin{cases} -e^{-rt}\{V(S_t,t) - S_t \frac{\partial V}{\partial x}(S_t,t)\}, & \text{if } t < \tau(\mathcal{E}^B), \\ 0, & \text{if } t \geq \tau(\mathcal{E}^B), \end{cases} \\
\gamma_*(t) &= \begin{cases} -\frac{\partial V}{\partial x}(S_t,t), & \text{if } t < \tau(\mathcal{E}^B), \\ 0, & \text{if } t \geq \tau(\mathcal{E}^B), \end{cases} \\
C_*(t) &= \int_0^{t\wedge\beta\wedge\tau_*} \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u),
\end{aligned}
\tag{7.8}
$$

where $\tau(\mathcal{E}^B)$ is the hitting time of the process S_t to the set $\mathcal{E}^B$. Then $(\tau_*, \pi_* = (\beta_*, \gamma_*), C_*)$ is an optimal hedging portfolio of the holder. It holds $R(t, \tau^*) = Z_{t\wedge\tau^*}^{-z_*,\pi_*,C_*}$ for any t.

Remark 1. The writer's optimal consumption $C^*(t)$ may be interpreted as the amount that he receives without any risk, provided that the holder does not ask the payment until time t. On the contrary if the holder ask the payment at the time τ_*, the amount of the consumption is 0. The holder's optimal consumption $C_*(t)$ may be interpreted as the amount that the holder receives without any risk provided that the writer does not cancel the option until time t. But the amount will be 0 if the writer cancel the option at the proper time σ^*.

Remark 2. The American call may be identified with the game call with sufficiently big penalty δ (See Theorem 2.2). Then by our definition, the upper hedging price of the American call with the exercise price K is equal to the infimum of z such that

$$
(S_t - K)^+ \leq X^{z,\pi,C}(t), \quad \forall t,
$$

since $\sigma^* = K$ holds a.s. Therefore it is equal to the definition of the upper hedging price of the American option given by Karatzas (1997). On the other hand, our definition of the lower hedging price seems to be new and it is different from his definition, where the endowment is involved. In our case, both consumptions and endowments are 0, so that the holder will not receive or loose any money without risk, provided that he stops the game option at τ_*.

Proof. Let σ^* be the stopping time in (7.7). Then the inequality

$$R(\sigma^*, t) \leq V(S_{t\wedge\sigma^*}, t \wedge \sigma^*), \quad \forall t \tag{7.9}$$

holds valid. Indeed, if $t < \sigma^*$ we have $R(\sigma^*, t) = (S_t - K)^+ \leq V(S_t, t)$, and if $t \geq \sigma^*$ we have $R(\sigma^*, t) = \delta = V(S_{\sigma^*}, \sigma^*)$, proving (7.9).

On the other hand, we have by Itô -Tanaka-Meyer's formula

$$\begin{aligned} & V(S_t, t) - V(S_0, 0) \\ &= \int_0^t \kappa S_u \frac{\partial V}{\partial x}(S_u, u) dW^0(u) + \int_0^t \mathcal{L}_0 V(S_u, u) du \\ &\quad + \int_s^t \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u), \end{aligned} \tag{7.10}$$

where

$$\begin{aligned} \mathcal{L}_0 U &= \frac{\partial U}{\partial s} + \frac{1}{2}\kappa^2 x^2 \frac{\partial^2 U}{\partial x^2} + (r-d)x\frac{\partial U}{\partial x} \\ &= rU + \mathcal{L}U \end{aligned}$$

and $L(t)$ is the local time of S_t at K. Therefore we have

$$\int_0^t \mathcal{L}_0 V(S_u, u) du = \int_0^t rV(S_u, u) du - \int_0^t (dS_u - rK) I_{S_u(x) < b(u)} du.$$

Consequently we have

$$\begin{aligned} V(S_t, t) &= V(S_0, 0) + \int_0^t \frac{\partial V}{\partial x}(S_u, u) \left\{ rS_u du + \kappa S_u dW^0(u)) \right\} \\ &\quad + \int_0^t \left\{ V(S_u, u) - \frac{\partial V}{\partial x}(S_u, u) S_u \right\} r du \\ &\quad - \int_0^t (dS_u - rK) I_{S_u \geq b(u)} du \\ &\quad + \int_s^t \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u) \\ &= V(S_0, 0) + \int_0^t \frac{\partial V}{\partial x}(S_u, u) dS_u \\ &\quad + \int_0^t \left\{ V(S_u, u) - \frac{\partial V}{\partial x}(S_u, u) S_u \right\} \frac{dB_u}{B_u} \\ &\quad - \int_0^{t\wedge\sigma^*} (dS_u - rK) I_{S_u > b(u)} du \\ &\quad + \int_s^t \left\{ \frac{\partial V}{\partial x}(K+, u) - \frac{\partial V}{\partial x}(K-, u) \right\} dL(u). \end{aligned}$$

