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# Robust slippage rank tests for $k$ location parameters in the presence of gross errors

By Itsuro Kakiuchi[a] and Miyoshi Kimura[b]

[a]Department of Computer and Systems Engineering, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan.
[b]Department of Mathematical Sciences, Nanzan University, 27 Seirei-cho, Seto-shi, Aichi 489-0863, Japan

## Abstract

A robust slippage test problem of $k$ location parameters in the presence of gross errors is formulated from the point of view of Huber's robust test theory. Under an asymptotic model of the robust slippage test problem an asymptotic level $\alpha$ slippage rank test based on $k$ linear rank statistics is constructed by applying majorization methods and its asymptotic minimum power is evaluated by applying weak majorization methods. It is also shown that the slippage rank test is asymptotically unbiased.

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*Keywords*: Robust slippage test problem of location; Slippage rank tests; Gross errors; Huber's robust test theory; Asymptotic maximum size; Asymptotic minimum power; Shurinking neighborhood; Majorizaton; Weak majorization; Schur-convex set; Schur-concave function

## 1 Introduction.

An important class of multiple decision problems is that of slippage problems. Because slippage problems have some symmetric structure of a null hypothesis and $k$ alternative hypotheses such as permutation equivariance, they have been treated in a manner similar to hypothesis testing. Slippage problems were first introduced by Mosteller (1948) as a problem of testing homogeneity of $k$ populations against $k$ slippage alternatives that exactly one of the $k$ populations is different. Paulson (1952), who treated the slippage problem of normal mean, was the first to formulate the problem satisfactorily. Since then, many contributions have been made to such slippage test problems. Among them, there are Traux (1953), Kudo (1956), Doornbos and Prins (1958), Karlin and Traux (1960), Hall and Kudo (1968a), Hall, Kudo and Yeh (1968b), Kimura and Kudo (1974), Kakiuchi and Kimura (1975), Kakiuchi, Kimura and Yanagawa (1977), Kimura (1984a) and so forth.

The robust slippage test problem was proposed by Kimura (1984b) from the point of view of robust test theory of Huber (1965), Huber and Strassen (1973) and Rieder (1977). He formulated it as a problem of testing a neighborhood of distributions against $k$ neighborhoods of distributions and derived a robust slippage version of the Neyman-Pearson's lemma. Kimura (1988b) considered a robust asymptotic test problem which was formulated by applying Rieder's (1978) asymptotic model with gross error neighborhoods shrinking at the rate of order $n^{-1/2}$. By using majorization and weak majorization inequalities, he constructed asymptotic level $\alpha$ slippage tests and gave lower bounds for their asymptotic minimum powers.

[1]E-*mail addresses:* kakiuchi@kobe-u.ac.jp (I. Kakiuchi), kimura@ms.nanzan-u.ac.jp (M. Kimura).

On the other hand, Kimura and Kakiuchi (1989) and Kakiuchi and Kimura (1995) developed some majorization methods on hyperplanes and studied their applications to various robust tests for approximate equality in the parametric setup. Kakiuchi and Kimura (2001) proposed a test problem of $k$-sample approximate equality in the nonparametric setup, which can be regarded as a generalization of Rieder's (1981) problem for the two-sample case. By using the majorization methods, under an asymptotic model with shrinking gross error neighborhoods they derived lower and upper bounds for the limiting probability that a random vector of $k$-sample rank statistics takes in a Schur convex set. As their applications, they constructed asymptotic level $\alpha$ rank tests for the $k$-sample approximate equality and obtained lower bounds for their asymptotic minimum powers, which were used for discussions of asymptotic relative efficiency.

The purpose of this paper is (1) to give a formulation of robust asymptotic slippage test problems of $k$ location parameters in the presence of gross errors, (2) to construct asymptotic level $\alpha$ slippage rank tests and (3) to derive lower bounds for their asymptotic minimum powers. To do this, we make use of Kakiuchi and Kimura's (2001) results and weak majorization inequalities.

In Section 2 we formulate a robust slippage test problem of $k$ location parameters in the presence of gross errors. In Section 3 we introduce a class of slippage rank tests based on certain score generating functions which are permutation equivariant. In Section 4 we give an asymptotic model with shrinking gross error neighborhoods for the robust slippage test problem of $k$ location parameters. In Section 5 we collect auxiliary results which are used to establish main results of this paper. In Section 6, by applying Kakiuchi and Kimura's (2001) majorization methods we construct asymptotic level $\alpha$ slippage rank tests and derive lower bounds for their asymptotic minimum powers. It is also shown that the constructed slippage rank tests are asymptotically unbiased. Finally, we recommend a score generating function.

## 2 The $k$-sample robust slippage problem of location.

Let $\mathsf{X}$ be the extended real line, $\mathsf{B}$ the $\sigma$-field of Borel subsets of $\mathsf{X}$ and $\mathsf{M}$ the set of all probability measures on $\mathsf{B}$. Let $\mathsf{M}_c$ denote the subset of $\mathsf{M}$ that corresponds to all continuous dstribution functions which assign probability zero to $-\infty$ and $+\infty$. A probablity measure $G$ is identified with its distribution function and expectation operator, that is, $G(x) = G([-\infty, x])$ for $x \in [-\infty, +\infty]$, $G(\{x\}) = G(x) - G(x-0)$ for $x \in [-\infty, +\infty]$, and $G(B) = G(I_B) = \int_B dG$ for $B \in \mathsf{B}$.

Let $\{ F_\theta; \theta \in \Theta\} \subset \mathsf{M}_c$ be a parametric family, whose parameter space $\Theta$ is a subset of $[-\infty, +\infty]$ and contains zero in its interior. We assume the following conditions:

(A1) $F_\theta$ is absolutely continuous with respect to $F_0$ for every $\theta \in \Theta$.

(A2) There exists a function $\Lambda \in L_2(dF_0)$ such that

$$\frac{f_\theta^{1/2} - 1}{\theta} \longrightarrow \frac{1}{2}\Lambda \quad \text{in} \quad L_2(dF_0) \quad \text{as} \quad \theta \to 0,$$

where $f_\theta$ denotes the density of $F_\theta$ with respect to $F_0$.

For given $\epsilon,\ \delta \in [0, 1)$ with $\epsilon + \delta < 1$ the gross error neighborhood $\mathsf{P}(\theta; \epsilon, \delta)$ of the center $F_\theta$ is defined as

$$\mathsf{P}(\theta; \epsilon, \delta) = \{ G \in \mathsf{M}_c\, ;\, G(B) \geq (1-\epsilon)F_\theta(B) - \delta \text{ for all } B \in \mathsf{B}\}. \tag{2.1}$$

This neighborhood is a generalization of $\epsilon$-contamination and total variation neighborhoods which was introduced by Rieder (1977).

