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Katsumi Sasaki and Shigeo Ohama

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The provability logic $\mathbf{R}^-$ introduced by Guaspari and Solovay

Katsumi Sasaki
sasaki@ms.nanzan-u.ac.jp
Nanzan University

Shigeo Ohama
ohama@toyota-ct.ac.jp
Toyota National College of Technology

Abstract. To discuss Rosser sentences, Guaspari and Solovay [GS79] enriched the modal language by adding, for each $\Box A$ and $\Box B$, the formulas $\Box A \prec \Box B$ and $\Box A \preceq \Box B$, with their arithmetic realizations the Σ_1-sentences "A^* is provable by a proof that is smaller than any proof of B^*", and "A^* is provable by a proof that is smaller than or equal to any proof of B^*". They axiomatized modal logic $\mathbf{R}^-$ complete for the above arithmetic interpretation. Here we introduce a sequent system for $\mathbf{R}^-$ with a kind of subformula property.

1 The logic $\mathbf{R}^-$

We use lower case Latin letters p, q, r, possibly with suffixes, for propositional variables. We use $\bot$ (contradiction), and logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\Box$ (provability), $\preceq$ (witness comparison), and $\prec$ (witness comparison).

Definition 1.1. Formulas are defined inductively as follows:
(1) every propositional variable is a formula,
(2) $\bot$ is a formula,
(3) if A and B are formulas, then so are $(A \wedge B), (A \vee B)$ and $(A \supset B)$,
(4) if A is a formula, then so is $(\Box A)$,
(5) if $\Box A$ and $\Box B$ are formulas, then so are $(\Box A \prec \Box B)$ and $(\Box A \preceq \Box B)$.

We use upper case Latin letters $A, B, C, \cdots$, possibly with suffixes, for formulas. The expression $\neg A$ denotes the formula $A \supset \bot$. A formula of the form $\Box A$ is said to be a $\Box$-formula. Also a formula of the form $\Box A \preceq \Box B$ ($\Box A \prec \Box B$) is said to be a $\preceq$-formula ($\prec$-formula).

Definition 1.2. Sigma-formulas are defined inductively as follows:
(1) formulas $\Box A, \Box B, \Box A \prec \Box B$ and $\Box A \preceq \Box B$ are Sigma-formulas,
(2) if A and B are Sigma-formulas, then so are $(A \wedge B)$ and $(A \vee B)$.

Definition 1.3. The modal system $\mathbf{R}^-$ is defined by the following axioms and inference rules:
Axioms:
$A1$: all tautologies,
$A2$: $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
$A3$: $\Box(\Box A \supset A) \supset \Box A$,
$A4$: $A \supset \Box A$, where A is a Sigma-formula,
$A5$: $(\Box A \preceq \Box B) \supset \Box A$,
$A6$: $(\Box A \preceq \Box B) \wedge (\Box B \preceq \Box C) \supset (\Box A \preceq \Box C)$,
$A7$: $(\Box A \vee \Box B) \supset (\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$,
$A8$: $(\Box A \prec \Box B) \supset (\Box A \preceq \Box B)$,
$A9$: $(\Box A \preceq \Box B) \wedge (\Box B \prec \Box A) \supset \bot$,

Inference rules:
MP : $A, A \supset B \in \mathbf{R}^-$ implies $B \in \mathbf{R}^-$,
N : $A \in \mathbf{R}^-$ implies $\Box A \in \mathbf{R}^-$.

In [GS79] and Symoriński [Sym85], the following two formulas are also axioms of $\mathbf{R}^-$, but they are redundant

$A10 : \Box A \supset (\Box A \preceq \Box A)$,

$A11 : (\Box A \wedge \neg \Box B) \supset (\Box A \prec \Box B)$.

Lemma 1.4. *$A10$ and $A11$ are provable in $\mathbf{R}^-$.*

Proof. For $A10$, we use the following axioms:

$A1 : \Box A \supset \Box A \vee \Box A$,

$A7 : (\Box A \vee \Box A) \supset (\Box A \preceq \Box A) \vee (\Box A \prec \Box A)$,

$A8 : (\Box A \prec \Box A) \supset (\Box A \preceq \Box A)$.

For $A11$, we use the following axioms:

$A1 : \Box A \supset \Box B \vee \Box A$,

$A7 : (\Box B \vee \Box A) \supset (\Box B \preceq \Box A) \vee (\Box A \prec \Box B)$,

$A8 : (\Box B \preceq \Box A) \supset \Box B$,

and obtain

$\Box A \supset (\Box B \vee (\Box A \prec \Box B))$.

