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# On the hub-and-spoke model with arc capacity constraints <sup>†</sup>

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## Abstract

In this paper, we present a new formulation of one-stop capacitated hub-and-spoke model as a natural extension of the uncapacitated one-stop model. The model involves arc capacity constraints as well as hub capacity constraints, which enables us to incorporate some practical factors into the model. We also present a branch-and-bound based exact solution method with Lagrangian relaxation bounding strategy, and report some results of numerical experiments using real aviation data. Computational results show the practical significance of the proposed capacitated model.

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# 1 Introduction

The hub location problem has attracted much attention since O’Kelly [17] formulated the single allocation hub-and-spoke model as a quadratic integer program. Campbell [4] classifies the  $p$ -hub location problem into four classes, and presented basic formulations and formulations with flow thresholds for spokes for each of them. The four classes are the  $p$ -hub median problem, the  $p$ -hub center problem, the uncapacitated hub location problem and the hub covering problem. Besides the basic single allocation and multiple allocation models, Campbell [4] presents models involving flow thresholds for spokes, and points out that the  $p$ -hub median problem and the uncapacitated hub location problem have mainly been studied and other classes of problems have not been considered seriously. Skorin-Kapov, Skorin-Kapov and O’Kelly [23] considered the uncapacitated  $p$ -hub median problem and developed linear programming formulations of both single and multiple allocation models that yield tight relaxations to the original problem. For the same class of problems, O’Kelly et al. [18] introduced a more compact formulation and discussed sensitivity analysis of the model with respect to various parameters. On the other hand, Aykin [1] formulated the capacitated hub-and-spoke model as a 0-1 integer programming problem and proposed two heuristics to solve the problem. Ebery et al. [11] also addressed the capacitated multiple allocation hub location problem that arises in a postal delivery system.

The above mentioned models are of two-stop type in the sense that a trip between each origin-destination (OD) pair uses at most two hubs. Therefore, the number of variables and constraints in the problem rapidly grows with the size of the model and hence we may expect to solve problems of practical size only approximately. Sasaki, Suzuki and Drezner [21, 22] considered the uncapacitated one-stop model, in which a trip between any OD pair uses only one hub. The one-stop model appears to be useful in certain situations and is simple enough to be solved exactly. In this paper, we propose a capacitated one-stop model as a natural extension of the uncapacitated one-stop model of [21, 22] and present a branch-and-bound based exact solution method with Lagrangian relaxation bounding strategy. Specifically, we introduce capacity constraints on both arcs and hubs. Similar arc capacitated network design problem have been considered in [13, 14]. However, studies on arc capacitated hub location problems are scarce. Hub capacity may be regarded as the maximum number of passengers a node can deal with when it is selected as a hub. For example, the number of runways and the number of slots assigned to the airline company may constitute the hub capacity. On the other hand, arc capacity may represent the number of available aircrafts for the airline company on that arc. By introducing arc capacity constraints as well as hub capacity constraints, we can enrich the model so as to take into account more practical situations. Bryan [2] first incorporated arc capacities into a hub location model. The model

is an extension of the model proposed by O’Kelly et al. [19], which uses a piecewise linear cost function. Assuming hub locations were given, they only focused on the allocation phase and examined how the arc capacity levels affect the total network cost.

This paper is organized as follows. In Section 2, we formulate the capacitated one-stop hub-and-spoke model. In Section 3, we present a branch-and-bound method for solving the proposed model and particularly describe Lagrangian relaxation strategy for finding good lower bounds for partial problems. In Section 4, we report some numerical experience. In Section 5, we give concluding remarks and briefly mention some future work.

## 2 Capacitated One-Stop Model

### 2.1 Model Description

Jaillet et al. [15] propose an airline network design problem. They examined the resultant network structure and figured out that the one-stop policy could bring results as good as those of two-stop policy, especially in the situation with relatively high level of demands. In some countries like Japan, a majority of domestic routes are provided by one-stop services via a hub. Even in the United States, a large number of trips seem to use one-stop routes. Motivated by these observations, an uncapacitated one-stop model has been proposed in [21, 22]. In this paper, we generalize this model to construct a capacitated one-stop model. We formulate the problem as a 0-1 mixed integer programming problem and propose a branch-and-bound method with Lagrangian relaxation bounding strategy.

Aykin [1] formulated the capacitated two-stop hub-and-spoke model and proposed two heuristics for the problem. In the model, capacity constraints on hubs are primarily considered. However, since airline companies may sometimes fail to provide flight service to all passengers who want to use some arc, it is also practically significant to consider capacity constraints on arcs as well as on hubs. Concerning route patterns, we suppose that nonstop services between non-hub nodes are not available, while passengers whose origin or destination is a hub are permitted to travel via another hub. Possible seven route patterns according to this rule are shown in Figure 1.

One of the major factors that characterize a hub location model is its allocation rule. Basically, there are two possible choices of allocation rules; the single allocation rule and the multiple allocation rule [3]. Allocating each demand node to more than one hubs is prohibited in the former rule and allowed in the latter rule. The first hub-and-spoke model presented by O’Kelly [17] adopts the single allocation rule. Recent papers often adopt the multiple allocation rule because it allows more flexible routing between OD pairs [5, 8, 18], while some others adopt single allocation rule [7, 10]. Skorin-Kapov et al. [23] and Ernst et

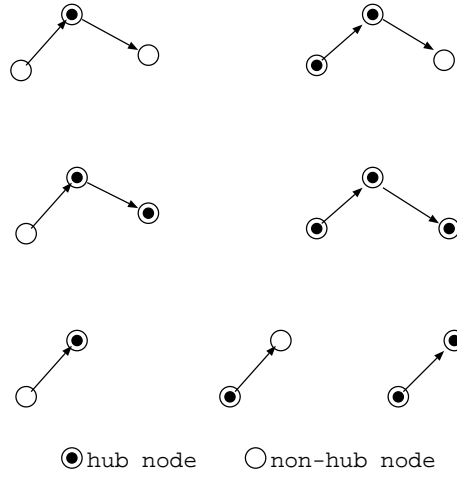


Figure 1: Route Patterns

al. [9] employ both rules in the uncapacitated  $p$ -hub median problem. These models are all two-stop models. The choice of the allocation rule makes a sensible difference in a two-stop model. However, in a one-stop model like the presented model, it is necessary to adopt the multiple allocation rule. Because, if we adopt the single allocation rule in a one-stop model, then any two demand nodes allocated to different hubs are not reachable.

The routing rule is another important factor in the model. There are also two possible choices of routing rules; the single routing rule and the multiple routing rule. Only one route service provided for each OD pair in the former rule and more than one route services are allowed in the latter rule. In other words, all passengers whose origin and destination are the same have to travel using the same route in the single routing rule. We note that the multiple routing rule is meaningful only in a capacitated problem, because in an uncapacitated problem, all passengers for each OD pair will use the least cost route even if multiple routes are available.

In this paper, we adopt the multiple routing rule. The objective of the model is to find hub locations and passenger routing for each OD pair that minimize the total transportation and location costs subject to the constraints derived from the assumed rules.

## 2.2 Model Formulation

We now present a mathematical formulation of the hub location problem with the multiple allocation and multiple routing rules. The following notation is employed:

$N$  : the set of demand nodes,  $|N| = n$ .

$A$  : the set of arcs.

$H \subseteq N$  : the set of hub candidates,  $|H| = h \leq n$ .

$\Pi$  : the set of OD pairs  $\pi = \langle i, j \rangle, i \in N, j \in N, i \neq j$ .  $|\Pi| = n(n-1)$ .

$A_i \subseteq \Pi$  : the set of OD pairs whose origin is  $i \in N$ .

$B_j \subseteq \Pi$  : the set of OD pairs whose destination is  $j \in N$ .

$p$  : the number of hubs to be selected.

$d_\pi$  : the trip demand (the number of passengers) for OD pair  $\pi \in \Pi$ .

$c_{\pi k}$  : the travel cost per passenger between OD pair  $\pi \in \Pi$  via hub  $k \in H$ .

$f_k$  : the fixed cost incurred by selecting hub  $k$ .

$w$  : weight parameter for the fixed cost.