Let $X_{i1}, \ldots, X_{in}$ $(i = 1, \ldots, k)$ be the $i$-th sample of size $n$ which are independent random variables distributed with $G_{i1}, \ldots, G_{in} \in \mathsf{M}_c$, respectively. We asumme that $G_{i1}, \ldots, G_{in}$ belong to $\mathsf{P}(\theta_i; \epsilon, \delta)$. The gross error neighborhood of all possible joint distributions of the $N(= kn)$ random variables $X_{11}, X_{12}, \ldots,$ $X_{kn}$ is defined as

$$\mathsf{P}^{(N)}(\boldsymbol{\theta}; \epsilon, \delta) = \{ \otimes_{i=1}^{k} \otimes_{j=1}^{n}\ G_{ij} \mid G_{ij} \in \mathsf{P}(\theta_i; \epsilon_n, \delta_n),\ i = 1, \ldots, k; j = 1, \ldots, n\}, \tag{2.2}$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ and $\otimes_{i=1}^{k} \otimes_{j=1}^{n} G_{ij}$ stands for the stochastic product of $G_{11}, \dots, G_{kn}$.

For any $\theta_0 \in \Theta$ and $\Delta > 0$ let

$$\begin{aligned} \boldsymbol{\theta}_0 &= (\underbrace{\theta_0, \dots, \theta_0}_{k}) \\ \boldsymbol{\theta}_i(\Delta) &= (\underbrace{\theta_0, \dots, \theta_0}_{i-1}, \theta_0 + \Delta, \underbrace{\theta_0, \dots, \theta_0}_{k-i}), \quad i = 1, \dots, k, \end{aligned} \tag{2.3}$$

and let $\boldsymbol{W}_N = \otimes_{i=1}^{k} \otimes_{j=1}^{n} G_{ij}$ denote the distribution of $\boldsymbol{X}_N = (X_{11}, \dots, X_{kn})$. In what follows, for simplicity we denote $\mathsf{P}^{(N)}(\boldsymbol{\theta}_0; \epsilon, \delta)$ and $\mathsf{P}^{(N)}(\boldsymbol{\theta}_i(\Delta); \epsilon, \delta)$ by $\mathsf{P}_0^{(N)}$ and $\mathsf{P}_i^{(N)}(\Delta)$, $i = 1, \dots, k$, respectively. We assume that $\mathsf{P}(\theta_0; \epsilon, \delta) \cap \mathsf{P}(\theta_0 + \Delta; \epsilon, \delta) = \phi$, that is, $\mathsf{P}_0^{(N)}$, and $\mathsf{P}_i^{(N)}(\Delta)$, $i = 1, \dots, k$ are disjoint.

Let us consider the following slippage test problem.

$$\begin{aligned} &H_{n0} : \boldsymbol{W}_N \in \mathsf{P}_0^{(N)} \\ &H_{ni}(\Delta) : \boldsymbol{W}_N \in \mathsf{P}_i^{(N)}(\Delta), \quad i = 1, \dots, k, \end{aligned} \tag{2.4}$$

where $\theta_0$ is unknown and $\Delta > 0$ denotes an amount of the slip to the right.

The problem (2.4) is also written as

$$\begin{aligned} &H_{n0} : \ G_{jl} \in \mathsf{P}(\theta_0; \epsilon, \delta), \quad j = 1, \dots, k; l = 1, \dots, n, \\ &H_{ni}(\Delta) : \begin{cases} G_{il} \in \mathsf{P}(\theta_0 + \Delta; \epsilon, \delta), & l = 1, \dots, n, \\ G_{jl} \in \mathsf{P}(\theta_0; \epsilon, \delta), & j = 1, \dots, k (j \neq i); l = 1, \dots, n, \end{cases} \\ &\qquad\qquad i = 1, \dots, k, \end{aligned} \tag{2.5}$$

where $\theta_0$ is unknown and $\Delta > 0$. This is a problem of testing approximate equality of $k$ location parameters against $k$ alternatives that all location parameters except exactly one parameter are approximate equal. We call the problem (2.4) or (2.5) a robust slippage test problem of $k$ location parameters.

A slippage test for (2.4) based on $\boldsymbol{x}_N$ is denoted by $\varphi_n(\boldsymbol{x}_N) = (\varphi_{n0}(\boldsymbol{x}_N), \varphi_{n1}(\boldsymbol{x}_N), \dots, \varphi_{nk}(\boldsymbol{x}_N))$ with $\sum_{j=0}^{k} \varphi_{nj}(\boldsymbol{x}_N) = 1$, where $\varphi_{nj}(\boldsymbol{x}_N)$ denotes the conditional probability that $\varphi_n$ takes $H_{ni}(\Delta)$ given $\boldsymbol{X}_N = \boldsymbol{x}_N$. For any $\varphi_n$ let

$$\alpha_n(\varphi_n) = \inf\{ E_{\mathsf{W}_{N0}}(\varphi_{n0}) ; \boldsymbol{W}_{N0} \in \mathsf{P}_0^{(N)} \}, \tag{2.6}$$

$$\beta_{ni}(\varphi_n) = \inf\{ E_{\mathsf{W}_{Ni}}(\varphi_{ni}) ; \boldsymbol{W}_{Ni} \in \mathsf{P}_i^{(N)}(\Delta) \}, \quad i = 1, \dots, k, \tag{2.7}$$

where $E_{\mathsf{W}}$ means the expectation under $\boldsymbol{W}$. The maximum size and minimum power of $\varphi_n$ are defined by

$$1 - \alpha_n(\varphi_n), \tag{2.8}$$

$$\beta_n(\varphi_n) = \sum_{i=1}^{k} \beta_{ni}(\varphi_n). \tag{2.9}$$

# 3 Slippage rank tests.

Let $R_{ij}$ be the rank of $X_{ij}$ among all $N$ random variables $X_{11}, X_{12}, \dots, X_{kn}$. We consider the following $k$-sample linear rank statistics

$$T_{Ni}(\boldsymbol{X}_N) = \frac{1}{n} \sum_{j=1}^{n} a_N(R_{ij}(\boldsymbol{X}_N)), \qquad i = 1, \dots, k. \tag{3.1}$$

The scores $a_N(r)$ are assumed to be generated by a scores generating function $a : (0,1) \to (-\infty, +\infty)$ in either one of the two ways,

$$
\begin{aligned}
a_N(r) &= a\left(\frac{r}{N+1}\right), \qquad & r = 1, \dots, N, \\
a_N(r) &= E\left(a(U_N^{(r)})\right), \qquad & r = 1, \dots, N,
\end{aligned}
\tag{3.2}
$$

where $U_N^{(r)}$ denotes the $r$-th order statistic in a random sample of size $N$ from the uniform distribution on $(0,1)$. Throughout this paper the scores generating function $a$ is assumed to satisfy the following conditions:

(A3) $a$ is nondecreasing and nonconstant, absolutely continuous inside $(0,1)$ and

$$
\int_0^1 t^{1/2}(1-t)^{1/2}\, da(t) < \infty.
$$

(A4) $a$ is Lipschitz bounded of order 1 on $[t_0, 1-t_0]$, concave on $(0, t_0)$ and convex on $(1-t_0, 1)$ for some $t_0 \in (0, 1/2]$.