Definition 1.5. A Kripke pseudo-model for $\mathbf{R}^-$ is a triple $\langle \mathbf{W}, <, \models \rangle$ where

(1) $\mathbf{W}$ is a non-empty finite set,

(2) $<$ is an irreflexive and transitive binary relation on $\mathbf{W}$ satisfying

$$\alpha < \gamma \text{ and } \beta < \gamma \text{ imply either one of } \alpha = \beta,\ \alpha < \beta \text{ or } \beta < \alpha,$$

(3) $\models$ is a valuation satisfying, in addition to the usual boolean laws,

$$\alpha \models \Box A \text{ if and only if for any } \beta \in \alpha{\uparrow}\ (= \{\gamma \mid \alpha < \gamma\}),\ \beta \models A.$$

Definition 1.6. A Kripke pseudo-model $\langle \mathbf{W}, <, \models \rangle$ for $\mathbf{R}^-$ is said to be a Kripke model for $\mathbf{R}^-$ if the following conditions hold, for any formula A, B, and C,

(1) $\alpha \models \Box A \preceq \Box B$ implies for any $\beta \in \alpha{\uparrow}, \beta \models \Box A \preceq \Box B$

(2) $\alpha \models \Box A \prec \Box B$ implies for any $\beta \in \alpha{\uparrow}, \beta \models \Box A \prec \Box B$,

(3) $\alpha \models \Box A \preceq \Box B$ implies $\alpha \models \Box A$,

(4) $\alpha \models \Box A \preceq \Box B$ and $\alpha \models \Box B \preceq \Box C$ imply $\alpha \models \Box A \preceq \Box C$,

(5) $\alpha \models \Box A \vee \Box B$ implies $\alpha \models (\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$,

(6) $\alpha \models \Box A \prec \Box B$ implies $\alpha \models \Box A \preceq \Box B$,

(7) $\alpha \models \Box A \prec \Box B$ implies $\alpha \not\models \Box B \preceq \Box A$.

De Jongh [Jon87] and Voorbraak [Vor90] showed simpler proofs for the completeness theorem. Also their axiomatization of $\mathbf{R}^-$ is slightly different form Definition 1.3, but equivalent. They use the following axioms instead of $A7, A8$ and $A9$:

$\Box A \supset (\Box A \preceq \Box B) \vee (\Box B \preceq \Box A)$,

$(\Box A \prec \Box B) \equiv (\Box A \preceq \Box B) \wedge \neg(\Box B \preceq \Box A)$,

where $X \equiv Y = (X \supset Y) \wedge (Y \supset X)$.

Lemma 1.7. *$A \in \mathbf{R}^-$ if and only if A is valid in any Kripke model for $\mathbf{R}^-$.*

2 A sequent system for $\mathbf{R}^-$

In this section we introduce a sequent system $\mathbf{GR}^-$ for $\mathbf{R}^-$. We use Greek letters, possibly with suffixes, for finite sets of formulas, especially we use Σ for a finite set of Sigma-formulas. The expression Γ_A

denotes the set $\Gamma - \{A\}$. The expression $\Box\Gamma$ denotes the set $\{\Box A \mid A \in \Gamma\}$. By a sequent, we mean the expression

$$\Gamma \to \Delta.$$

For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

By $\mathsf{Sub}(A)$, we mean the set of subformulas of A. We put

$$\mathsf{Sub}^+(A) = \mathsf{Sub}(A) \cup \{\Box B \preceq \Box C | \Box B, \Box C \in \mathsf{Sub}(A)\} \cup \{\Box B \prec \Box C | \Box B, \Box C \in \mathsf{Sub}(A)\},$$

$$\mathsf{Sub}(\Gamma \to \Delta) = \bigcup_{B \in \Gamma \cup \Delta} \mathsf{Sub}(B),$$

$$\mathsf{Sub}^+(\Gamma \to \Delta) = \bigcup_{B \in \Gamma \cup \Delta} \mathsf{Sub}^+(B).$$

The system $\mathbf{GR}^-$ is defined from the following axioms and inference rules in the usual way.

Axioms of $\mathbf{GR}^-$

$$A \to A$$

$$\bot \to$$

$$\Box A \preceq \Box B, \Box B \preceq \Box C \to \Box A \preceq \Box C$$

$$\Box A \to \Box A \preceq \Box B, \Box B \prec \Box A$$

$$\Box B \to \Box A \preceq \Box B, \Box B \prec \Box A$$

$$\Box A \prec \Box B \to \Box A \preceq \Box B$$

$$\Box A \preceq \Box B, \Box B \prec \Box A \to$$

Inference rules of $\mathbf{GR}^-$

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(W \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to W)$$

$$\frac{\Gamma \to \Delta, A \qquad A, \Gamma \to \Delta}{\Gamma \to \Delta}(\text{cut})$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \qquad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \qquad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}(\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to\supset)$$

$$\frac{\Box A, \Sigma, \Gamma, \Box\Gamma \to A}{\Sigma, \Box\Gamma \to \Box A}(\Box)$$

$$\frac{\Box A, \Gamma \to \Delta}{\Box A \preceq \Box B, \Gamma \to \Delta}(\preceq\to) \qquad \frac{\Gamma \to \Delta, \Box A}{\Gamma \to \Delta, \Box A \preceq \Box A}(\to\preceq)$$

By $\mathbf{GR}_1^-$, we mean the system obtained by restricting a cut to the following two forms:

$$\frac{\Gamma \to \Delta, \Box A \preceq \Box B \qquad \Box A \preceq \Box B, \Gamma \to \Delta}{\Gamma \to \Delta}$$

$$\frac{\Gamma \to \Delta, \Box A \prec \Box B \qquad \Box A \prec \Box B, \Gamma \to \Delta}{\Gamma \to \Delta}$$

where $\Box A$ and $\Box B$ are subformulas of some formula occurring in the lower sequent.