Now suppose $d > 0$ and set $t = t \wedge \sigma^*$. Then the last term of the above is 0 since the local time $L(t)$ increases only at t such that $S_t = K$. Therefore we have

$$\begin{aligned} V(S_{t\wedge\sigma^*}, t\wedge\sigma^*) &= V(S_0,0) + \int_0^{t\wedge\sigma^*} \frac{\partial V}{\partial x}(S_u,u)dS_u \\ &\quad + \int_0^{t\wedge\sigma^*} \left\{ V(S_u,u) - \frac{\partial V}{\partial x}(S_u,u)S_u \right\} \frac{dB_u}{B_u} \\ &\quad - \int_0^{t\wedge\sigma^*} (dS_u - rK) I_{S_u > b(u)} du. \end{aligned}$$

If $d = 0$, the last term of the above is 0, since $b(u) = \infty$. The right hand side is written as $Z^{z^*,\pi^*,C^*}_{t\wedge\sigma^*}$ where $z^* = V(S_0,0)$ and the triple (σ^*,π^*,C^*) is given by (7.7). Then it is an optimal upper hedging portfolio of the writer. We have further $R(\sigma^*,t) = X^{z^*,\pi^*,C^*}$ for any t.

We next consider the lower hedging problem. Let τ^* be the stopping time of (7.8). Since z coincides with $V(S_0,0) = h_{up}$, it is an optimal hedging portfolio of the writer. Then the inequality

$$R(t,\tau_*) \geq V(S_{t\wedge\tau_*}, t\wedge\tau_*), \quad \forall t$$

holds. Indeed, if $t < \tau^*$ we have $R(t,\tau^*) \geq V(S_t,t)$ and if $t \geq \tau^*$ we have $R(t,\tau^*) = V(S_{\tau^*},\tau^*)$. We have further,

$$\begin{aligned} V(S_{t\wedge\tau_*}, t\wedge\tau_*) &= V(S_0,0) + \int_0^{t\wedge\tau_*} \frac{\partial V}{\partial x}(S_u,u)dS_u \\ &\quad + \int_0^{t\wedge\tau^*} \left\{ V(S_u,u) - \frac{\partial V}{\partial x}(S_u,u)S_u \right\} \frac{dB_u}{B_u} \\ &\quad + \int_0^{t\wedge\beta\wedge\tau_*} \left\{ \frac{\partial V}{\partial x}(K+,u) - \frac{\partial V}{\partial x}(K-,u) \right\} dL(u). \end{aligned}$$

The right hand side is written as $-Z^{-z_*,\pi_*,C_*}$, where $z_* = V(S_0,0)$ and (π_*,C_*) is given by (7.8). It is a lower hedging wealth process with (τ_*,π_*,C_*) starting from $V(S_0,0)$. This yields $h_{low} \geq V(S_0,0)$. Then we have $h_{low} = V(S_0,0)$ and the triple (τ_*,π_*,C_*) is an optimal hedging portfolio of the holder. We have further $R(t,\tau^*) = -X^{-z_*,\pi_*,C_*}$ for any t. □

8 Summary for game put option

Finally we shall discuss briefly the game put option. The value function of the game put option with the exercise price K and the penalty δ is defined by

$$\hat{V}(x,s) = \hat{V}_T(x,s) = \inf_{\sigma\in\mathcal{T}_{s,T}} \sup_{\tau\in\mathcal{T}_{s,T}} \hat{J}^x_s(\sigma,\tau). \tag{8.1}$$