Now, for the problem (2.4) we are interested in the following rank tests $\varphi_n = (\varphi_{n0}, \varphi_{n1}, \dots, \varphi_{nk})$ based on $\boldsymbol{T}_N = (T_{N1}, \dots, T_{Nk})$ with $T_{Ni}$ in (3.1):

$$
\begin{aligned}
\varphi_{n0}(\boldsymbol{x}_N) &= \begin{cases} 1, & \text{if } \max\limits_{1 \le j \le k} T_{Nj}(\boldsymbol{x}_N) \le \lambda_n, \\ 0, & \text{if } \max\limits_{1 \le j \le k} T_{Nj}(\boldsymbol{x}_N) > \lambda_n, \end{cases} \\
\varphi_{ni}(\boldsymbol{x}_N) &= \begin{cases} \dfrac{1}{m(\boldsymbol{x}_N)}, & \text{if } T_{Ni}(\boldsymbol{x}_N) = \max\limits_{1 \le j \le k} T_{Nj}(\boldsymbol{x}_N) > \lambda_n, \\ 0, & \text{otherwise}, \end{cases} \\
i &= 1, \dots, k,
\end{aligned}
\tag{3.3}
$$

where $m(\boldsymbol{x}_N)$ is the number of times $\max_{1 \le j \le k} T_{nj}(\boldsymbol{x}_N)$ is attained, and $\lambda_n$ is a critical value.

**Lemma 3.1** *The test $\varphi_n$ in (3.3) satisfies the following.*

(i) $\inf\{E_{\mathsf{W}_{Ni}}(\varphi_{ni}(\boldsymbol{X}_N));\, \boldsymbol{W}_{Ni} \in \mathsf{P}_i^{(N)}(\Delta)\} = \inf\{E_{\mathsf{W}_{Nj}}(\varphi_{nj}(\boldsymbol{X}_N));\, \boldsymbol{W}_{Nj} \in \mathsf{P}_j^{(N)}(\Delta)\}$,

(ii) $\sup\{E_{\mathsf{W}_{Ni}}(\varphi_{n0}(\boldsymbol{X}_N));\, \boldsymbol{W}_{Ni} \in \mathsf{P}_i^{(N)}(\Delta)\} = \sup\{E_{\mathsf{W}_{Nj}}(\varphi_{n0}(\boldsymbol{X}_N));\, \boldsymbol{W}_{Nj} \in \mathsf{P}_j^{(N)}(\Delta)\}$,

(iii) $\sup\{E_{\mathsf{W}_{N0}}(\varphi_{ni}(\boldsymbol{X}_N));\, \boldsymbol{W}_{Ni} \in \mathsf{P}_0^{(N)}(\Delta)\} = \sup\{E_{\mathsf{W}_{N0}}(\varphi_{nj}(\boldsymbol{X}_N));\, \boldsymbol{W}_{N0} \in \mathsf{P}_0^{(N)}(\Delta)\}$,

(iv) $\sup\{E_{\mathsf{W}_{Ni}}(\varphi_{nj}(\boldsymbol{X}_N));\, \boldsymbol{W}_{Ni} \in \mathsf{P}_i^{(N)}(\Delta)\} = \sup\{E_{\mathsf{W}_{Ni'}}(\varphi_{nj'}(\boldsymbol{X}_N));\, \boldsymbol{W}_{Ni'} \in \mathsf{P}_{i'}^{(N)}(\Delta)\}$,

$i, i', j, j' \; (i \ne j,\ i' \ne j') = 1, \dots, k.$

**Proof.** Since all the proofs are similar, we give only the proof of (i). Let $\Pi$ $(\pi \in \Pi : i \to \pi(i))$ be the symmetric group of all permutations on $\{1, \dots, k\}$. For any $\pi \in \Pi$ let $g_\pi$ be a transformation on $\mathsf{X}^N$ defined by $g_\pi(x_{11}, x_{12}, \dots, x_{kn}) = (x_{\pi^{-1}(1)1}, x_{\pi^{-1}(1)2}, \dots, x_{\pi^{-1}(k)n})$, where $\pi^{-1}$ is the inverse of $\pi$. Then we have

$$
T_{Ni}(\boldsymbol{x}_N) = T_{N\pi(i)}(g_\pi(\boldsymbol{x}_N)), \quad i = 1, \dots, k,\ \forall \pi \in \Pi,
$$

and hence

$$
\varphi_{ni}(\boldsymbol{x}_N) = \varphi_{n\pi(i)}(g_\pi(\boldsymbol{x}_N)), \quad i = 1, \dots, k,\ \forall \pi \in \Pi.
$$

Therefore, for any $\boldsymbol{W}_{Ni} \in \mathsf{P}_i^{(N)}(\varDelta)$

$$E_{\mathsf{W}_{Ni}}(\varphi_{ni}(\boldsymbol{X}_N)) = E_{\mathsf{W}_{Ni}}(\varphi_{n\pi(i)}(g_\pi(\boldsymbol{X}_N))) = E_{\mathsf{W}_{Ni}g_\pi^{-1}}(\varphi_{n\pi(i)}(\boldsymbol{X}_N)), \tag{3.4}$$

where $\boldsymbol{W}_{Ni}g_\pi^{-1}$ denotes the distribution of $g_\pi(\boldsymbol{X}_N)$ when the ditribution of $\boldsymbol{X}_N$ is $\boldsymbol{W}_{Ni}$. It is easy to see that

$$\mathsf{P}_i^{(N)}(\varDelta)g_\pi^{-1} = \mathsf{P}_{\pi(i)}^{(N)}(\varDelta), \quad i = 1, \dots, k, \ \forall \pi \in \Pi, \tag{3.5}$$

where $\mathsf{P}_i^{(N)}(\varDelta)g_\pi^{-1} = \{\boldsymbol{W}_{Ni}g_\pi^{-1} \mid \boldsymbol{W}_{Ni} \in \mathsf{P}_i^{(N)}(\varDelta)\}$. Taking $\pi$ such that $\pi(i) = j$, the assertion (i) readily follows from (3.4) and (3.5).

# 4 The asymptotic model.

Let $\tau \in (0, +\infty)$ be a constant satisfying

$$0 < \frac{\epsilon + 2\delta}{\tau} < \int \Lambda^+ \, dF_0, \tag{4.1}$$

where $\Lambda$ is given in (A2) and $x^+ = \max(x, 0)$. Let

$$\mathsf{W}_0 = \left\{(\boldsymbol{W}_N) \mid \boldsymbol{W}_N \in \mathsf{P}^{(N)}(\boldsymbol{\theta}_0; \epsilon_n, \delta_n) \quad \text{for } \forall n \in \mathbf{N}\right\},$$
$$\mathsf{W}_i(\varDelta) = \left\{(\boldsymbol{W}_N) \mid \boldsymbol{W}_N \in \mathsf{P}^{(N)}(\boldsymbol{\theta}_i(\varDelta_n); \epsilon_n, \delta_n) \quad \text{for } \forall n \in \mathbf{N}\right\}, \quad i = 1, \dots, k,$$

where $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_i(\varDelta_n)$ are given in (2.3) with $\varDelta_n = n^{-1/2}\varDelta$, $\epsilon_n = n^{-1/2}\epsilon$ and $\delta_n = n^{-1/2}\delta$. In this asymptotic setup we rewrite $\mathsf{P}^{(N)}(\boldsymbol{\theta}_0; \epsilon_n, \delta_n)$ and $\mathsf{P}^{(N)}(\boldsymbol{\theta}_i(\varDelta_n); \epsilon_n, \delta_n)$ as $\mathsf{P}_0^{(N)}$ and $\mathsf{P}_i^{(N)}(\varDelta_n)$, respectively.