Example. A proof figure in $\mathbf{GR}_1^-$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot \to}{\to \top}\,(\to\supset)}{\to \Box\top}\,(\Box)}{\to \Box\top \preceq \Box\top}\,(\to\preceq) \qquad \dfrac{\dfrac{\Box\top \to \Box\top \prec \Box\bot, \Box\bot \preceq \Box\top \qquad \dfrac{\dfrac{\dfrac{\bot \to}{\bot \to \Box\top \prec \Box\bot}\,(\to W)}{\Box\bot \to \Box(\Box\top \prec \Box\bot)}\,(\Box)}{\Box\bot \preceq \Box\top \to \Box(\Box\top \prec \Box\bot)}\,(\preceq\to)}{\Box\top \to \Box(\Box\top \prec \Box\bot), \Box\top \prec \Box\bot}\,(cut)}{\Box\top \preceq \Box\top \to \Box(\Box\top \prec \Box\bot), \Box\top \prec \Box\bot}\,(\preceq\to)}{\to \Box(\Box\top \prec \Box\bot), \Box\top \prec \Box\bot}\,(cut) \qquad \dfrac{\Box\top \prec \Box\bot \to \Box\top \prec \Box\bot}{\Box\top \prec \Box\bot \to \Box(\Box\top \prec \Box\bot)}\,(\Box)}{\to \Box(\Box\top \prec \Box\bot)}\,(cut)$$

where $\top = \neg\bot$.

Theorem 2.1. *The following conditions are equivalent:*
(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_1^-$,
(2) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}^-$,
(3) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n \in \mathbf{R}^-$,
(4) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ *is valid in any Kripke model for* $\mathbf{R}^-$.

"(1) implies (2)" is clear. From Lemma 1.7, it follows that "(3) implies (4)". "(2) implies (3)" is shown by checking the corresponding formula of each axiom in $\mathbf{GR}^-$ is provable in $\mathbf{R}^-$ and each inference rule in $\mathbf{GR}^-$ preserves the provability of $\mathbf{R}^-$. The former can be easily seen and the latter can be shown in the usual way using Lemma 1.4. To prove "(4) implies (1)", we need some preparations.

Definition 2.2. A sequent $\Gamma \to \Delta$ is said to be saturated if the following conditions hold:
(1) if $A \wedge B \in \Gamma$, then $A, B \in \Gamma$,
(2) if $A \vee B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$,
(3) if $A \supset B \in \Gamma$, then $A \in \Delta$ or $B \in \Gamma$,
(4) if $A \wedge B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$,
(5) if $A \vee B \in \Delta$, then $A, B \in \Delta$,
(6) if $A \supset B \in \Delta$, then $A \in \Gamma$ and $B \in \Delta$,
(7) if $\Box A \preceq \Box B \in \Gamma$, then $\Box A \in \Gamma$,
(8) if $\Box A \preceq \Box A \in \Delta$, then $\Box A \in \Delta$.
(9) if $\Box A, \Box B \in \mathsf{Sub}(\Gamma \to \Delta)$, then $\Box A \preceq \Box B, \Box A \prec \Box B \in \Gamma \cup \Delta$.

Lemma 2.3. *If* $\Gamma \to \Delta \notin \mathbf{GR}_1^-$, *then there exists a sequent* $\Gamma' \to \Delta'$ *satisfying the following four conditions:*
(1) $\Gamma' \to \Delta' \notin \mathbf{GR}_1^-$,
(2) $\Gamma' \to \Delta'$ *is saturated,*
(3) $\Gamma \subseteq \Gamma' \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$,
(4) $\Delta \subseteq \Delta' \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$.

Proof. Since $\mathsf{Sub}^+(\Gamma \to \Delta)$ is finite, there exist formulas $A_0, A_1, \cdots, A_{n-1}$ such that

$$\mathsf{Sub}^+(\Gamma \to \Delta) = \{A_0, A_2, \cdots, A_{n-1}\}.$$

We define a sequence of sequents

$$\Gamma_0 \to \Delta_0, \Gamma_1 \to \Delta_1, \cdots, \Gamma_k \to \Delta_k, \cdots$$

inductively as follows.
Step 1: $(\Gamma_0 \to \Delta_0) = (\Gamma \to \Delta)$.