$a_k$  : capacity of hub  $k \in H$ .

$b_{ij}$  : capacity of arc  $(i, j) \in A$ .

Decision variables in the model are as follows:

$x_{\pi k}$  : the number of passengers who travel between OD pair  $\pi \in \Pi$  via hub  $k \in H$ .

$y_k$  : binary variable such that  $y_k = 1$  if node  $k$  is selected as a hub, and 0 otherwise.

We note that  $\langle i, j \rangle$  denotes the OD pair whose origin is node  $i$  and destination is node  $j$ , while  $(i, j)$  denotes the arc connecting node  $i$  and node  $j$ . For an OD pair  $\pi = \langle i, j \rangle$ , if  $i \in H$ , then  $x_{\pi i}$  is the number of passengers traveling directly from hub  $i$  to node  $j$ . Similarly, if  $j \in H$ , then  $x_{\pi j}$  is the number of passengers traveling directly from node  $i$  to hub  $j$ . Moreover, if  $i \in H$  and  $j \in H$  hold simultaneously, then both  $x_{\pi i}$  and  $x_{\pi j}$  are the number of passengers from  $i$  to  $j$ . In this case, the total number of passengers on the route consisting of the single arc  $(i, j)$  is given by  $x_{\pi i} + x_{\pi j}$ .

Now we are ready to formulate the model as the following mixed 0–1 integer programming problem.

$$P_0: \quad \text{minimize} \quad \sum_{\pi \in \Pi} \sum_{k \in H} c_{\pi k} x_{\pi k} + \sum_{k \in H} f_k y_k \quad (1)$$

$$\text{subject to} \quad \sum_{k \in H} x_{\pi k} = d_\pi, \quad \pi \in \Pi, \quad (2)$$

$$\sum_{\pi \in A_i} x_{\pi j} + \sum_{\pi \in B_j} x_{\pi i} \leq b_{ij}, \quad i, j \in H, i \neq j, \quad (3)$$

$$\sum_{\pi \in A_i} x_{\pi j} \leq b_{ij}, \quad i \notin H, j \in H, i \neq j, \quad (4)$$

$$\sum_{\pi \in B_j} x_{\pi i} \leq b_{ij}, \quad i \in H, j \notin H, i \neq j, \quad (5)$$

$$\sum_{k \in H} y_k = p, \quad (6)$$

$$\begin{aligned}
\sum_{\pi \in \Pi} x_{\pi k} &\leq a_k y_k, & k \in H, \\
x_{\pi k} &\geq 0, & \pi \in \Pi, k \in H, \\
y_k &\in \{0, 1\}, & k \in H.
\end{aligned} \tag{7}$$

The objective function (1) is the sum of the total travel costs and fixed costs. Constraints (2) imply that each passenger between OD pair  $\pi$  travels via one of the hubs. Constraints (3)–(5) represent arc capacity constraints; (3) applies to the case where both end points of arc  $(i, j)$  belong to the hub candidate set, while (4) and (5) apply to the case where one of the end points belongs to the hub candidate set. Constraint (6) requires that exactly  $p$  hubs have to be selected. Constraints (7) represent capacity constraints for hubs, implying that, when hub  $k$  is selected, the number of passengers who use hub  $k$  cannot exceed the hub capacity  $a_k$ . Constraints (7) also assure that any node cannot be connected to non-hub nodes. Namely,  $x_{\pi k}$  for all  $\pi \in \Pi$  are forced to vanish unless  $y_k = 1$ .

### 3 Branch-and-Bound Method

#### 3.1 Lagrangian Relaxation

We note that  $P_0$  can naturally be decomposed into two problems by fixing an arbitrary  $y_k$  at 0 or 1. We call such variable  $y_k$  and the generated problems a branching variable and partial problems, respectively. A partial problem  $P_\nu$  may have one of the following two properties:

1. An optimal solution of  $P_\nu$  is obtained.
2. It is detected that  $P_\nu$  cannot produce an optimal solution of  $P_0$ .

In either case, we can fathom  $P_\nu$ , because it is not necessary to decompose  $P_\nu$  further. To check whether or not  $P_\nu$  has property 2, a lower bound test is applied to  $P_\nu$ . The branch-and-bound method repeatedly applies decomposition and lower bound tests until there remain no partial problems that are active, i.e., generated but neither fathomed nor decomposed.

To describe a partial problem  $P_\nu$ , we partition the hub candidate set  $H$  as  $H = H_\nu^F \cup H_\nu^1 \cup H_\nu^0$ , where  $H_\nu^F$ ,  $H_\nu^1$  and  $H_\nu^0$  are specified as follows.

$H_\nu^F$  : the set of indices  $k$  such that  $y_k$  is a free variable.

$H_\nu^1$  : the set of indices  $k$  such that  $y_k$  is fixed as  $y_k = 1$ .

$H_\nu^0$  : the set of indices  $k$  such that  $y_k$  is fixed as  $y_k = 0$ .

For the partial problem  $P_\nu$ , either  $|H_\nu^1| = p$  or  $|H_\nu^1| < p$  holds. If  $|H_\nu^1| = p$ , then  $P_\nu$  becomes a linear programming problem by fixing all free variables to be zero. We can then obtain an

upper bound of  $P_0$  by solving this problem, since its optimal solution is feasible to  $P_0$ . If this upper bound is smaller than an incumbent value, we let it be a new incumbent value of  $P_0$ .

If  $|H_\nu^1| < p$ , then we attempt to obtain a good lower bound for the optimal value of  $P_\nu$ . Let us denote  $H_\nu^+ = H_\nu^F \cup H_\nu^1$ . Note that we eliminate variables  $x_{\pi k}, y_k, k \in H_\nu^0$  from  $P_\nu$  because it immediately implies that  $x_{\pi k} = 0$  for all  $\pi \in \Pi$  if  $k \in H_\nu^0$ . Then partial problem  $P_\nu$  can be written as follows.

$$\begin{aligned}
P_\nu : \quad & \text{minimize} \quad \sum_{\pi \in \Pi} \sum_{k \in H_\nu^+} c_{\pi k} x_{\pi k} + \sum_{k \in H_\nu^1} f_k + \sum_{k \in H_\nu^F} f_k y_k \\
& \text{subject to} \quad \sum_{k \in H_\nu^+} x_{\pi k} = d_\pi, \quad \pi \in \Pi, \\
& \sum_{\pi \in A_i} x_{\pi j} + \sum_{\pi \in B_j} x_{\pi i} \leq b_{ij}, \quad i, j \in H_\nu^+, i \neq j, \quad (8) \\
& \sum_{\pi \in A_i} x_{\pi j} \leq b_{ij}, \quad i \notin H_\nu^+, j \in H_\nu^+, \quad (9) \\
& \sum_{\pi \in B_j} x_{\pi i} \leq b_{ij}, \quad i \in H_\nu^+, j \notin H_\nu^+, \quad (10) \\
& \sum_{k \in H_\nu^F} y_k = p - |H_\nu^1|, \\
& \sum_{\pi \in \Pi} x_{\pi k} - a_k y_k \leq 0, \quad k \in H_\nu^F, \quad (11) \\
& \sum_{\pi \in \Pi} x_{\pi k} \leq a_k, \quad k \in H_\nu^1, \\
& x_{\pi k} \geq 0, \quad \pi \in \Pi, k \in H_\nu^+, \\
& y_k \in \{0, 1\}, \quad k \in H_\nu^F.
\end{aligned}$$

We consider relaxing all the arc capacity constraints and the hub capacity constraints (11) by bringing them into the objective function. To achieve this, we introduce Lagrange multipliers  $\lambda = (\lambda_{ij})$  and  $\mu = (\mu_k)$  corresponding to the arc capacity constraints (8)–(10) and the hub capacity constraints (11), respectively. Then we obtain the following Lagrangian relaxation problem  $RP_\nu[\lambda, \mu]$  for partial problem  $P_\nu$ .