We consider the following test problem which is an asymptotic version of (2.4).

$$\begin{aligned} &H_0 : \ (\boldsymbol{W}_N) \in \mathsf{W}_0 \\ &H_i(\varDelta) : \ (\boldsymbol{W}_N) \in \mathsf{W}_i(\varDelta), \quad i = 1, \dots, k, \end{aligned} \tag{4.2}$$

where $\varDelta \geq 2\tau$ is unknown. We call this problem (4.2) a robust asymptotic slippage test problem of $k$ location parameters.

The asymptotic maximum size and the asymptotic minimum power of a sequence $(\varphi_n)$ of tests are defined as

$$1 - \liminf_{n\to\infty} \alpha_n(\varphi_n), \tag{4.3}$$

$$\liminf_{n\to\infty} \beta_n(\varphi_n). \tag{4.4}$$

A sequence $(\varphi_n)$ of tests is called an asymptotic level $\alpha$ test for the problem (4.2) if

$$\liminf_{n\to\infty} \alpha_n(\varphi_n) \geq 1 - \alpha, \quad 0 < \alpha < 1. \tag{4.5}$$

We define an asymptotic unbiased test as follows:
An asymptotic level $\alpha$ test $(\varphi_n)$ is called asymptotically unbiased if

$$\liminf_{n\to\infty} \beta_n(\varphi_n) \geq \alpha. \tag{4.6}$$

**Remark 4.1.**

(i) The condition (4.1) is equivalent to that if $|\theta_1 - \theta_2| = \tau$, then $\mathsf{P}(n^{-1/2}\theta_1; \epsilon_n, \delta_n) \cap \mathsf{P}(n^{-1/2}\theta_2; \epsilon_n, \delta_n) = \emptyset$ holds for large $n$, which is an asymptotic disjointness condition (see Rieder, 1977, 1978).

(ii) The condition $\varDelta \geq 2\tau$ guarantees that $\mathsf{W}_0, \mathsf{W}_1(\varDelta), \dots, \mathsf{W}_k(\varDelta)$ are distinguishable one another.

## 5 Auxiliary results.

We hereafter assume $\theta_0 = 0$ without loss of generality, since the distributions of the $k-$sample rank statistics $T_{Ni},\ i = 1, \ldots, k$, given by (3.1) do not depend on $\theta_0$. We wish to construct an asymptotic level $\alpha$ slippage rank test for problem (4.2) and to derive its asymptotic minimum power. To do this we need the following three lemmas and two propositions concering majorization and weak majorization.

**Lemma 5.1 (Kakiuchi and Kimura, 2001, Theorem 2.1)** *For any* $(\boldsymbol{W}_N) \in \mathsf{W}_0 \cup \left(\cup_{i=1}^k \mathsf{W}_i(\Delta)\right)$, *the random vector*

$$n^{1/2}(T_{N1} - \mu_{N1}, \ldots, T_{Nk} - \mu_{Nk})/A \tag{5.1}$$

*has the limiting normal distribution* $N(\mathbf{0}, \boldsymbol{\Sigma})$, *where*

$$\mu_{Ni} = \int_{-\infty}^{+\infty} a(H_N(x))\, dH_{ni}(x), \quad i = 1, \ldots, k,$$

$$A^2 = \int_0^1 \left(a(t) - \bar{a}\right)^2 dt,$$

$$\boldsymbol{\Sigma} = (\sigma_{ij})\,;\sigma_{ij} = \begin{cases} 1 - \dfrac{1}{k}, & (i = j), \\ -\dfrac{1}{k}, & (i \neq j), \end{cases}$$

$$H_{ni}(x) = n^{-1} \sum_{j=1}^n G_{ij}(x),\ H_N(x) = k^{-1} \sum_{i=1}^k H_{ni}(x) \ \textit{and}\ \bar{a} = \int_0^1 a(t)\, dt.$$

The following lemma is easily obtained from Lemma 5.1.

**Lemma 5.2** *For any* $(\boldsymbol{W}_N) \in \mathsf{W}_0 \cup \left(\cup_{i=1}^k \mathsf{W}_i(\Delta)\right)$, *the random vector*

$$n^{1/2}\left((T_{N1} - T_{Nk}) - (\mu_{N1} - \mu_{Nk}), \ldots, (T_{Nk-1} - T_{Nk}) - (\mu_{Nk-1} - \mu_{Nk})\right)/A \tag{5.2}$$

*has the limiting* $(k-1)$*-dimensional normal distribution* $N(\mathbf{0}, \tilde{\boldsymbol{\Sigma}})$, *where*

$$\tilde{\boldsymbol{\Sigma}} = (\tilde{\sigma}_{ij}); \tilde{\sigma}_{ij} = \begin{cases} 2, & (i = j), \\ 1, & (i \neq j). \end{cases}$$

**Definitions.**

(i) A vector $\boldsymbol{x} \in \mathsf{R}^k$ is said to be majorized by a vector $\boldsymbol{y} \in \mathsf{R}^k$, written in symbol $\boldsymbol{x} \prec \boldsymbol{y}$, if

$$\sum_{i=1}^k x_i = \sum_{i=1}^k y_i \quad \text{and} \quad \sum_{i=1}^j x_{[i]} \leq \sum_{i=1}^j y_{[i]},\ \ j = 1, \ldots, k-1,$$

where $x_{[1]} \geq \cdots \geq x_{[k]}$ and $y_{[1]} \geq \cdots \geq y_{[k]}$ denote the components of $\boldsymbol{x} = (x_1, \ldots, x_k)$ and $\boldsymbol{y} = (y_1, \ldots, y_k)$ in decreasing order. A vector $\boldsymbol{x} \in \mathsf{R}^k$ is said to be weakly majorized by a vector $\boldsymbol{y} \in \mathsf{R}^k$, written in symbol $\boldsymbol{x} \prec\prec \boldsymbol{y}$, if

$$\sum_{i=1}^j x_{[i]} \leq \sum_{i=1}^j y_{[i]},\ \ j = 1, \ldots, k.$$

(ii) A real valued function $\psi$ is said to be Schur-concave (Schur-convex), if $\boldsymbol{x} \prec \boldsymbol{y} \Rightarrow \psi(\boldsymbol{x}) \geq\ (\leq)\ \psi(\boldsymbol{y})$. A subset $\boldsymbol{D}$ of $\mathsf{R}^k$ is said to be Schur-convex, if $\boldsymbol{y} \in \boldsymbol{D}$ and $\boldsymbol{y} \succ \boldsymbol{x} \Rightarrow \boldsymbol{x} \in \boldsymbol{D}$. A subset $\boldsymbol{D}$ of $\mathsf{R}^k$ is said to be decreasing if its indicator function $I_D$ is decreasing.