Step $k+1$: $(\Gamma_{k+1} \to \Delta_{k+1})$

$$= \begin{cases} (B, C, \Gamma_k \to \Delta_k) & \text{if } A_{(k+1) \bmod n} = B \wedge C \in \Gamma_k - \Delta_k \\ (\Gamma_k \to \Delta_k, B) & \text{if } A_{(k+1) \bmod n} = B \wedge C \in \Delta_k - \Gamma_k \text{ and } (\Gamma_k \to \Delta_k, B) \notin \mathbf{GR}_1^- \\ (\Gamma_k \to \Delta_k, C) & \text{if } A_{(k+1) \bmod n} = B \wedge C \in \Delta_k - \Gamma_k, (\Gamma_k \to \Delta_k, B) \in \mathbf{GR}_1^- \text{ and } (\Gamma_k \to \Delta_k, C) \notin \mathbf{GR}_1^- \\ (B, \Gamma_k \to \Delta_k) & \text{if } A_{(k+1) \bmod n} = B \vee C \in \Gamma_k - \Delta_k \text{ and } (B, \Gamma_k \to \Delta_k) \notin \mathbf{GR}_1^- \\ (C, \Gamma_k \to \Delta_k) & \text{if } A_{(k+1) \bmod n} = B \vee C \in \Gamma_k - \Delta_k (B, \Gamma_k \to \Delta_k) \in \mathbf{GR}_1^- \text{ and } (C, \Gamma_k \to \Delta_k) \notin \mathbf{GR}_1^- \\ (\Gamma_k \to \Delta_k, B, C) & \text{if } A_{(k+1) \bmod n} = B \vee C \in \Delta_k - \Gamma_k \\ (\Gamma_k \to \Delta_k, B) & \text{if } A_{(k+1) \bmod n} = B \supset C \in \Gamma_k - \Delta_k \text{ and } (\Gamma_k \to \Delta_k, B) \notin \mathbf{GR}_1^- \\ (C, \Gamma_k \to \Delta_k) & \text{if } A_{(k+1) \bmod n} = B \supset C \in \Gamma_k - \Delta_k (\Gamma_k \to \Delta_k, B) \in \mathbf{GR}_1^- \text{ and } (C, \Gamma_k \to \Delta_k) \notin \mathbf{GR}_1^- \\ (B, \Gamma_k \to \Delta_k, C) & \text{if } A_{(k+1) \bmod n} = B \supset C \in \Delta_k - \Gamma_k \\ (\Gamma_k \to \Delta_k, \Box B \preceq \Box C) & \text{if } A_{(k+1) \bmod n} = \Box B \preceq \Box C, (\Gamma_k \to \Delta_k, \Box B \preceq \Box C) \notin \mathbf{GR}_1^- \text{ and } B \neq C \\ (\Gamma_k \to \Delta_k, \Box B \preceq \Box B, \Box B) & \text{if } A_{(k+1) \bmod n} = \Box B \preceq \Box B \text{ and } (\Gamma_k \to \Delta_k, \Box B \preceq \Box B) \notin \mathbf{GR}_1^- \\ (\Box B, \Box B \preceq \Box C, \Gamma_k \to \Delta_k) & \text{if } A_{(k+1) \bmod n} = \Box B \preceq \Box C, (\Gamma_k \to \Delta_k, \Box B \preceq \Box C) \in \mathbf{GR}_1^- \text{ and } (\Box B \preceq \Box C, \Gamma_k \to \Delta_k) \notin \mathbf{GR}_1^- \\ (\Gamma_k \to \Delta_k) & \text{otherwise.} \end{cases}$$

By an induction on k, it is not hard to show that $\Gamma_k \to \Delta_k$ satisfies the conditions (1), (3) and (4). Also in the usual way, we can prove that

$$\bigcup_{i=1}^{\infty} \Gamma_i \to \bigcup_{i=1}^{\infty} \Delta_i$$

is a sequnt and satisfies the conditions (1),(2),(3) and (4). $\dashv$

For $\Gamma \to \Delta \notin \mathbf{GR}_1^-$, we fix a sequent satisfying the four conditions in the above lemma and call it *a saturation* of $\Gamma \to \Delta$, write $sat(\Gamma \to \Delta)$.[1] For $\Gamma \to \Delta \in \mathbf{GR}_1^-$, we put $sat(\Gamma \to \Delta) = (\Gamma \to \Delta)$.

Definition 2.4. A sequence of formulas is defined as follows:
(1) [ ] is a sequence of formulas,
(2) if $[A_1, \cdots, A_n]$ is a sequence of formulas, then so is $[A_1, \cdots, A_n, B]$.

We call the sequence [ ] the empty sequence and use λ to express the empty sequence. A binary operator $\circ$ is defined by

$$[A_1, \cdots, A_m] \circ [B_1, \cdots, B_n] = [A_1, \cdots, A_m, B_1, \cdots, B_n]$$

We use τ and σ, possibly with suffixes, for sequences of formulas.