Here,

$$\hat{J}^x_s(\sigma,\tau) = E^0\left[e^{-r(\sigma\wedge\tau - s)}\left\{(K - S_{s,\sigma}(x))^+ + \delta)1_{\sigma<\tau} + (K - S_{s,\tau}(x))^+ 1_{\tau\le\sigma}\right\}\right],$$

and $\mathcal{T}_{s,T}$ denotes the totality of stopping times with values in the interval $[s,T]$. Then the inequality

$$(K-x)^+ \le \hat{V}(x,s) \le (K-x)^+ + \delta, \quad \forall (x,s) \in \mathbf{R}^+ \times [0,T) \tag{8.2}$$

holds. We will define subsets of $\mathbf{R}^+ \times [0,T)$ concerning the game put by

$$\begin{array}{rcl} \hat{\mathcal{C}} & = & \{(x,s) \in \mathbf{R}^+ \times [0,T); (x-K)^+ < \hat{V}(x,s) < (x-K)^+ + \delta\}, \\ \hat{\mathcal{E}}^A & = & \{(x,s) \in \mathbf{R}^+ \times [0,T); \hat{V}(x,s) = (x-K)^+ + \delta\}, \\ \hat{\mathcal{E}}^B & = & \{(x,s) \in \mathbf{R}^+ \times [0,T); \hat{V}(x,s) = (x-K)^+\}. \end{array} \tag{8.3}$$

The set $\hat{\mathcal{C}}$ is called the *continuation region* of the game put option, the set $\hat{\mathcal{E}}^B$ is called the *exercise region* of the holder of the option and the set $\hat{\mathcal{E}}^A$ is called the *cancellation region* of the writer of the option. Then the infimum and the supremum of $\hat{J}^x_s$ are attained by the following two stopping times $\hat{\sigma}^x_s = \tau^x_s(\hat{\mathcal{E}}^A)$ and $\hat{\tau}^x_s = \tau^x_s(\hat{\mathcal{E}}^B)$ of $\mathcal{T}_{s,T}$.

$$\hat{V}(x,s) = \hat{J}^x_s(\hat{\sigma}^x_s, \hat{\tau}^x_s), \quad \forall (x,s) \in \mathbf{R}^+ \times [0,T). \tag{8.4}$$

The upper hedging price and the lower hedging price of the game put are defined similarly as those of the game call. A part of the following theorem is obtained by Kühn and Kyprianou (2004), where the game put option is called the Israeli put option. The proof of the theorem can be done similarly as in the case of the game call option. Details will be discussed elsewhere.

The theorem can be applied for both cases $d = 0$ and $d > 0$.

Theorem 8.1. *1) The holder's exercise region $\hat{\mathcal{E}}^B$ is nonempty. Its section $\hat{\mathcal{E}}^B_s$ is an interval*

$$\hat{\mathcal{E}}^B_s = \{x; 0 < x \le \hat{b}(s)\}, \tag{8.5}$$

where $(\hat{b}(s), s \in [0,T))$ is a nondecreasing function satisfying

$$0 < \hat{b}(s) < K. \tag{8.6}$$

$$\lim_{s\to T} \hat{b}(s) = K \tag{8.7}$$

2) Let $\hat{\beta}$ be the infimum of t satisfying $\hat{V}(K,t) < \delta$. Then it holds $0 \le \beta < T$. Sections of the writer's cancellation region $\hat{\mathcal{E}}^A$ are given by

$$\hat{\mathcal{E}}^A_s = \begin{cases} \{K\}, & \textit{if } \ s \le \hat{\beta}, \\ \phi, & \textit{if } \ s > \hat{\beta}. \end{cases} \tag{8.8}$$

3) The value function $\hat{V}(x,s)$ is positive and locally Lipschitz continuous in (x,s). For any s, $\hat{V}(x,s)$ is convex and strictly decreasing with respect to x. For any x, it is nonincreasing with respect to s.

4) If $s \geq \hat{\beta}$, we have $\hat{V}(x,s) = \hat{V}_A(x,s)$ for any x, where $\hat{V}_A(x,s)$ is the value function of the American put with the exercise price K. If $s < \hat{\beta}$, we have $\hat{V}(x,s) < \hat{V}_A(x,s)$ for any x.

In particular, if $\hat{V}_A(K,0) \leq \delta$, then $\hat{\beta} = 0$ and the value of the game put option and that of the American put option coincide each other.