$RP_\nu[\lambda, \mu] :$

$$\begin{aligned}
& \text{minimize} \quad \sum_{\pi \in \Pi} \sum_{k \in H_\nu^+} c_{\pi k} x_{\pi k} + \sum_{k \in H_\nu^F} f_k y_k \\
& \quad + \sum_{i \in H_\nu^+} \sum_{j \in H_\nu^+, j \neq i} \lambda_{ij} \left( \sum_{\pi \in A_i} x_{\pi j} + \sum_{\pi \in B_j} x_{\pi i} \right) \\
& \quad + \sum_{i \notin H_\nu^+} \sum_{j \in H_\nu^+} \lambda_{ij} \sum_{\pi \in A_i} x_{\pi j} + \sum_{i \in H_\nu^+} \sum_{j \notin H_\nu^+} \lambda_{ij} \sum_{\pi \in B_j} x_{\pi i} \\
& \quad + \sum_{k \in H_\nu^F} \mu_k \left( \sum_{\pi \in \Pi} x_{\pi k} - a_k y_k \right) + L_{\nu 0}(\lambda)
\end{aligned}$$



$$\begin{aligned}
\text{subject to} \quad & \sum_{k \in H_\nu^\dagger} x_{\pi k} = d_\pi, & \pi \in \Pi, \\
& \sum_{k \in H_\nu^F} y_k = p - |H_\nu^1|, \\
& \sum_{\pi \in \Pi} x_{\pi k} \leq a_k, & k \in H_\nu^1, \\
& x_{\pi k} \geq 0, & \pi \in \Pi, k \in H_\nu^+, \\
& y_k \in \{0, 1\}, & k \in H_\nu^F,
\end{aligned}$$

where

$$L_{\nu 0}(\lambda) = \sum_{k \in H_\nu^1} f_k - \sum_{i \notin H_\nu^0} \sum_{j \notin H_\nu^0, j \neq i} \lambda_{ij} b_{ij}.$$

It is well-known that the optimal value of  $RP_\nu[\lambda, \mu]$  provides a lower bound of the optimal value of  $P_\nu$  for arbitrary  $\lambda \geq 0$  and  $\mu \geq 0$ . We can obtain an optimal solution of  $RP_\nu[\lambda, \mu]$  easily by decomposing it into some simple problems. First, note that the objective function of  $RP_\nu[\lambda, \mu]$  can be rewritten as

$$\begin{aligned}
& \sum_{\pi \in \Pi} \sum_{k \in H_\nu^\dagger} c_{\pi k} x_{\pi k} + \sum_{\pi \in \Pi} \sum_{k \in H_\nu^F} \mu_k x_{\pi k} + \sum_{i \in H} \sum_{j \in H_\nu^\dagger, j \neq i} \lambda_{ij} \sum_{\pi \in A_i} x_{\pi j} + \sum_{i \in H_\nu^\dagger} \sum_{j \in H, j \neq i} \lambda_{ij} \sum_{\pi \in B_j} x_{\pi i} \\
& + \sum_{k \in H_\nu^F} (f_k - \mu_k a_k) y_k + L_{\nu 0}(\lambda). \tag{12}
\end{aligned}$$

Let  $\alpha_\pi$  and  $\beta_\pi$  denote the origin and the destination of OD pair  $\pi$ , respectively. Then the third term of (12) can be rewritten as

$$\sum_{\pi \in \Pi} \sum_{j \in H_\nu^\dagger, j \neq \alpha_\pi} \lambda_{\alpha_\pi j} x_{\pi j}. \tag{13}$$

Similarly, the fourth term of (12) can be rewritten as

$$\sum_{\pi \in \Pi} \sum_{i \in H_\nu^\dagger, i \neq \beta_\pi} \lambda_{i \beta_\pi} x_{\pi i}. \tag{14}$$

By substituting (13) and (14) into (12), we may rewrite  $RP_\nu[\lambda, \mu]$  as follows.

$RP_\nu[\lambda, \mu]$  :

$$\begin{aligned}
\text{minimize} \quad & \sum_{\pi \in \Pi} \left( \sum_{k \in H_\nu^\dagger} c_{\pi k} x_{\pi k} + \sum_{k \in H_\nu^\dagger, k \neq \alpha_\pi} \lambda_{\alpha_\pi k} x_{\pi k} + \sum_{k \in H_\nu^\dagger, k \neq \beta_\pi} \lambda_{k \beta_\pi} x_{\pi k} + \sum_{k \in H_\nu^F} \mu_k x_{\pi k} \right) \\
& + \sum_{k \in H_\nu^F} (f_k - \mu_k a_k) y_k + L_{\nu 0}(\lambda) \\
\text{subject to} \quad & \sum_{k \in H_\nu^\dagger} x_{\pi k} = d_\pi, & \pi \in \Pi, \\
& \sum_{\pi \in \Pi} x_{\pi k} \leq a_k, & k \in H_\nu^1,
\end{aligned}$$

$$\begin{aligned}
\sum_{k \in H_\nu^F} y_k &= p - |H_\nu^1|, \\
x_{\pi k} &\geq 0, & \pi \in \Pi, k \in H_\nu^+, \\
y_k &\in \{0, 1\}, & k \in H_\nu^F.
\end{aligned}$$

Let us define  $C_{\pi k}$  to represent the aggregate coefficient of  $x_{\pi k}$ :

$$C_{\pi k} = \begin{cases} c_{\pi k} + \lambda_{\alpha_\pi k} + \lambda_{k\beta_\pi} + \mu_k, & \text{if } k \in H_\nu^F, k \neq \alpha_\pi, k \neq \beta_\pi \\ c_{\pi k} + \lambda_{\alpha_\pi k} + \mu_k, & \text{if } k \in H_\nu^F, k = \beta_\pi \\ c_{\pi k} + \lambda_{k\beta_\pi} + \mu_k, & \text{if } k \in H_\nu^F, k = \alpha_\pi \\ c_{\pi k} + \lambda_{\alpha_\pi k}, & \text{if } k \in H_\nu^1, k = \beta_\pi \\ c_{\pi k} + \lambda_{k\beta_\pi}, & \text{if } k \in H_\nu^1, k = \alpha_\pi. \end{cases}$$

Then, problem  $RP_\nu[\lambda, \mu]$  can be decomposed into the linear programming problem:

$$\begin{aligned}
&\text{minimize} && \sum_{\pi \in \Pi} \sum_{k \in H_\nu^+} C_{\pi k} x_{\pi k} \\
&\text{subject to} && \sum_{k \in H_\nu^+} x_{\pi k} = d_\pi, \quad \pi \in \Pi, \\
&&& \sum_{\pi \in \Pi} x_{\pi k} \leq a_k, \quad k \in H_\nu^1, \\
&&& x_{\pi k} \geq 0, \quad k \in H_\nu^+,
\end{aligned} \tag{15}$$

and the 0–1 integer programming problem:

$$\begin{aligned}
&\text{minimize} && \sum_{k \in H_\nu^F} (f_k - \mu_k a_k) y_k \\
&\text{subject to} && \sum_{k \in H_\nu^F} y_k = p - |H_\nu^1|, \\
&&& y_k \in \{0, 1\}, \quad k \in H_\nu^F.
\end{aligned} \tag{16}$$

Note that the linear programming problem (15) reduces to a transportation problem, if we regard the nodes  $k \in H_\nu^+$  as source nodes and OD pairs  $\pi \in \Pi$  as sink nodes. Moreover, an optimal solution  $\bar{y}_k, k \in H_\nu^F$ , of the 0–1 integer programming problem (16) is obtained in the following simple way: Order the indices  $k \in H_\nu^F$  as  $k_1, k_2, \dots, k_{|H_\nu^F|}$  in the increasing order of the coefficients  $f_k - \mu_k a_k$ , and let

$$\bar{y}_k = \begin{cases} 1, & \text{if } k \in K, \\ 0, & \text{if } k \notin K, \end{cases} \tag{17}$$

where  $K = \{k_1, k_2, \dots, k_{p-|H_\nu^1|}\}$ .