Definition 2.5. Let S_0 be a sequent, which is not provable in $\mathbf{GR}_1^-$. We define the set $\mathbf{W}(S_0)$ of pairs of a sequent and a sequence of formulas as follows:
(1) $(sat(S_0); \lambda) \in \mathbf{W}(S_0)$,
(2) if a pair $(\Gamma \to \Delta, \Box A; \tau)$ belongs to $\mathbf{W}(S_0)$, then so does the pair

$$(sat(\Box A, \{D \mid \Box D \in \Gamma\}, \{D \mid D \in \Gamma,\ D \text{ is a Sigma-formula}\} \to A); \tau \circ [\Box A]).$$

Lemma 2.6. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_1^-$ and let $(S; \tau)$ be a pair in $\mathbf{W}(S_0)$. Then*
(1) $S \notin \mathbf{GR}_1^-$,
(2) *S consists of only formulas in $\mathsf{Sub}^+(S_0)$,*
(3) *τ consists of only $\Box$-formulas in $\mathsf{Sub}(S_0)$.*

Proof. We use an induction on $(S; \tau)$ as an element in $\mathbf{W}(S_0)$. If $(S; \tau) = (sat(S_0); \lambda)$, then the lemma is clear. Suppose that $(S; \tau) \neq (sat(S_0); \lambda)$. Then by Definition 2.5, there exists a pair $(\Gamma \to \Delta, \Box A; \sigma) \in \mathbf{W}(S_0)$ such that S is the saturation of

$$\Box A, \{D \mid \Box D \in \Gamma\}, \{D \mid D \in \Gamma,\ D \text{ is a Sigma-formula}\} \to A.$$

[1] Note that a sequence $A_0, A_1, \cdots A_n$ is not unique, and neither is $\bigcup_{i=1}^{\infty} \Gamma_i \to \bigcup_{i=1}^{\infty} \Delta_i$ in the proof of Lemma 2.3.

and τ is the sequence $\sigma \circ [\Box A]$. By the induction hypothesis, we have the following three:

(4) $\Gamma \to \Delta, \Box A \notin \mathbf{GR}_1^-$,

(5) $\Gamma \to \Delta, \Box A$ consists only formulas in $\mathsf{Sub}^+(S_0)$

(6) σ consists of only $\Box$-formulas in $\mathsf{Sub}(S_0)$.

From (5) and (6), we obtain (3). By Lemma 2.3 and (5), we have (2). Also we consider the following figure.

$$\cfrac{\cfrac{\Box A, \{D \mid \Box D \in \Gamma\}, \{D \mid D \in \Gamma,\ D \text{ is a Sigma-formula}\} \to A}{\{\Box D \mid \Box D \in \Gamma\}, \{D \mid D \in \Gamma,\ D \text{ is a Sigma-formula}\} \to \Box A}(\Box)}{\cfrac{\text{using weakening rules, possibly several times}}{\Gamma \to \Delta, \Box A}}$$

The figure says that if the sequent at the top of the figure is provable in $\mathbf{GR}_1^-$, then so is the sequent $\Gamma \to \Delta, \Box A$, and so, we have a contradiction. Hence the sequent at the top of the figure is not provable in $\mathbf{GR}_1^-$, and using Lemma 2.3, neither is S. We have (1). ⊣

Lemma 2.7. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_1^-$. Then*

(1) $S_1 = S_2$ *for any* $(S_1; \tau), (S_2; \tau) \in \mathbf{W}(S_0)$,

(2) *if* $(\Gamma_1 \to \Delta_1; \tau), (\Gamma_2 \to \Delta_2; \tau \circ \sigma) \in \mathbf{W}(S_0)$, *then each Sigma-formula in* Γ_1 *is a member of* Γ_2,

(3) *if* $(\Gamma \to \Delta; \tau \circ \sigma) \in \mathbf{W}(S_0)$, *then* $(\Gamma_1 \to \Delta_1; \tau) \in \mathbf{W}(S_0)$ *for some* $\Gamma_1 \to \Delta_1$,

(4) *if* $(\Gamma \to \Delta; \tau \circ [\Box A] \circ \sigma) \in \mathbf{W}(S_0)$, *then* $\Box A \in \Gamma$,

(5) $(S; \tau_1 \circ [\Box A] \circ \tau_2 \circ [\Box A] \circ \tau_3) \notin \mathbf{W}(S_0)$, *for any A and S*,

(6) $\mathbf{W}(S_0)$ *is finite.*

Proof. For (1). We use an induction on τ. If $\tau = \lambda$, then by Definition 2.5, we have $S_1 = sat(S_0)$. Similarly, we also have $S_2 = sat(S_0)$.