5) The value function $\hat{V}(x,s)$ is twice continuously differentiable in $(0,\hat{b}(s)) \cup (\hat{b}(s),K) \cup (K,\infty)$. The derivative $\frac{\partial \hat{V}}{\partial x}$ is strictly decreasing and satisfies $-1 < \frac{\partial \hat{V}}{\partial x} < 0$ at $(\hat{b}(s),K) \cup (K,\infty)$. Further, at the boundary points $\hat{b}(s)$, we have

$$\frac{\partial \hat{V}}{\partial x}(\hat{b}(s)-,s) = \frac{\partial \hat{V}}{\partial x}(\hat{b}(s)+,s) = -1, \quad \forall s \in [0,T), \tag{8.9}$$

and at $x = K$, we have

$$\begin{array}{rcll} -1 < \dfrac{\partial \hat{V}}{\partial x}(K-,s) & < & \dfrac{\partial \hat{V}}{\partial x}(K+,s) < 0, & \textit{if } \ s < \hat{\beta}, \\ -1 < \dfrac{\partial \hat{V}}{\partial x}(K-,s) & = & \dfrac{\partial \hat{V}}{\partial x}(K+,s) < 0, & \textit{if } \ s \geq \hat{\beta}. \end{array} \tag{8.10}$$

6) If $s < \beta$, the function $\hat{V}(x,s)$ is represented by

$$\begin{aligned} \hat{V}(x,s) &= \hat{V}_E(x,s) + E^0\left[\int_s^T e^{-r(u-s)} rK 1_{S_{s,u}(x) < \hat{b}(u)} du\right] \\ &\quad - E^0\left[\int_s^{\hat{\beta}} e^{-r(u-s)} \left(\frac{\partial \hat{V}}{\partial x}(K+,u) - \frac{\partial \hat{V}}{\partial x}(K-,u)\right) dL(u)\right], \end{aligned}$$

where $\hat{V}_E(x,s)$ is the value function of the European put option with the exercise price K and L is the local time of the price process.

7) The upper hedging price $\hat{h}_{up}$ and the lower hedging price $\hat{h}_{low}$ coincide with $\hat{V}(S_0,0)$. Further the writer's optimal hedging portfolio $(\hat{\sigma}^*, \hat{\pi}^*, \hat{C}^*)$ is given by

$$\begin{aligned} \hat{\sigma}^* &= \tau(\hat{\mathcal{E}}^A), \\ \hat{\beta}^*(t) &= \begin{cases} e^{-rt}\{\hat{V}(S_t,t) - S_t \frac{\partial \hat{V}}{\partial x}(S_t,t)\}, & \textit{if } \ t < \hat{\tau}(\hat{\mathcal{E}}^A), \\ 0, & \textit{if } \ t \geq \hat{\tau}(\hat{\mathcal{E}}^A), \end{cases} \\ \hat{\gamma}^*(t) &= \begin{cases} \frac{\partial \hat{V}}{\partial x}(S_t,t), & \textit{if } \ t < \hat{\tau}(\hat{\mathcal{E}}^A), \\ 0, & \textit{if } \ t \geq \hat{\tau}(\hat{\mathcal{E}}^A), \end{cases} \\ \hat{C}^*(t) &= \int_0^{t \wedge \hat{\sigma}^*} rK I_{S_u \leq \hat{b}(u)} du. \end{aligned} \tag{8.11}$$

The holder's optimal portfolio $(\hat{\tau}_*, \hat{\pi}_*, \hat{C}_*)$ *is given by*

$$
\begin{aligned}
\hat{\tau}_* &= \tau(\hat{\mathcal{E}}^B), \\
\hat{\beta}_*(t) &= \begin{cases} -e^{-rt}\{\hat{V}(S_t,t) - S_t\frac{\partial \hat{V}}{\partial x}(S_t,t)\}, & \text{if } t < \tau(\hat{\mathcal{E}}^B), \\ 0, & \text{if } t \geq \tau(\hat{\mathcal{E}}^B), \end{cases} \\
\hat{\gamma}_*(t) &= \begin{cases} -\frac{\partial \hat{V}}{\partial x}(S_t,t), & \text{if } t < \tau(\hat{\mathcal{E}}^B), \\ 0, & \text{if } t \geq \tau(\hat{\mathcal{E}}^B), \end{cases} \\
\hat{C}_*(t) &= \int_0^{t\wedge\beta\wedge\hat{\tau}_*} \left\{ \frac{\partial \hat{V}}{\partial x}(K+,u) - \frac{\partial \hat{V}}{\partial x}(K-,u) \right\} dL(u).
\end{aligned}
\tag{8.12}
$$

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