### 3.2 Subgradient Method

It is practically important to use appropriate values of Lagrange multipliers so as to obtain a good lower bound for the optimal value of the partial problem  $P_\nu$ . The optimal Lagrange multipliers can be obtained by solving the following Lagrangian dual problem for  $P_\nu$ .

$$\begin{aligned} LD_\nu : \quad & \text{maximize} \quad G_\nu(\lambda, \mu) \\ & \text{subject to} \quad \lambda, \mu \geq 0, \end{aligned}$$

where  $G_\nu(\lambda, \mu)$  denotes the optimal value of  $RP_\nu[\lambda, \mu]$ . This is a problem of maximizing a nondifferentiable concave function. Since it is expensive to obtain an optimal solution of  $LD_\nu$  exactly, we use the subgradient method, which is practically useful because it can find an approximate solution of  $LD_\nu$  conveniently.

The subgradient method for solving  $LD_\nu$  consists of the following iterative process: Choose initial Lagrange multipliers  $\lambda^{(0)} \geq 0, \mu^{(0)} \geq 0$ , and then successively update  $(\lambda^{(l)}, \mu^{(l)})$  by

$$\begin{aligned} \lambda^{(l+1)} &= \max\{0, \lambda^{(l)} + \theta^{(l)} g_\lambda^{(l)}\}, \\ \mu^{(l+1)} &= \max\{0, \mu^{(l)} + \theta^{(l)} g_\mu^{(l)}\}, \end{aligned} \tag{18}$$

where  $\theta^{(l)} > 0$  is a stepsize and  $(g_\lambda^{(l)}, g_\mu^{(l)})$  is a subgradient of  $G_\nu$  at  $(\lambda^{(l)}, \mu^{(l)})$ . Using an optimal solution  $(\bar{x}^{(l)}, \bar{y}^{(l)})$  of  $RP_\nu[\lambda^{(l)}, \mu^{(l)}]$ , where  $\bar{x}^{(l)}$  is an optimal solution of the linear programming problem (15) with  $(\lambda, \mu) = (\lambda^{(l)}, \mu^{(l)})$  and  $y^{(l)}$  is determined by (17), we may compute each element of the subgradient  $(g_\lambda^{(l)}, g_\mu^{(l)})$  as

$$\left[ g_\lambda^{(l)} \right]_{ij} = \begin{cases} \sum_{\pi \in A_i} \bar{x}_{\pi j}^{(l)} + \sum_{\pi \in B_j} \bar{x}_{\pi i}^{(l)} - b_{ij}, & \text{if } i \in H_\nu^+ \text{ and } j \in H_\nu^+, \\ \sum_{\pi \in A_i} \bar{x}_{\pi j}^{(l)} - b_{ij}, & \text{if } i \notin H_\nu^+ \text{ and } j \in H_\nu^+, \\ \sum_{\pi \in B_j} \bar{x}_{\pi i}^{(l)} - b_{ij}, & \text{if } i \in H_\nu^+ \text{ and } j \notin H_\nu^+, \end{cases} \tag{19}$$

$$\left[ g_\mu^{(l)} \right]_k = \sum_{\pi \in \Pi} \bar{x}_{\pi k}^{(l)} - a_k \bar{y}_k^{(l)}, \quad k \in H_\nu^F.$$

Stepsize  $\theta^{(l)}$  is determined according to the rule

$$\theta^{(l)} = \frac{\xi(\hat{G}_\nu - G_\nu(\lambda^{(l)}, \mu^{(l)}))}{\|(g_\lambda^{(l)}, g_\mu^{(l)})\|^2}, \tag{20}$$

where  $\hat{G}_\nu$  is an estimate of the optimal value of  $LD_\nu$ , and  $\xi$  is a parameter satisfying  $\epsilon \leq \xi \leq 2 - \epsilon$  for some  $\epsilon > 0$  [20]. If we use the exact optimal value  $G_\nu^*$  of  $LD_\nu$  instead of  $\hat{G}_\nu$ , the

sequence generated by (18) converges to an optimal solution of  $LD_\nu$ . However, the optimal value of  $LD_\nu$  is usually unknown. So we iterate (18) with (20) using an appropriate estimate  $\hat{G}_\nu$  of the optimal value  $G_\nu^*$ . By using an appropriate adjusting strategy, we may expect that  $\hat{G}_\nu$  approaches  $G_\nu^*$  and  $(\lambda^{(l)}, \mu^{(l)})$  converges to an optimal solution of  $LD_\nu$ .

As a matter of fact, the convergence rate of subgradient method significantly depends on the strategy for updating an estimate of the optimal value. For simplicity, we take  $\xi = 1$  in the following discussion. Let  $\hat{G}_\nu > G_\nu^*$  and  $0 < \zeta < 1$ . Then it can be shown that there is a positive integer  $S$  such that

$$l \geq S \implies G_\nu^{\max} - (2G_\nu^* - \hat{G}_\nu) \geq \zeta \left[ G_\nu(\lambda^{(l+1)}, \mu^{(l+1)}) - (2G_\nu^* - \hat{G}_\nu) \right], \quad (21)$$

where  $G_\nu^{\max}$  is the maximum value of  $LD_\nu$  obtained up to the  $l$ -th iteration with the initial point  $(\lambda^{(0)}, \mu^{(0)})$  [16]. Based on this fact, we propose to update the estimated value  $\hat{G}_\nu$  at every  $S$  iterations. With this strategy,  $\hat{G}_\nu$  is expected to decrease steadily and converge to  $G_\nu^*$ . A procedure to update the estimated value  $\hat{G}_\nu$  is described as follows:

[Updating  $\hat{G}_\nu$  ]

**Input:**  $\zeta \in (0, 1), S \in \{1, 2, \dots\}$ .

**Step 0:** Let  $\hat{G}_\nu$  be an arbitrary upper bound of  $G_\nu^*$ . Select an initial point  $(\lambda^{(0)}, \mu^{(0)})$ . Set  $l := 0$ .

**Step 1:** Set  $G_\nu^{\max} := -\infty$ .

**Step 2:** If  $\text{mod}(l+1, S) \neq 0$  then go to Step 3. Otherwise, go to Step 4.

**Step 3:** If  $G_\nu(\lambda^{(l)}, \mu^{(l)}) > G_\nu^{\max}$ , then update  $G_\nu^{\max}$  as  $G_\nu^{\max} := G_\nu(\lambda^{(l)}, \mu^{(l)})$ , and let  $\lambda^{\max} := \lambda^{(l)}$  and  $\mu^{\max} := \mu^{(l)}$ . Compute  $\lambda^{(l+1)}, \mu^{(l+1)}$  according to (18)–(20). Set  $l := l+1$  and go to Step 2.

**Step 4:** Update the estimated value  $\hat{G}_\nu$  as  $\hat{G}_\nu := \frac{G_\nu^{\max} + (1 - \zeta)\hat{G}_\nu - \zeta G_\nu(\lambda^{(0)}, \mu^{(0)})}{2 - 2\zeta}$ . Set  $\lambda^{(l+1)} := \lambda^{\max}, \mu^{(l+1)} := \mu^{\max}, l := l+1$  and go to Step 1.

If  $G_\nu^{\max} - \hat{G}_\nu$  becomes smaller than a predetermined tolerance, then we stop the iterations. In general, it is impossible to obtain the exact values of  $\zeta$  and  $S$ . However, it is seen that  $S$  gets smaller as  $\zeta$  gets larger because of (21). From some preliminary computations, we have decided to set  $\zeta = 0.5$  and  $S = 5$  in our computational experiments.