Suppose that $\tau = \sigma \circ [A]$. Then by Definition 2.5, there exists $(\Gamma_1 \to \Delta_1, \Box A; \sigma) \in \mathbf{W}(S_0)$ such that $S_1 = sat(\Box A, \{D \mid \Box D \in \Gamma_1\}, \{D \mid D \in \Gamma_1,\ D \text{ is a Sigma-formula}\} \to A)$. Similarly, there exists $(\Gamma_2 \to \Delta_2, \Box A; \sigma) \in \mathbf{W}(S_0)$ such that $S_2 = sat(\Box A, \{D \mid \Box D \in \Gamma_2\}, \{D \mid D \in \Gamma_2,\ D \text{ is a Sigma-formula}\} \to A)$. By the induction hypothesis, $(\Gamma_1 \to \Delta_1, \Box A) = (\Gamma_2 \to \Delta_2, \Box A)$, and so, we have $S_1 = S_2$.

For (2). We use an induction on σ. If $\sigma = \lambda$, then by (1), we have $\Gamma_1 = \Gamma_2$. Suppose that $\sigma = \sigma' \circ [\Box A]$. Then by Definition 2.5, there exists $(\Gamma_3 \to \Delta_3, \Box A; \tau \circ \sigma') \in \mathbf{W}(S_0)$ such that $(\Gamma_2 \to \Delta_2) = sat(\Box A, \{D \mid \Box D \in \Gamma_3\}, \{D \mid D \in \Gamma_3,\ D \text{ is a Sigma-formula}\} \to A)$. By the induction hypothesis, each Sigma-formula in Γ_1 is also a member of Γ_3. On the other hand, by Lemma 2.3(3), $\{D \mid D \in \Gamma_3,\ D \text{ is a Sigma-formula}\} \subseteq \Gamma_2$. Hence each Sigma-formula in Γ_1 is also a member of Γ_2.

For (3). We use an induction on σ. If $\sigma = \lambda$, then the lemma is clear. Suppose that $\sigma = \sigma' \circ [A]$. Then by Definition 2.5, there exists $(\Gamma_1 \to \Delta_1, \Box A; \tau \circ \sigma') \in \mathbf{W}(S_0)$ such that $(\Gamma \to \Delta) = sat(\Box A, \{D \mid \Box D \in \Gamma_1\}, \{D \mid D \in \Gamma_1,\ D \text{ is a Sigma-formula}\} \to A)$. By the induction hypothesis, $(\Gamma_2 \to \Delta_2; \tau) \in \mathbf{W}(S_0)$ for some $\Gamma_2 \to \Delta_2$.

For (4). By (3), $(\Gamma_1 \to \Delta_1; \tau \circ [\Box A]) \in \mathbf{W}(S_0)$ for some $\Gamma_1 \to \Delta_1$. Using Definition 2.5 and Lemma 2.3, we have $\Box A \in \Gamma_1$. Using (2), we have $\Box A \in \Gamma$.

For (5). Suppose that $(S; \tau_1 \circ [\Box A] \circ \tau_2 \circ [\Box A] \circ \tau_3) \in \mathbf{W}(S_0)$. Then by (3), $(\Gamma \to \Delta; \tau_1 \circ [\Box A] \circ \tau_2 \circ [\Box A]) \in \mathbf{W}(S_0)$ for some $\Gamma \to \Delta$. By Definition 2.5, there exists $(\Gamma_1 \to \Delta_1, \Box A; \tau_1 \circ [\Box A] \circ \tau_2) \in \mathbf{W}(S_0)$. Using (4), $\Box A \in \Gamma_1$. So, $\Gamma_1 \to \Delta_1, \Box A \in \mathbf{GR}_1^-$. This is contradictory to Lemma 2.6.

For (6). By (5), $\{\tau \mid (S; \tau) \in \mathbf{W}(S_0)\}$ contains only sequences of $\Box$-formulas in $\mathsf{Sub}(S_0)$, in which no formulas occurs twice. So, $\{\tau \mid (S; \tau) \in \mathbf{W}(S_0)\}$ is finite, and by (1), so is $\mathbf{W}(S_0)$. ⊣

Definition 2.8. Let S_0 be a sequent, which is not provable in $\mathbf{GR}_1^-$. We define a structure $\mathcal{K}(S_0) = \langle \mathbf{W}(S_0), <, \models \rangle$ as follows:

(1) $(\Gamma_1 \to \Delta_1; \tau_1) < (\Gamma_2 \to \Delta_2; \tau_2)$ if and only if $\tau_2 = \tau_1 \circ \sigma$ for some non-empty sequence σ,

(2) $\models$ is a valuation, in addition to the conditions in Definition 1.5(3), satisfying,

(2.1) $p \in \Gamma$ if and only if $(\Gamma \to \Delta; \tau) \models p$, for any propositional variable p,
(2.2) $\Box A \preceq \Box B \in \Gamma$ if and only if $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box B$, for any $\Box A, \Box B \in \mathsf{Sub}(S_0)$,
(2.3) $\Box A \prec \Box B \in \Gamma$ if and only if $(\Gamma \to \Delta; \tau) \models \Box A \prec \Box B$, for any $\Box A, \Box B \in \mathsf{Sub}(S_0)$.