**Proposition 5.1 (Kimura and Kakiuchi, 1989, Theorem)** *Let $Z_1, \dots, Z_k$ $(k \geq 3)$ be exchangeable random variables with $\sum_{i=1}^{k} Z_i = c$ such that $(Z_1, \dots, Z_{k-1})$ has a joint Schur-concave density. Here $c$ is a constant. Let $\boldsymbol{D}$ be a Schur-convex set. Then, $P(\boldsymbol{Z} + \boldsymbol{\mu} \in \boldsymbol{D})$ is a Schur-concave function of $\boldsymbol{\mu}$, where $\boldsymbol{Z} = (Z_1, \dots, Z_k)$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$.*

**Proposition 5.2 (Tong, 1980, Theorem 6.3.8)** *Let $\boldsymbol{X} = (X_1, \dots, X_k)$ be a $k$-dimensional random vector with a Schur-concave density function. Let $\boldsymbol{D}$ be a Schur-convex and decreasing set. If $\boldsymbol{\theta} \prec\prec \boldsymbol{\eta}$, then*

$$P(\boldsymbol{X} + \boldsymbol{\theta} \in \boldsymbol{D}) \geq P(\boldsymbol{X} + \boldsymbol{\eta} \in \boldsymbol{D}).$$

**Proposition 5.3 (Kakiuchi and Kimura, 2001, Lemma 6.1)** *It holds that for any integer $m(1 \leq m \leq k)$*

$$\text{(i)} \lim_{n\to\infty} n^{1/2} \sup \left\{ \sum_{i\in A(m)} (\mu_{Ni} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}^{(N)}(\boldsymbol{\theta}_n; \epsilon_n, \delta_n) \right\}$$
$$= \sum_{i\in A(m)} (\theta_i - \bar{\theta}) \int_0^1 \Lambda\big(F_0^{-1}(t)\big) a(t)\, dt + \frac{m(k-m)}{k} (\epsilon + 2\delta)\big(a(1) - a(0)\big),$$

$$\text{(ii)} \lim_{n\to\infty} n^{1/2} \inf \left\{ \sum_{i\in A(m)} (\mu_{Ni} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}^{(N)}(\boldsymbol{\theta}_n; \epsilon_n, \delta_n) \right\}$$
$$= \sum_{i\in A(m)} (\theta_i - \bar{\theta}) \int_0^1 \Lambda\big(F_0^{-1}(t)\big) a(t)\, dt - \frac{m(k-m)}{k} (\epsilon + 2\delta)\big(a(1) - a(0)\big),$$

*where $A(m)$ is any element of the family of all subsets consisting of $m$ elements of the set $\{1, 2, \dots, k\}$.*

We let $\boldsymbol{\nu}_N = (\nu_{N1}, \dots, \nu_{Nk-1})$ be the vector with

$$\nu_{Ni} = \mu_{Ni} - \mu_{Nk}, \quad i = 1, \dots, k-1 \tag{5.3}$$

**Lemma 5.3** *For any integer $m(1 \leq m \leq k-1)$ it holds that*

$$\text{(i)} \lim_{n\to\infty} n^{1/2} \sup \left\{ \sum_{i=1}^{m} \nu_{N[i]} \mid \boldsymbol{W}_N \in \mathsf{P}_k^{(N)}(\Delta_n) \right\}$$
$$\leq -m\Delta \int_0^1 \Lambda(F_0^{-1}(t)) a(t) dt + \frac{m(2k-m-1)}{k} (\epsilon + 2\delta)(a(1) - a(0)),$$

$$\text{(ii)} \lim_{n\to\infty} n^{1/2} \inf \left\{ \sum_{i=1}^{m} \nu_{N[i]} \mid \boldsymbol{W}_N \in \mathsf{P}_1^{(N)}(\Delta_n) \right\}$$
$$\geq \Delta \int_0^1 \Lambda(F_0^{-1}(t)) a(t) dt - \frac{m(2k-m-1)}{k} (\epsilon + 2\delta)(a(1) - a(0)),$$

*where $\nu_{N[1]} \geq \cdots \geq \nu_{N[k-1]}$ denote the components of $\boldsymbol{\nu}_N$ in decreasing order.*

**Proof.** We first note that

$$\sup \left\{ \sum_{i=1}^{m} \nu_{N[i]} \mid \boldsymbol{W}_N \in \mathsf{P}_k^{(N)}(\Delta_n) \right\}$$
$$\leq \sup \left\{ \sum_{i=1}^{m} (\mu_{N[i]} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_k^{(N)}(\Delta_n) \right\} - m \inf \left\{ (\mu_{Nk} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_k^{(N)}(\Delta_n) \right\}, \tag{5.4}$$

where $\mu_{N[1]} \geq \cdots \geq \mu_{N[k-1]}$ denote the components of $(\mu_{N1}, \cdots, \mu_{Nk-1})$ in decreasing order. By substituting $\theta_i = 0,\ i = 1, \ldots, k-1,\ \theta_k = \Delta$ and $\bar{\theta} = \Delta/k$ into (i) and (ii) of Proposition 5.3, we also obtain

$$\lim_{n\to\infty} n^{1/2} \sup \left\{ \sum_{i=1}^{m} (\mu_{N[i]} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_k^{(N)}(\Delta_n) \right\}$$
$$= -\frac{m\Delta}{k} \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt + \frac{m(k-m)}{k}(\epsilon + 2\delta)(a(1) - a(0)), \tag{5.5}$$
$$\lim_{n\to\infty} n^{1/2} \inf \left\{ (\mu_{Nk} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_k^{(N)}(\Delta_n) \right\}$$
$$= \frac{(k-1)\Delta}{k} \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt - \frac{(k-1)}{k}(\epsilon + 2\delta)(a(1) - a(0)). \tag{5.6}$$

These facts (5.4), (5.5) and (5.6) imply that the assertion (i) holds.