Although we may use an arbitrary upper bound of  $G_\nu^*$  as an initial estimate of the optimal value of  $LD_\nu$  in Step 0, it is not always easy to obtain it. In the above procedure, we use an arbitrary upper bound for  $P_\nu$  rather than  $G_\nu^*$ , because any upper bound for  $P_\nu$  is greater

than or equal to  $G_\nu^*$ . We note that the uncapacitated one-stop hub location problem can be regarded as a shortest path problem if any  $p$  hubs are fixed. Therefore, for an uncapacitated problem, we can easily obtain an upper bound by solving the shortest path problem. For a hub capacitated problem, we can also find a feasible flow by solving a shortest path problem, if we select  $p$  hubs so that the total hub capacity is greater than the total demand. Thus, we can obtain an upper bound of a hub capacitated problem by allocating flows appropriately to each arc. However, it is not necessarily easy to obtain an upper bound for  $P_\nu$  in the case where both hubs and arcs are capacitated. The reason is that an arc capacitated problem often becomes infeasible even if we select  $p$  hubs so as to satisfy the above-mentioned total capacity condition. In general, any feasible solution of  $P_\nu$  is also feasible to  $P_0$ , though the converse does not always hold. However, if a feasible solution of  $P_0$  is still feasible to  $P_\nu$ , its objective value also provides an upper bound of  $P_\nu$ . Therefore we keep a list of any feasible solutions to  $P_0$  obtained during the branch-and-bound computation, and try to find a feasible solution of  $P_\nu$  from the list so as to use it as an initial estimated value of  $LD_\nu$ . If there are more than one feasible solutions to  $P_\nu$ , then we may choose the one with smallest objective value. On the other hand, if the list contains no such feasible solution for  $P_\nu$ , then we try to obtain a new upper bound by solving a new shortest path problem that is constructed from  $P_\nu$  by adding  $p - |H_\nu^1|$  hubs arbitrarily chosen from  $H_\nu^F$ . If the new shortest path problem is feasible, we use its objective value as an initial estimated value and keep the solution for the subsequent computation. Otherwise, we let  $\hat{G}_\nu$  be a tentative estimated value, which is chosen to be greater than the current incumbent value  $z^*$  of  $P_0$ . We note that there is no guarantee that such a tentative estimated value is an upper bound of  $G_\nu^*$ . If  $\hat{G}_\nu < G_\nu^*$ , then  $G_\nu(\lambda^{(l)}, \mu^{(l)})$  will converge to a smaller value than  $G_\nu^*$  [16]. Nevertheless, we may fathom the partial problem  $P_\nu$  if we can find a  $(\lambda^{(l)}, \mu^{(l)})$  such that  $G_\nu(\lambda^{(l)}, \mu^{(l)})$  is greater than the incumbent value  $z^*$ . After fathoming a partial problem and backtracking to another partial problem, some feasible solutions of  $P_0$  in the list may become invalid in the sense that they can never be feasible to any partial problem generated in the subsequent computation. By removing such invalid data, we can reduce the area of searching for feasible solutions of a partial problem.

### 3.3 Upper Bound

Upper bounds for the optimal value of problem  $P_0$  play an important role in developing a branch-and-bound method. An upper bound can be obtained by finding any feasible solution of  $P_0$ , which is not always easy as described in the previous subsection. However, since we cannot expect to fathom any partial problem without knowing an upper bound for  $P_0$ , we need to obtain it before getting into a lower bound computation. In addition, whether we

can obtain a good upper bound in the early stage of iterations has a significant effect on the performance of the branch-and-bound method.

First we note that any partial problem  $P_\nu$  as well as  $P_0$  reduces to an arc capacitated network flow problem if we select  $p$  hubs. In general, it is more likely to be able to find a feasible flow of the arc capacitated network flow problem by selecting hub candidates with larger capacities. However, selecting such hubs is generally costly because hubs with large capacities usually incur a high fixed cost. On the other hand, to obtain a good lower bound, it seems effective to select those hub candidates which are expected to be selected at optimal solutions. Consequently, we use the following strategy to obtain an upper bound. First we select  $p$  hubs according to an appropriate criterion, then solve the resultant arc capacitated network flow problem. If this problem is feasible, the obtained objective value is an upper bound. Here we introduce the following criterion as a likelihood of each hub candidate to be selected as a hub at an optimal solution:

$$h(k) = \gamma \frac{f_k}{a_k} + \sum_{\pi \in \Pi} c_{\pi k}, \quad k \in H,$$

where  $\gamma > 0$  is a parameter. The first term represents the fixed cost per unit capacity multiplied by a positive parameter  $\gamma$  and the second term represents the total travel cost when all passengers use the hub  $k$ . Note that we do not consider capacity constraints here. By introducing this criterion, a hub candidate associated with small  $h(k)$  is likely to be selected as a hub at an optimal solution. We sort  $h(k)$  in increasing order and express the ordered indices of  $h(k)$  as  $\kappa_1, \kappa_2, \dots, \kappa_{|H|}$ . If the arc capacitated network flow problem in which hubs  $\{\kappa_1, \kappa_2, \dots, \kappa_p\}$  are selected is feasible, we use the objective value as an initial upper bound. Otherwise, we select  $p$  hubs with  $p$  largest  $a_k$ 's and solve the resultant network flow problem. If this problem is also infeasible, we proceed to branch-and-bound iterations without knowing any upper bound at this stage.

### 3.4 Branching Strategies

The computational efficiency of branch-and-bound method significantly depends on a branching strategy to select a branching variable in each stage to decompose the current partial problem into new partial problems. Selecting a variable whose value is likely to be 1 at an optimal solution is one of the practical strategies. However, when we proceed to branch-and-bound iterations without finding an upper bound for  $P_0$ , we need to obtain at least one feasible solution of  $P_0$  as fast as possible. In such a case, we therefore employ a different strategy.

When we do not have any feasible solution of  $P_0$ , we select the index

$$\tilde{k} = \arg \max_{k \in H_\nu^f} a_k \tag{22}$$

as the next branching variable with the hope to obtain a feasible solution. In this case, we just decompose the current partial problem  $P_\nu$  by fixing  $y_{\tilde{k}}$  to 0 and 1, and generate two new partial problems. We do not compute a lower bound because an upper bound is unknown. If  $|H_\nu^1| = p$ , this partial problem  $P_\nu$  becomes an arc capacitated network flow problem and we solve it exactly.

Once we find a feasible solution of  $P_0$ , we select the index

$$\tilde{k} = \arg \max_{k \notin K} \sum_{\pi \in \Pi} \bar{x}_{\pi k} \quad (23)$$

as the next branching variable. Note that when we solve a Lagrangian relaxation problem  $RR_\nu[\lambda, \mu]$  of  $P_\nu$ , we obtain as a byproduct an exact solution  $\bar{y}_k$  as shown in (17). If some  $k \notin K$  violate the capacity constraints (11) of  $P_\nu$ , selecting  $y_k$  that corresponds to one of such hubs  $k$  as a branching variable may yield a good feasible solution to  $P_\nu$ . By (23), we actually select  $y_k$  such that hub  $k \notin K$  violates the capacity constraints (11) of  $P_0$  most seriously.

Now we summarize the branch-and-bound method for solving  $P_0$ , which uses the depth-first search rule. We let  $\mathcal{A}$  denote the set of active partial problems, i.e., partial problems that have been generated but not tested yet.

### [Branch-and-Bound Method ]

**Step 0:** (Initialize) Find an optimal solution  $(x^*, y^*)$  of the arc capacitated network flow problem, in which candidates  $\kappa_1, \kappa_2, \dots, \kappa_p$  are selected as hubs. Let the optimal value of this problem be the incumbent value  $z^*$ . If this problem is infeasible, then find an optimal solution  $(x^*, y^*)$  of another arc capacitated network flow problem, in which candidates with  $p$  largest  $a_k$ 's are selected as hubs. Let the optimal value of this problem be the incumbent value  $z^*$ . If this problem is infeasible, set  $z^* = \infty$ . Set  $\nu := 0, H_\nu^F := H, H_\nu^1 := \emptyset, H_\nu^0 := \emptyset$  and  $\mathcal{A} := \{P_0\}$ . Select an integer parameter  $S$ , and real parameters  $0 < \zeta < 1, \eta > 0$  and  $\epsilon > 0$ .

**Step 1:** (Search) If  $\mathcal{A} = \emptyset$ , then go to Step 7. Otherwise, select an active partial problem  $P_{\nu+1} \in \mathcal{A}$  according to the depth-first search strategy. Set  $\nu := \nu + 1$ . If  $H_\nu^1 = p$  or  $H_\nu^0 = n - p$ , then go to Step 2. If a feasible solution of  $P_0$  is unknown, then go to Step 5. If  $H_\nu^1 < p$ , then go to Step 3.

**Step 2:** (Compute upper bound) Find an optimal solution  $(x_\nu^*, y_\nu^*)$  of  $P_\nu$  with the objective value  $z_\nu^*$ . If  $z_\nu^* < z^*$ , then let  $z^* := z_\nu^*, x^* := x_\nu^*, y^* := y_\nu^*$  and go to Step 5.