Lemma 2.9. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_1^-$. Then for any $A \in \mathsf{Sub}^+(S_0)$ and for any $(\Gamma \to \Delta; \tau) \in \mathbf{W}(S_0)$,*
(1) $A \in \Gamma$ implies $(\Gamma \to \Delta; \tau) \models A$,
(2) $A \in \Delta$ implies $(\Gamma \to \Delta; \tau) \not\models A$.

Proof. We use an induction on A.

If $A = \bot$, then by Lemma 2.6(1), $A \notin \Gamma$. So we have (1). On the other hand, $(\Gamma \to \Delta; \tau) \not\models A$, and so, we have (2).

If A is a propositional variable, then (1) is clear. Suppose that $p \in \Delta$. By Lemma 2.6(1), $p \notin \Gamma$, and so, we have (2).

Suppose that A is not a propositional variable. If A is a $\preceq$-formula or a $\prec$-formula, then the lemma can be shown similarly to the case that A is a propositional variable. Other cases can be shown in the usual way (cf. Avron [Avr84]). Here we show only the case that $A = \Box B$.

For (1). Suppose that $\Box B \in \Gamma$ and $(\Gamma \to \Delta; \tau) < (\Gamma_1 \to \Delta_1; \tau_1)$. Then $\tau_2 = \tau_1 \circ \sigma \circ [\Box C]$ for some σ and C. Hence there exists $(\Gamma_2 \to \Delta_2, \Box C; \tau_1 \circ \sigma) \in \mathbf{W}(S_0)$ such that $(\Gamma_1 \to \Delta_1) = sat(\Box C, \{D \mid \Box D \in \Gamma_2\}, \{D \mid D \in \Gamma_2,\ D \text{ is a Sigma-formula}\} \to C)$. By Lemma 2.7(2), we have $\Box B \in \Gamma_2$. Using Lemma 2.3, $B \in \Gamma_1$. By the induction hypothesis, we have $(\Gamma_1 \to \Delta_1; \tau_1) \models B$. Hence $(\Gamma \to \Delta; \tau) \models \Box B$.

For (2). Suppose that $\Box B \in \Delta$. Then

$$(\Gamma \to \Delta; \tau) < (sat(\Box B, \{D \mid \Box D \in \Gamma\}, \{D \mid D \in \Gamma,\ D \text{ is a Sigma-formula}\} \to B); \tau \circ [\Box B]) \in \mathbf{W}(S_0).$$

By Lemma 2.3, B belongs to the succeedent of the above saturation. By the induction hypothesis, B is false at the new pair above. Hence $(\Gamma \to \Delta; \tau) \not\models \Box B$. $\dashv$

Corollary 2.10. *Let $A_1, \cdots, A_m \to B_1, \cdots, B_n$ be a sequent, which is not provable in $\mathbf{GR}_1^-$. Then in $\mathcal{K}(A_1, \cdots, A_m \to B_1, \cdots, B_n)$,*

$$(sat(A_1, \cdots, A_m \to B_1, \cdots, B_n); \lambda) \not\models A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n.$$

Lemma 2.11. *Let S_0 be a sequent, which is not provable in $\mathbf{GR}_1^-$. Then $\mathcal{K}(S_0)$ is a Kripke pseudo-model for $\mathbf{R}^-$ satisfying the seven conditions in Definition 1.6 for any $\Box A, \Box B, \Box C \in \mathsf{Sub}(S_0)$.*

Proof. By Lemma 2.7(6), $\mathbf{W}(S_0)$ is finite. The irreflexivity and the transitivity of $<$ can be shown easily. We show

$$\alpha < \gamma \text{ and } \beta < \gamma \text{ imply either one of } \alpha = \beta,\ \alpha < \beta \text{ or } \beta < \alpha.$$

Suppose that $(S_1; \tau_1) < (S_3; \tau_3)$ and $(S_2; \tau_2) < (S_3; \tau_3)$. Then $\tau_3 = \tau_1 \circ \sigma_1 = \tau_2 \circ \sigma_2$ for some non-empty sequences σ_1 and σ_2. Hence either $\tau_1 = \tau_2 \circ \sigma_2'$ or $\tau_1 \circ \sigma_1' = \tau_2$ holds. Hence we have either one of $(S_1; \tau_1) = (S_2; \tau_2)$, $(S_1; \tau_1) < (S_2; \tau_2)$ or $(S_2; \tau_2) < (S_1; \tau_1)$.

We show the seven conditions in Definition 1.6 for any $\Box$-formulas. Let be that $\Box A, \Box B, \Box C \in \mathsf{Sub}(S_0)$.

For (1). Suppose that $(\Gamma_1 \to \Delta_1; \tau_1) \models \Box A \preceq \Box B$ and $(\Gamma_1 \to \Delta_1; \tau_1) < (\Gamma_2 \to \Delta_2; \tau_2)$. Then $\Box A \preceq \Box B \in \Gamma_1$. By Lemma 2.7(2), $\Box A \preceq \Box B \in \Gamma_2$. Hence $(\Gamma_2 \to \Delta_2; \tau_2) \models \Box A \preceq \Box B$.