Similarly, we can see

$$\inf \left\{ \sum_{i=1}^{m} \nu_{N[i]} \mid \boldsymbol{W}_N \in \mathsf{P}_1^{(N)}(\Delta_n) \right\}$$
$$\geq \inf \left\{ \sum_{i=1}^{m} (\mu_{N[i]} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_1^{(N)}(\Delta_n) \right\} - m \sup \left\{ (\mu_{Nk} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_1^{(N)}(\Delta_n) \right\}. \tag{5.7}$$

By substituting $\theta_1 = \Delta,\ \theta_i = 0,\ i = 2, \ldots, k$ and $\bar{\theta} = \Delta/k$ into (i) and (ii) of Proposition 5.3, we obtain

$$\lim_{n\to\infty} n^{1/2} \inf \left\{ \sum_{i=1}^{m} (\mu_{N[i]} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_1^{(N)}(\Delta_n) \right\}$$
$$= \frac{(k-m)\Delta}{k} \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt - \frac{m(k-m)}{k}(\epsilon + 2\delta)(a(1) - a(0)), \tag{5.8}$$
$$\lim_{n\to\infty} n^{1/2} \sup \left\{ (\mu_{Nk} - \bar{a}) \mid \boldsymbol{W}_N \in \mathsf{P}_1^{(N)}(\Delta_n) \right\}$$
$$= -\frac{\Delta}{k} \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt + \frac{(k-1)}{k}(\epsilon + 2\delta)(a(1) - a(0)). \tag{5.9}$$

Thus the assertion (ii) follows from (5.7), (5.8) and (5.9).

Let

$$\Omega_0 = \left\{ \lim_{n\to\infty} n^{1/2}(\mu_{N1} - \bar{a}, \ldots, \mu_{Nk} - \bar{a}) \mid (\boldsymbol{W}_N) \in \mathsf{W}_0 \right\},$$
$$\Omega_k(\Delta) = \left\{ \lim_{n\to\infty} n^{1/2}(\mu_{N1} - \bar{a}, \ldots, \mu_{Nk} - \bar{a}) \mid (\boldsymbol{W}_N) \in \mathsf{W}_k(\Delta) \right\},$$
$$\Gamma_1(\Delta) = \left\{ \lim_{n\to\infty} n^{1/2}\boldsymbol{\nu}_N \mid (\boldsymbol{W}_N) \in \mathsf{W}_1(\Delta) \right\},$$
$$\Gamma_k(\Delta) = \left\{ \lim_{n\to\infty} n^{1/2}\boldsymbol{\nu}_N \mid (\boldsymbol{W}_N) \in \mathsf{W}_k(\Delta) \right\},$$

and let $\boldsymbol{\mu}^M = (\mu_1^M, \ldots, \mu_k^M)$ and $\boldsymbol{\nu}(\Delta) = (\nu_1(\Delta), \ldots, \nu_{k-1}(\Delta))$ be the vectors defined as

$$\mu_i^M = \frac{1}{k}(k - 2i + 1)(\epsilon + 2\delta)\big(a(1) - a(0)\big), \quad i = 1, \ldots, k, \tag{5.10}$$

$$\nu_i(\Delta) = -\Delta \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt + \frac{2(k-i)}{k}(\epsilon + 2\delta)(a(1) - a(0)), \quad i = 1, \ldots, k-1. \tag{5.11}$$

Now we consider the following condition.

(A5) $$\tau \int_0^1 \Lambda(F_0^{-1}(t))a(t)\, dt \geq \frac{k}{2}(\epsilon + 2\delta)(a(1) - a(0)).$$

**Lemma 5.4** *It holds that*

(i) $\boldsymbol{\mu}^M \succ \boldsymbol{\mu}$ *for every* $\boldsymbol{\mu} \in \Omega_0$,

(ii) *if* (A5) *is satisfied,* $\boldsymbol{\mu}^M \prec \boldsymbol{\mu}$ *for every* $\boldsymbol{\mu} \in \Omega_k(\varDelta)$.

**Proof.** By substituting $\theta_i = 0,\ i = 1, \dots, k,\ \bar{\theta} = 0$ and $\theta_i = 0,\ i = 1, \dots, k-1,\ \theta_k = \varDelta,\ \bar{\theta} = \varDelta/k$ into (i) and (ii) of Proposition 5.3, we easily obtain (i) and (ii) from the assumption (A5), respectively.

**Lemma 5.5** *It holds that*

(i) $\boldsymbol{\nu}(\varDelta) \succ\succ \boldsymbol{\nu}$ *for every* $\boldsymbol{\nu} \in \Gamma_k(\varDelta)$,

(ii) *if* (A5) *is satisfied,* $\boldsymbol{\nu}(\varDelta) \prec\prec \boldsymbol{\nu}$ *for every* $\boldsymbol{\nu} \in \Gamma_1(\varDelta)$.

**Proof.** We first note that for every $m(1 \le m \le k-1)$

$$\sum_{i=1}^{m} \nu_{[i]}(\varDelta) = -m\varDelta \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt + \frac{m(2k-m-1)}{k}(\epsilon + 2\delta)(a(1) - a(0)).$$

From (i) of Lemma 5.3, (i) is easily obtained. From (ii) of Lemma 5.3 and (A5), for every $\boldsymbol{\nu} \in \Gamma_1(\varDelta)$

$$\begin{aligned}\sum_{i=1}^{m} \nu_{[i]} - \sum_{i=1}^{m} \nu_{[i]}(\varDelta) &\ge (m+1)\varDelta \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt - \frac{2m(2k-m-1)}{k}(\epsilon + 2\delta)(a(1) - a(0)) \\ &> 2\left\{\varDelta \int_0^1 \Lambda(F_0^{-1}(t))a(t)dt - (k-1)(\epsilon + 2\delta)(a(1) - a(0))\right\}.\end{aligned}$$

From the assumtion (A5) and $\varDelta \ge 2\tau$ the right-hand side of the above inequality is nonnegative, which completes the proof of (ii).

# 6 Asymptotic slippage rank tests

We construct an asymptotic level $\alpha$ test. Let

$$\boldsymbol{D}_0(\lambda) = \left\{(x_1, \dots, x_k) \mid \max_{1 \le i \le k} x_i \le \lambda\right\} \tag{6.1}$$

and for any $\alpha \in (0,1)$ let $\lambda_\alpha$ be a constant determined by

$$P\left(\boldsymbol{Z} + \frac{\boldsymbol{\mu}^M}{A} \in \boldsymbol{D}_0\left(\frac{\lambda_\alpha}{A}\right)\right) = 1 - \alpha, \tag{6.2}$$

where $\boldsymbol{Z}$ denotes a random vector distributed with $N(\boldsymbol{0}, \Sigma)$ given in Lemma 5.1. The following theorem gives an asymptotic level $\alpha$ slippage rank test.