**Step 3:** (Subgradient method)

- (a) Set  $\lambda^{(0)} := 0$ ,  $\mu^{(0)} := 0$ , and  $l := 0$ . Let  $\hat{G}_\nu$  be upper bound of  $P_\nu$ . If an upper bound is not known, set  $\hat{G}_\nu := \eta z^*$ .
- (b) Set  $G_\nu^{\max} := -\infty$ .
- (c) Solve  $RP_\nu[\lambda^{(l)}, \mu^{(l)}]$  to obtain an optimal solution  $(\bar{x}, \bar{y})$ , and optimal value  $G_\nu^*$ . If  $G_\nu^* > G_\nu^{\max}$ , then update  $G_\nu^{\max} := G_\nu^*$ ,  $\lambda^{\max} := \lambda^{(l)}$  and  $\mu^{\max} := \mu^{(l)}$ . If  $G_\nu^* > z^*$ , then go to Step 6.
- (d) If  $\text{mod}(l + 1, S) = 0$ , then go to Step 3(e). Otherwise, compute Lagrange multipliers  $\lambda^{(l+1)}$  and  $\mu^{(l+1)}$  according to (18)–(20). Set  $l := l + 1$  and go to Step 3(c).
- (e) Update  $\hat{G}_\nu$  as  $\hat{G}_\nu := \frac{G_\nu^{\max} + (1 - \zeta)\hat{G}_\nu - \zeta G_\nu(\lambda^{(l-S+1)}, \mu^{(l-S+1)})}{2 - 2\zeta}$ . If  $\hat{G}_\nu < z^*$  or  $|(\hat{G}_\nu - G_\nu^{\max})/\hat{G}_\nu| < \epsilon$ , then go to Step 5. Otherwise, set  $\lambda^{(l+1)} := \lambda^{\max}$ ,  $\mu^{(l+1)} := \mu^{\max}$ ,  $l := l + 1$  and go to Step 3(b).

**Step 5:** (Branching) If a feasible solution of  $P_0$  is unknown, select the next branching variable  $\tilde{k}$  according to (22). Otherwise, select the next branching variable  $\tilde{k}$  according to (23).

Set  $\mathcal{A} := \mathcal{A} \cup \{(H_\nu^1 \cup \{\tilde{k}\}, H_\nu^0, H_\nu^F \setminus \{\tilde{k}\}), (H_\nu^1, H_\nu^0 \cup \{\tilde{k}\}, H_\nu^F \setminus \{\tilde{k}\})\}$  and go to Step 1.

**Step 6:** (Backtrack) Set  $\mathcal{A} := \mathcal{A} \setminus \{P_\nu\}$  and go to Step 1.

**Step 7:** (Terminate)  $x^*$  and  $y^*$  are optimal solution of  $P_0$ , and  $z^*$  is the optimal value. If  $z^* = \infty$ , then  $P_0$  is infeasible.

## 4 Numerical Experiments

In this section, we report some computational results. All programs were coded in MATLAB R12.1 (version 6.1) with optimization toolbox [6]. They were run on a DELL DIMENSION 4400 computer with Pentium4 2.0AGHz processor operated under Windows XP professional with 512 Mb DDR-SDRAM memory. We used the function `linprog` included in the optimization toolbox, which solves a linear programming problem using the interior point method.

We prepared the demand data based on the well-known U.S. 25 cities data evaluated in 1970 by CAB (Civil Aeronautics Board). The data set, however, does not contain node capacities and fixed costs, so we generated them based on the data reported by FAA (Federal Aviation Administration) [12]. They report the passenger enplanements in the fiscal year of



Table 1: Scaled fixed cost  $\tilde{f}_k$  and capacity  $\tilde{a}_k$ 

hub#	City	(a)FY2000 Enplanements $\times 10^3$	(b)FY2012 Forecast $\times 10^3$	(b)–(a) $\times 10^3$	Fixed Cost $\tilde{f}_k$	Capacity $\tilde{a}_k$
1	Atlanta	39,375	59,353	19,978	11.69	8.98
2	Baltimore	10,618	16,497	5,879	3.44	2.50
3	Boston	13,816	18,695	4,879	2.86	2.83
4	Chicago	34,153	46,178	12,025	7.04	6.99
5	Cincinnati	9,186	18,749	9,563	5.60	2.84
6	Cleveland	6,746	10,935	4,189	2.45	1.65
7	Dallas-FW	28,066	41,759	13,693	8.01	6.32
8	Denver	18,884	27,541	8,657	5.07	4.17
9	Detroit	17,874	28,287	10,413	6.09	4.28
10	Houston	16,564	26,510	9,944	5.82	4.01
11	Kansas City	5,889	7,978	2,089	1.22	1.21
12	Los Angeles	32,332	47,294	14,962	8.76	7.16
13	Memphis	6,234	8,488	2,254	1.32	1.28
14	Miami	16,716	24,156	7,440	4.35	3.65
15	Minneapolis	17,203	25,914	8,711	5.10	3.92
16	New Orleans	4,900	6,609	1,709	1.00	1.00
17	New York	16,081	22,991	6,910	4.04	3.48
18	Philadelphia	13,023	20,748	7,725	4.52	3.14
19	Phoenix	18,652	31,252	12,600	7.37	4.73
20	Pittsburgh	10,521	14,319	3,798	2.22	2.17
21	St. Louis	1,4553	21,631	7,078	4.14	3.27
22	San Francisco	18,499	25,634	7,135	4.18	3.88
23	Seattle	14,225	20,854	6,629	3.88	3.16
24	Tampa	8,200	11,384	3,184	1.86	1.72
25	WashingtonDC	8,502	15,146	6,644	3.89	2.29

2000 and the passenger forecast in the fiscal year of 2012. In Table 1, these data are shown in the column labeled “(a) FY2000 Enplanements” and the column labeled “(b) FY2012 Forecast”, respectively, and the growth forecast of passenger enplanements from 2000 to 2012 is shown in the column labeled “(b)–(a)”. Moreover,  $\tilde{f}_k$  denotes a scaled fixed cost obtained by setting the fixed cost of New Orleans to be 1.0. Similarly,  $\tilde{a}_k$  denotes a scaled capacity obtained by setting the capacity of New Orleans to be 1.0.

Since the data of node capacities are not available in the published documents, we determined the node capacity  $a_k$  for each candidate by

$$a_k = \frac{\sum_{\pi \in \Pi} \tilde{d}_\pi}{p \cdot \min_{k \in H} \tilde{a}_k}, \quad \forall k \in H, \quad (24)$$

where  $\tilde{d}_\pi$  is a scaled trip demand between OD pair  $\pi$ , that is  $\tilde{d}_\pi = d_\pi / \min_{\pi \in \Pi} d_\pi$ . In a similar

way, we determined arc capacities as follows. First we calculate the scaled arc capacities by assigning node capacities  $a_k$  equally to all arcs connected to it. In the computational experiments, we assume that  $H = N$ , i.e., all nodes are candidate hubs. So the number of arcs connected to each hub candidate is  $2(n - 1)$ . Consequently,  $\tilde{b}_{ij}$  are calculated as

$$\tilde{b}_{ij} = \frac{a_i + a_j}{2(n - 1)}, \quad (i, j) \in A,$$

from which we determined arc capacities  $b_{ij}$  as

$$b_{ij} = \delta \tilde{b}_{ij}, \quad (i, j) \in A,$$

where  $\delta \geq 1$  is a parameter. It is clear that a problem with small  $\delta$  has tight arc capacity constraints. Similarly, we determined  $f_k$ , the fixed cost of hub candidate  $k$ , as

$$f_k = \rho \tilde{f}_k, \quad k \in H,$$

where  $\rho > 0$  is a parameter. To see how the arc capacity constraints and fixed costs affect the total optimal cost and hub locations, we solved the problem with different values of  $\delta$  and  $\rho$ .