For (2). (2) can be shown similarly to (1).

(3) Suppose that $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box B$. Then $\Box A \preceq \Box B \in \Gamma$. Since $\Gamma \to \Delta$ is a saturation, we have $\Box A \in \Gamma$. Using Lemma 2.9, we obtain (3).

(4) Suppose that $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box B$ and $(\Gamma \to \Delta; \tau) \models \Box B \preceq \Box C$. Then $\Box A \preceq \Box B, \Box B \preceq \Box C \in \Gamma$. Since $\Gamma \to \Delta$ is a saturation, we have $\Box A \preceq \Box C \in \Gamma \cup \Delta$. By Lemma 2.6, $\Box A \preceq \Box C \notin \Delta$ and so, $\Box A \preceq \Box C \in \Gamma$. Hence $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box C$.

(5) Suppose that $(\Gamma \to \Delta; \tau) \not\models (\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$. Then $(\Gamma \to \Delta; \tau) \not\models \Box A \preceq \Box B$ and $(\Gamma \to \Delta; \tau) \not\models \Box B \prec \Box A$, and so, $\Box A \preceq \Box B \notin \Gamma$ and $\Box B \prec \Box A \notin \Gamma$. Since $\Gamma \to \Delta$ is a saturation, $\Box A \preceq \Box B, \Box B \prec \Box A \in \Delta$.

Also $\Box A \preceq \Box A \in \Gamma \cup \Delta$. If $\Box A \preceq \Box A \in \Gamma$, then $\Box A \in \Gamma$ since $\Gamma \to \Delta$ is a saturation. Using $\Box A \preceq \Box B, \Box B \prec \Box A \in \Delta$, we have $\Gamma \to \Delta \in \mathbf{GR}_1^-$, which is contradictory to Lemma 2.6. If $\Box A \preceq \Box A \in \Delta$, then $\Box A \in \Delta$ since $\Gamma \to \Delta$ is a saturation. Using Lemma 2.9, $(\Gamma \to \Delta; \tau) \not\models \Box A$.

Similarly, we also have $(\Gamma \to \Delta; \tau) \not\models \Box B$.

(6) Suppose that $(\Gamma \to \Delta; \tau) \models \Box A \prec \Box B$. Then $\Box A \prec \Box B \in \Gamma$. Since $\Gamma \to \Delta$ is a saturation, $\Box A \preceq \Box B \in \Gamma \cup \Delta$. If it belongs to Δ, then we have $\Gamma \to \Delta \in \mathbf{GR}_1^-$, which is contradictory to Lemma 2.6. Hence it belongs to Γ, and so, $(\Gamma \to \Delta; \tau) \models \Box A \preceq \Box B$.

(7) Suppose that $(\Gamma \to \Delta; \tau) \models \Box A \prec \Box B$ and $(\Gamma \to \Delta; \tau) \models \Box B \preceq \Box A$. Then $\Box A \prec \Box B, \Box B \preceq \Box A \in \Gamma$, and so, $\Gamma \to \Delta \in \mathbf{GR}_1^-$, which is contradictory to Lemma 2.6. $\dashv$

Theorem 2.12. *Let $A_1, \cdots, A_m \to B_1, \cdots, B_n$ be a sequent, which is not provable in $\mathbf{GR}_1^-$. Then there exists a Kripke model $\mathbf{K}$ for $\mathbf{R}^-$, in which the formula $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ is not valid.*

Proof. Let $\mathbf{S}$ be a set of formulas satisfying

$$A \in \mathbf{S} \text{ implies } \mathsf{Sub}^+(A) \subseteq \mathbf{S}$$

and Let $\mathcal{K}^*$ be a Kripke pseudo-model for $\mathbf{R}^-$ satisfying the seven conditions in Definition 1.6 for any $\Box A, \Box B, \Box C \in \mathbf{S}$. [GS79] showed that there exists a Kripke model $\mathcal{K}$ for $\mathbf{R}^-$ such that for any $A \in \mathbf{S}$,

$$A \text{ is valid in } \mathcal{K}^* \text{ if and only if } A \text{ is valid in } \mathcal{K}$$

So, by Corollary 2.10 and Lemma 2.11, we obtain the theorem. $\dashv$

Corollary 2.13. *If a sequent $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ is valid in any Kripke model for $\mathbf{R}^-$, then $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_1^-$.*

From the above corollary, we obtain the proof of "(4) implies (1)" in Theorem 2.1, and hence, we obtain Theorem 2.1.

Corollary 2.14. *If a sequent S is provable in $\mathbf{GR}^-$, then there exists a proof figure $\mathcal{P}$ for S such that each formula occurring in $\mathcal{P}$ belongs to $\mathsf{Sub}^+(S)$.*

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