**Theorem 6.1** *Let* $(\varphi_n)$ *be a sequence of tests* (3.3) *with* $\lambda_n = \bar{a} + n^{-1/2}\lambda_\alpha$. *Then*

$$\liminf_{n\to\infty} \alpha_n(\varphi_n) \ge 1 - \alpha. \tag{6.3}$$

**Proof.** From the definition of $\alpha_n(\varphi_n)$, for any $\eta > 0$ and any $n \in \mathsf{N}$ there exists $\boldsymbol{W}^*_{N0} \in \mathsf{P}_0^{(N)}$ such that

$$E_{\mathsf{W}^*_{N0}}(\varphi_{n0}(\boldsymbol{X}_N)) < \alpha_n(\varphi_n) + \eta. \tag{6.4}$$

From the definition of $\varphi_{n0}$

$$\begin{aligned}
&E_{\mathsf{W}^*_{N0}}(\varphi_{n0}(\boldsymbol{X}_N)) \\
&= \boldsymbol{W}^*_{N0}\left(\max_{1\le i\le k} T_{Ni} \le \lambda_n\right) \\
&= \boldsymbol{W}^*_{N0}\left(\max_{1\le i\le k} \left\{n^{1/2}\left(T_{Ni} - \mu_{Ni}\right)/A + n^{1/2}(\mu_{Ni} - \bar{a})/A\right\} \le n^{1/2}(\lambda_n - \bar{a})/A\right).
\end{aligned}$$

Hence, by Lemma 5.1

$$\lim_{n\to\infty} E_{\mathsf{W}^*_{N0}}(\varphi_{n0}(\boldsymbol{X}_N)) = P\left(\boldsymbol{Z} + \boldsymbol{\mu}^*/A \in \boldsymbol{D}_0(\lambda_\alpha/A)\right), \tag{6.5}$$

where $\boldsymbol{\mu}^* = \lim_{n\to\infty} n^{1/2}(\mu_{N1} - \bar{a}, \ldots, \mu_{Nk} - \bar{a})$ is the limiting vector under $\boldsymbol{W}^*_{N0}$. Since $\boldsymbol{D}_0(\lambda)$ is Schur-convex, it follows from Proposition 5.1 that the right-hand side of (6.5) is Schur-concave in $\boldsymbol{\mu}^*$. From (i) of Lemma 5.4 We have

$$P\left(\boldsymbol{Z} + \boldsymbol{\mu}^*/A \in \boldsymbol{D}_0(\lambda_\alpha/A)\right) \ge P\left(\boldsymbol{Z} + \boldsymbol{\mu}^M/A \in \boldsymbol{D}_0(\lambda_\alpha/A)\right). \tag{6.6}$$

Therefore, from (6.2), (6.4), (6.5) and (6.6) it follows that

$$\liminf_{n\to\infty} \alpha_n(\varphi_n) + \eta \ge P\left(\boldsymbol{Z} + \boldsymbol{\mu}^M/A \in \boldsymbol{D}_0(\lambda_\alpha/A)\right) = 1 - \alpha.$$

This implies that (6.3) holds.

**Remark 6.1.**

(i) The asymptotic critical value $\lambda_\alpha$ dose not depend on $F_0$ which is an asymptotic distribution freeness of $(\varphi_N)$.

(ii) Althought the asymptotic maximum size of $(\varphi_N)$ in Theorem 6.1 is not exactly equal to $\alpha$, the approximation cannot be improved any more by the majorization method.

The following theorem gives a lower bound of the asymptotic minimum power of $(\varphi_n)$ in Theorem 6.1.

**Theorem 6.2** *Let $(\varphi_n)$ be a sequence of tests (3.3) with $\lambda_n = \bar{a} + n^{-1/2}\lambda_\alpha$. Then*

$$\liminf_{n\to\infty} \beta_n(\varphi_n) \ge kP\left(\boldsymbol{U} + \frac{\boldsymbol{\nu}(\Delta)}{A} \in \boldsymbol{D}_k\left(\frac{\lambda_\alpha}{A}\right)\right), \tag{6.7}$$

*where $\boldsymbol{U} = (U_1, \ldots, U_{k-1})$ denotes a random vector distributed with $N(\mathbf{0}, \tilde{\boldsymbol{\Sigma}})$ in Lemma 5.2 and*

$$\boldsymbol{D}_k(\lambda) = \left\{(x_1, \ldots, x_{k-1}) \mid \max_{1\le i\le k-1} x_i \le 0,\ \sum_{i=1}^{k-1} x_i \le -k\lambda\right\}. \tag{6.8}$$

**Proof.** By (i) of Lemma 3.1 we have $\beta_n(\varphi_n) = k\beta_{nk}(\varphi_n)$. Hence we only need to evaluate the limiting value of $\beta_{nk}(\varphi_n)$. From the definition of $\beta_{nk}(\varphi_n)$, for any $\eta$ and any $n \in \mathsf{N}$ there exists $\boldsymbol{W}^*_{Nk} \in \mathsf{P}_k^N(\Delta_n)$ such that

$$E_{\mathsf{W}^*_{Nk}}(\varphi_{nk}(\boldsymbol{X}_N)) < \beta_{nk}(\varphi_n) + \eta. \tag{6.9}$$

Noting that

$$\frac{1}{k}\sum_{i=1}^{N} T_{Ni} = \frac{1}{N}\sum_{i=1}^{N} a\left(\frac{i}{N+1}\right) \longrightarrow \bar{a}, \quad \text{as} \quad n \to \infty,$$

it follows from the definition of $\varphi_{nk}$ that

$$\begin{aligned}
&E_{\mathsf{W}^*_{Nk}}(\varphi_{nk}(\boldsymbol{X}_N)) \\
&\geq \boldsymbol{W}^*_{Nk}\left(T_{Nk} = \max_{1\leq i\leq k} T_{Ni},\ T_{Nk} > \lambda_n\right) \\
&= \boldsymbol{W}^*_{Nk}\Bigg(\max_{1\leq i\leq k-1}\left\{n^{1/2}\left((T_{Ni}-T_{Nk})-\nu_{Ni}\right)+n^{1/2}\nu_{Ni}\right\} \leq 0\ , \\
&\sum_{i=1}^{k-1}\left\{n^{1/2}\left((T_{Ni}-T_{Nk})-\nu_{Ni}\right)+n^{1/2}\nu_{Ni}\right\} \leq n^{1/2}\left\{-\lambda_N + N^{-1}\sum_{i=1}^{N} a\left(i/(N+1)\right)\right\}\Bigg).
\end{aligned}$$

Hence, by Lemma 5.2

$$\begin{aligned}
&\lim_{n\to\infty} E_{\mathsf{W}^*_{Nk}}(\varphi_{nk}(\boldsymbol{X}_N)) \\
&= P\left(\max_{1\leq i\leq k-1}\left\{U_i + \lim_{n\to\infty} n^{1/2}\nu_{Ni}/A\right\} \leq 0, \sum_{i=1}^{k-1}\left\{U_i + \lim_{n\to\infty} n^{1/2}\nu_{Ni}/A\right\} \leq -k\lambda_\alpha/A\right) \\
&= P\left(\boldsymbol{U} + \boldsymbol{\nu}^*/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right),
\end{aligned} \tag{6.10}$$

where $\boldsymbol{\nu}^* = \lim_{n\to\infty} n^{1/2}(\nu_{N1},\ldots,\nu_{Nk-1})$ is the limiting vector under $\boldsymbol{W}^*_{Nk}$. It is easy to check that $\boldsymbol{D}_k(\lambda)$ is Schur-convex and decreasing. Then, from (i) of Lemma 5.5 and Proposition 5.2 we obtain

$$P\ \left(\boldsymbol{U} + \boldsymbol{\nu}(\Delta)/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right) \leq P\ \left(\boldsymbol{U} + \boldsymbol{\nu}/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right) \quad \text{for every } \boldsymbol{\nu} \in \Gamma_k(\Delta). \tag{6.11}$$

Therefore, from (6.9), (6.10) and (6.11) it follows that

$$\liminf_{n\to\infty} \beta_{nk}(\varphi_n) + \eta \geq P\ \left(\boldsymbol{U} + \boldsymbol{\nu}(\Delta)/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right).$$

This implies that

$$\liminf_{n\to\infty} \beta_{nk}(\varphi_n) \geq P\ \left(\boldsymbol{U} + \boldsymbol{\nu}(\Delta)/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right),$$

which completes the proof.