The computational results are given in Tables 2 and 3. Table 2 shows the results with  $\rho = 1000$ , where the ratio of fixed cost to the total cost is slightly less than 5%. Table 3 shows the results with  $\rho = 10000$ , where the ratio of fixed cost to the total cost is approximately 10%. The column labeled “optimal hubs” and “optimal cost” show the set of optimal hubs and the optimal total cost, respectively. The column labeled “# problems solved” shows the number of partial problems  $P_\nu$  solved exactly, i.e., the partial problems with  $|H_\nu| = p$ . The column labeled “B&B CPU time” shows the total CPU time required to solve the problem by the branch-and-bound method. The column labeled “% cost reduced” shows how the optimal cost reduces as  $\delta$  is increased from  $\delta = 1$ .

A problem with small  $\delta$  often contains many infeasible partial problems because of its tight arc capacity constraints. In such a case, it is generally difficult to obtain a good upper bound in the early stage of branch-and-bound iterations, and hence it takes much computation time. In fact, Tables 2 and 3 indicate that the computational time and the number of partial problems solved increase as  $\delta$  decreases. The optimal cost decreases monotonically as  $\delta$  increases, because arc capacity constraints become loose as  $\delta$  increases. However, when  $\rho = 1000$ , the decrease of the optimal cost is less significant. We can observe that the difference of the optimal cost between problems with  $\delta = 1$  and  $\delta = 3$  is less than 2% regardless of the value of  $p$ . In particular, when  $p = 4$ , the difference is much smaller than that in the cases of  $p = 2$  and  $p = 3$ . The reason for this phenomenon may be resorted to the fact that  $p$  is relatively large and the ratio of fixed cost to total cost is relatively small.

Regardless of the value of  $\delta$ , Kansas City(#11) is always selected as a hub when  $p = 2$ , and St. Louis(#21) and San Francisco(#22) are always selected when  $p = 3$ . Another selected hub changes in the order of New York(#17), Philadelphia(#18), Baltimore(#2) as  $\delta$  increases. Note that these three cities are located closely to each other. These results indicate that the optimal hub location does not change substantially when  $\rho = 1000$ .

On the other hand, when  $\rho = 10000$ , hub locations are significantly affected by the value of  $\delta$ . Table 3 shows that the maximum difference of the optimal cost between the two cases  $\delta = 1$  and  $\delta = 4$  is nearly 5%. Although the difference of 5% seems small, it brings a considerable amount of cost reduction when we deal with a large network. Figures 2 ~ 4 display the results for problems with  $n = 25$  and  $p = 2, 3, 4$ . The thickness of each arc represents flow congestion on that arc. These figures show that the optimal hub location varies with the value of  $\delta$ . In particular, since the ratio of fixed costs to the total cost is relatively large compared with the case of  $\delta = 1000$ , a candidate with lower fixed cost is likely to be selected. As  $\delta$  increases, the hub location changes from New York(#17) to Pittsburgh(#20), from Memphis(#13) to New Orleans(#16) and from New York(#17) to Boston(#3). These changes occur in such a way that a candidate is replaced by another candidate with lower fixed cost. However, when  $p = 4$ , the selected hub location is drastically changed from Boston(#3) to San Francisco(#22) in spite of the expensive fixed cost of San Francisco. Note that, when  $p \leq 3$ , no candidate located in the western area is selected. It becomes possible to select such a candidate when  $\delta = 4$ , because of capacity relaxation. By selecting a candidate located in the western area, we can reduce the cost of travel which originates from and destines for the western area. This travel cost reduction is large enough to compensate for the increase of fixed cost; thereby it induces the total cost reduction. Figure 4 shows that, when  $p \leq 3$ , the trips originating from Seattle, San Francisco and Los Angeles are long and the links emanating from these cities are congested.

From the above observation, we see that the total cost is affected by arc capacities, especially for problems in which the ratio of the fixed cost to the total cost is large, i.e.,  $\delta$  is large. Taking into account the fact that the arc capacities are likely to be changed frequently by various surroundings, it is important to select a robust or stable candidate. To select such hubs, airlines may expect to have a constant revenue. When  $\rho = 10000$ , Kansas City is always selected regardless of the values of  $p$  and  $\delta$ . Its good location in the central area and the cheap fixed cost may be the main reason why it is selected constantly. On the other hand, when  $\rho = 1000$ , there is no such stable hub candidate which is chosen independently of the value of  $p$  and  $\delta$ . In this case, almost all candidates except for some very expensive ones are likely to become hubs, because the ratio of the fixed cost to the total cost is small. When  $p = 2$ , Kansas City(#11) is always selected regardless of the value of  $\delta$ .

Similarly, when  $p = 3$ , St. Louis(#21) is always selected. Moreover, when  $p = 4$ , selected hubs remain the same completely. We may regard such hubs as stable for each  $p$ .

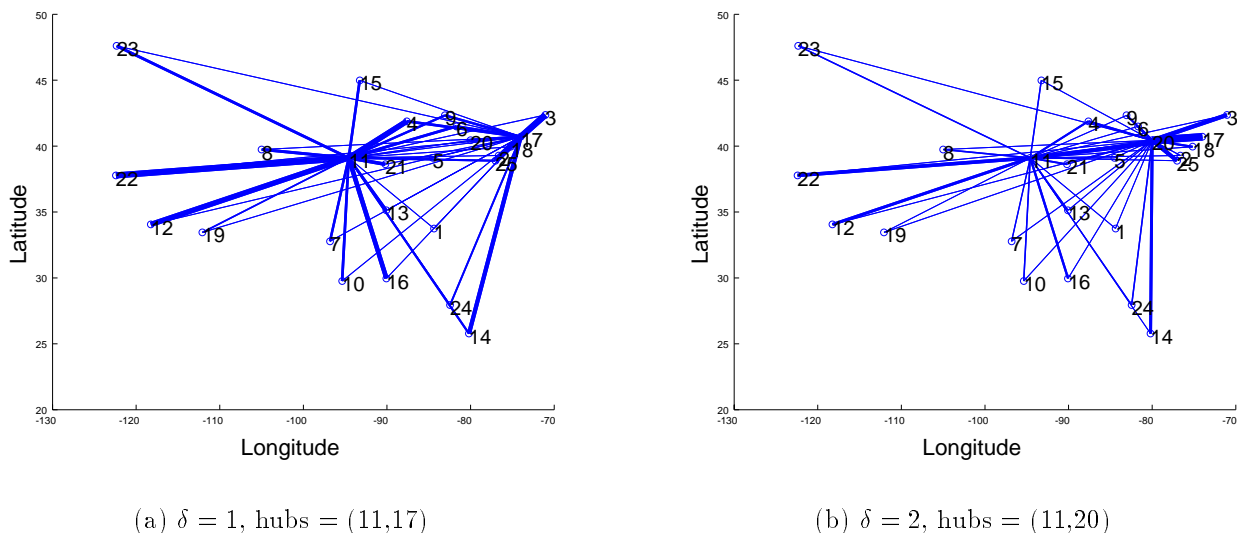


Figure 2: Results for  $n = 25, p = 2, \rho = 10000$

## 5 Conclusion

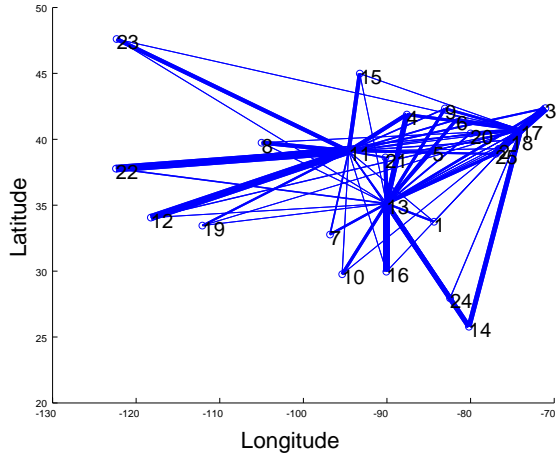
In this paper, we have considered a one-stop capacitated hub location problem in which both hubs and arcs have capacity constraints. We formulated the problem as a mixed 0–1 integer programming problem and solve the problem using branch-and-bound method with Lagrangian relaxation bounding strategy. We also made some computational experiments using actual aviation data based on the well-known CAB data set and the future enplanements forecast reported by FAA. From our computational results, we see that the total cost is affected by arc capacities, especially for problems in which the ratio of the fixed cost to the total cost is large. We also observe that the maximum difference in the optimal cost between the case of loose capacity constraints and that of tight capacity constraints is nearly 5 %. These results indicate that it is important to incorporate arc capacities into the model when we deal with a large network. Taking into account the fact that the arc capacities are likely to be changed frequently by various surroundings, it is also important to select hubs that are insensitive to those changes. In our computational experiments, some hub candidates are always chosen independently of the number of selected hubs and arc capacities. It is an interesting future research to develop an effective method of selecting such a stable hub set by using sensitivity analysis.

Table 2: Computational Results for  $\rho = 1000$ 

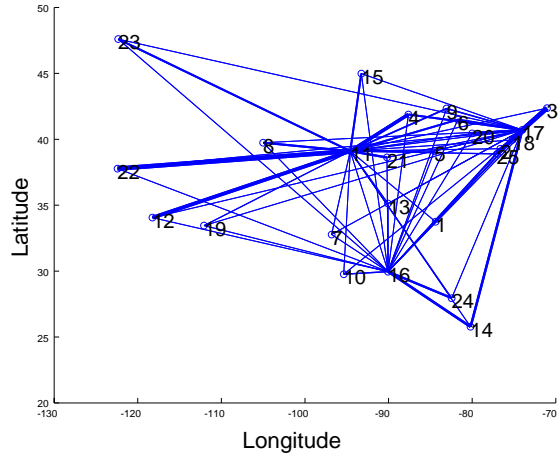
$n$	$p$	$\delta$	Optimal hubs	Optimal cost( $\times 10^5$ )	#Problems solved	B & B CPU time	% Cost reduced
25	2	1	11,17	3.3289	171	102.31	–
		2	11,18	3.3044	174	62.59	0.74
		3	2,11	3.2746	172	61.28	1.63
25	3	1	17,21,22	2.9741	890	848.16	–
		2	17,21,22	2.9541	904	573.50	0.67
		3	18,21,22	2.9448	890	574.88	0.98
25	4	1	6,12,13,17	2.8762	3287	4433.56	–
		2	6,12,13,17	2.8571	3463	3400.27	0.66
		3	6,12,13,17	2.8566	3272	3248.89	0.68

Table 3: Computational Results for  $\rho = 10000$ 

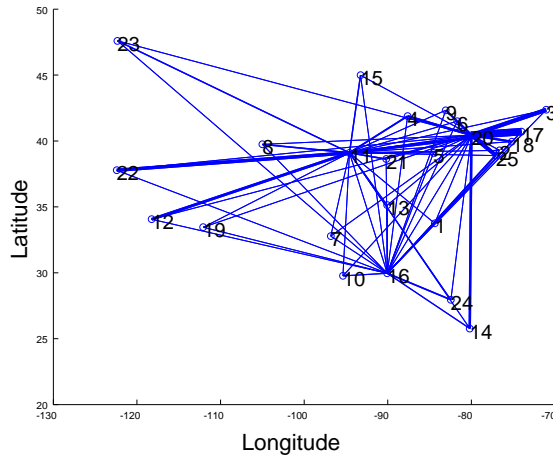
$n$	$p$	$\delta$	Optimal hubs	Optimal cost( $\times 10^5$ )	#Problems solved	B & B CPU time	% Cost reduced
25	2	1	11,17	3.8023	174	108.64	–
		2	11,20	3.6991	166	59.86	2.72
		3	11,20	3.6374	160	57.88	4.34
		4	11,20	3.6373	157	57.45	4.34
25	3	1	11,13,17	3.8237	1047	1041.38	–
		2	11,16,17	3.7882	867	550.58	0.93
		3	11,16,20	3.6916	796	501.55	3.45
		4	11,16,20	3.6535	772	489.38	4.45
25	4	1	11,16,17,20	3.8790	2272	3630.61	–
		2	3,11,16,20	3.8569	2027	2148.84	0.57
		3	3,11,16,20	3.8286	1630	1714.69	1.30
		4	11,16,20,22	3.7695	1593	1648.42	2.82



(a)  $\delta = 1$ , hubs = (11,13,17)

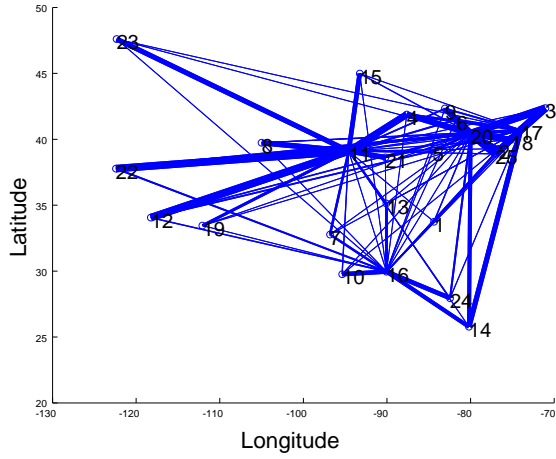


(b)  $\delta = 2$ , hubs = (11,16,17)

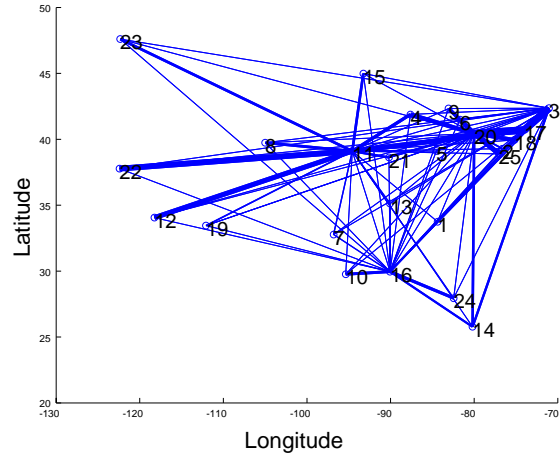


(c)  $\delta = 3$ , hubs = (11,16,20)

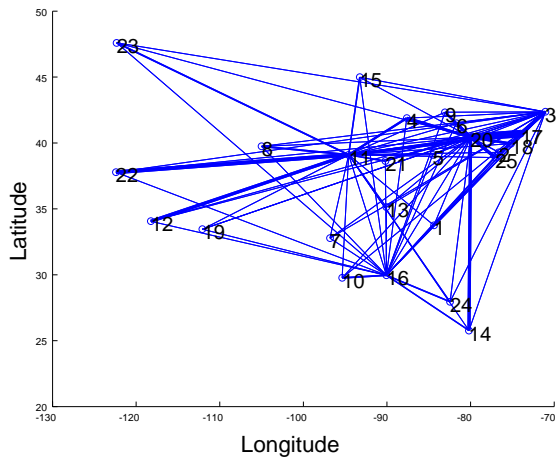
Figure 3: Results for  $n = 25, p = 3, \rho = 10000$



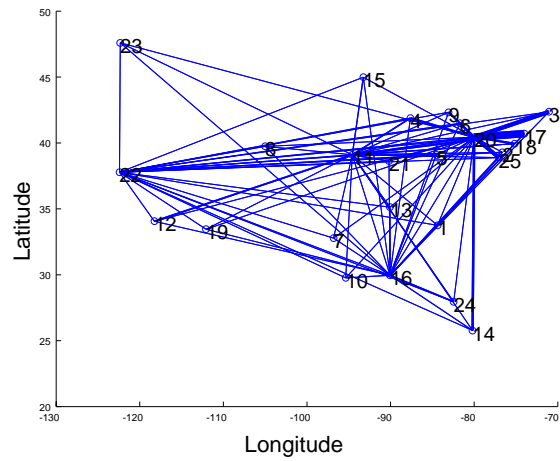
(a)  $\delta = 1$ , hubs = (11,16,17,20)



(b)  $\delta = 2$ , hubs = (3,11,16,20)



(c)  $\delta = 3$ , hubs = (3,11,16,20)



(d)  $\delta = 4$ , hubs = (11,16,20,22)

Figure 4: Results for  $n = 25, p = 4, \rho = 10000$

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