The following theorem states that $(\varphi_n)$ is asymptotically unbiased.

**Theorem 6.3** *Let* $(\varphi_n)$ *be a sequence of tests* (3.3) *with* $\lambda_n = \bar{a} + n^{-1/2}\lambda_\alpha$. *Assume that* (A5) *is satisfied. Then*

$$\liminf_{n\to\infty}\ \beta_n(\varphi_n) \geq \alpha. \tag{6.12}$$

**Proof.** By (i) of Lemma 3.1, (6.12) is equivalent to

$$\liminf_{n\to\infty}\ \beta_{nk}(\varphi_n) \geq \frac{\alpha}{k}. \tag{6.13}$$

It is sufficient for (6.13) that for any $(\boldsymbol{W}_{Nk}) \in \mathsf{W}_k(\Delta)$ we have

$$\lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{nk}(\boldsymbol{X}_N)) \geq \frac{\alpha}{k}. \tag{6.14}$$

Obviously, the following two facts are sufficient for (6.14):

$$\lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{n0}(\boldsymbol{X}_N)) \leq 1-\alpha \tag{6.15}$$

$$\lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{nk}(\boldsymbol{X}_N)) \geq \lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{ni}(\boldsymbol{X}_N)), \quad i=1,\dots,k-1. \tag{6.16}$$

First we show (6.15). As in the proof of Theorem 6.1, for any $(\boldsymbol{W}_{Nk}) \in \mathsf{W}_k(\Delta)$ we have

$$\lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{n0}(\boldsymbol{X}_N)) = P\left(\boldsymbol{Z}+\boldsymbol{\mu}^*/A \in \boldsymbol{D}_0(\lambda_\alpha/A)\right), \tag{6.17}$$

where $\boldsymbol{\mu}^* = \lim_{n\to\infty} n^{1/2}(\mu_{N1}-\bar{a},\dots,\mu_{Nk}-\bar{a})$ is the limiting vector under $(\boldsymbol{W}_{Nk})$. Therefore, from (6.2) and (6.17) we obtain

$$\lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{n0}(\boldsymbol{X}_N)) \leq P\left(\boldsymbol{Z}+\boldsymbol{\mu}^M/A \in \boldsymbol{D}_0(\lambda_\alpha/A)\right) = 1-\alpha.$$

Next, we show (6.16). From (6.10) and (6.11) it follows that for any $(\boldsymbol{W}_{Nk}) \in \mathsf{W}_k(\Delta)$

$$\lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{nk}(\boldsymbol{X}_N)) \geq P\left(\boldsymbol{U}+\boldsymbol{\nu}(\Delta)/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right). \tag{6.18}$$

On the other hand, as in the proof of Lemma 3.1, taking a permutation $\pi$ on $\{1,\dots,k\}$ such that $\pi(i)=k$ and $\pi(k)=1$, we have for any $(\boldsymbol{W}_{Nk}) \in \mathsf{W}_k(\Delta)$

$$E_{\mathsf{W}_{Nk}}(\varphi_{ni}(\boldsymbol{X}_N)) = E_{\mathsf{W}_{N1}}(\varphi_{nk}(\boldsymbol{X}_N)), \tag{6.19}$$

where $\boldsymbol{W}_{N1} = \boldsymbol{W}_{Nk} g_\pi^{-1} \in \mathsf{P}_1(\Delta)$. We note that

$$\lim_{n\to\infty} E_{\mathsf{W}_{N1}}(\varphi_{nk}(\boldsymbol{X}_N)) = P\left(\boldsymbol{U}+\boldsymbol{\nu}^*/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right), \tag{6.20}$$

where $\boldsymbol{\nu}^* = \lim_{n\to\infty} n^{1/2}(\nu_{N1},\dots,\nu_{Nk})$ is the limiting vector under $(\boldsymbol{W}_{N1})$. Thus, by (6.19), (6.20), Proposition 5.2 and (ii) of Lemma 5.5

$$\lim_{n\to\infty} E_{\mathsf{W}_{Nk}}(\varphi_{ni}(\boldsymbol{X}_N)) \leq P\left(\boldsymbol{U}+\boldsymbol{\nu}_k(\Delta)/A \in \boldsymbol{D}_k(\lambda_\alpha/A)\right). \tag{6.21}$$

Therefore, (6.16) is obtained from (6.18) and (6.21). This completes the proof of the theorem.

We wish to get an asymptotic slippage rank test $(\varphi_n)$ whose asymtotic minimum power (6.7) is as large as posible. Taking (A.5) into account, we can recommend $(\varphi_n)$ based on a score generating function $a^*$ given by

$$a^*(t) = d_0 \vee \Lambda(F_0^{-1}(t)) \wedge d_1,$$

where the truncation points $d_0$ and $d_1$ are real numbers determined by

$$\int_0^1 \left(d_0 - \Lambda(F_0^{-1}(t))\right)^+ dt = \frac{k(\epsilon+2\delta)}{2\tau} = \int_0^1 \left(\Lambda(F_0^{-1}(t)) - d_1\right)^+ dt,$$

and $x \vee y \wedge z = \max(x, \min(y,z))$. We should notice that when $k=2$, $a^*$ is the same as that of Rieder (1981) for the two sample case.

**Remark 6.2.**

(i) The lower bound of the asymptotic minimum power (6.7) of $(\varphi_n)$ with $\lambda_n$ is increasing in $\Delta$, because $\boldsymbol{\nu}(\Delta)$ is decreasing in $\Delta$ with respect to the order of weak majorization.

(ii) Our tentative simulation shows that the inequality (6.7) based on the majorization methods seems to be satisfactory for practical use on the whole. When $\epsilon$, $\delta$ or $k$ increase, the accuracy of the inequality tends to go down slightly.

(iii) The asumption (A5) is not restrictive. It is satisfied when $F_0$ is normal. For $k = 2$ it coincides with Rieder's(1981) asymptotic unbiasedness condition in the two-sample case.

(iv) $d_0$ and $d_1$ are uniquely determined, if $k(\epsilon + 2\delta)/2\tau < \int_0^1 \Lambda(F_0^{-1}(t)))^+ \, dt$. When $F_0$ is normal, $d_0 = -d_1$ is determined by $\varphi(d_1) - d_1(1 - \Phi(d_1))$.

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