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On bimodal provability logics **MOS** and $\mathbf{PRL}_1$

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Abstract. Provability logic **GL** is one of the normal modal logics, which is obtained from the smallest normal modal logic **K** by adding Löb's axiom. The name "*provability logic*" derives from Solovay's theorem in Solovay [4]. He proved that **GL** is complete for the formal provability interpretation in Peano arithmetic **PA**. By his theorem, we might as well say a formula $\Box A$ asserts that A^* is provable in **PA**, where A^* is an arithmetic sentence corresponding to A.

Smoryński [3] considered bimodal provability logics by extending **GL** with another modal operator ∇. An arithmetic interpretation of ∇A is a provability of another arithmetic theory. After that, bimodal provability logics for other arithmetic theories have been considered in several papers, especially Beklemishev [2] gives a detailed survey of this topic.

Here we treat two bimodal provability logics **MOS** and $\mathbf{PRL}_1$ among the logics in [3] and give cut-free sequent systems for these two logics.

1. MOS and $\mathbf{PRL}_1$

In this section, we introduce bimodal provability logics **MOS** and $\mathbf{PRL}_1$ and their Kripke semantics(cf. [3]). The language of the bimodal provability logics consists of propositional variables, $\bot$, $\wedge$, $\vee$, $\supset$, $\Box$ and ∇. Formulas are defined as usual. We use upper case Latin letters $A, B, C, \cdots$, possibly with suffixes, for formulas.

Definition 1.1. The modal logic **MOS** is the smallest set of formulas containing
all tautologies,
K: $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
L: $\Box(\Box A \supset A) \supset \Box A$,
$A1$: $\nabla A \supset \Box\nabla A$,
$A2$: $\Box A \supset \nabla A$ and
$A3$: $\Box(A \supset B) \supset (\nabla A \supset \nabla B)$
and closed under modus ponens and necessitation.

Definition 1.2. The modal logic $\mathbf{PRL}_1$ is the smallest set of formulas containing all tautologies, K, L, $A1$, $A2$ and
$A4$: $\nabla(A \supset B) \supset (\nabla A \supset \nabla B)$
and closed under modus ponens and necessitation.

The following lemmas are useful for our investigations.

Lemma 1.3.([3])
(1) $\mathbf{MOS} \subseteq \mathbf{PRL}_1$
(2) $\Box A \supset \Box\Box A \in \mathbf{MOS}$
(3) $(\Box A \wedge \Box B) \supset \Box(A \wedge B) \in \mathbf{MOS}$
(4) $\nabla(\nabla A \supset A) \supset \nabla A \in \mathbf{MOS}$

Lemma 1.4.

$$(C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset A \in \mathbf{MOS}$$

implies

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \Box A \in \mathbf{MOS}.$$

Proof. By the assumption and necessitation,

$$\Box((C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset A) \in \mathbf{MOS}$$

Using K,

$$\Box(C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \triangledown D_1 \wedge \cdots \wedge \triangledown D_m) \supset \Box A \in \mathbf{MOS}$$

Using Lemma 1.3(3),

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \Box\Box C_1 \wedge \cdots \wedge \Box\Box C_n \wedge \Box\triangledown D_1 \wedge \cdots \wedge \Box\triangledown D_m) \supset \Box A \in \mathbf{MOS}$$

Using Lemma 1.3(2) and $A1$, we obtain the lemma.

Definition 1.5. A Kripke model for **MOS** is a tuple $\mathcal{K} = \langle \mathbf{W}, <, F, \models \rangle$, where
(1) $\mathbf{W}$ is a non-empty finite set,
(2) $<$ is an irreflexive and transitive binary relation on $\mathbf{W}$,
(3) F is a mapping from $\mathbf{W}$ to $\mathcal{P}(\mathcal{P}(\mathbf{W}))$ satisfying
  (3.1) $X \in F(\alpha)$ implies $X \subseteq \{\gamma \mid \alpha < \gamma\}$,
  (3.2) $\{\gamma \mid \alpha < \gamma\} \in F(\alpha)$ and
  (3.3) $\alpha < \beta$ and $X \in F(\alpha)$ imply $X \cap \{\gamma \mid \beta < \gamma\} \in F(\beta)$,
(4) $\models$ is a valuation satisfying, in addition to the usual boolean laws,
  (4.1) $\alpha \models \Box A$ iff for any $\beta \in \{\gamma \mid \alpha < \gamma\}$, $\beta \models A$ and
  (4.2) $\alpha \models \triangledown A$ iff there exists $X \in F(\alpha)$ such that for any $\beta \in X$, $\beta \models A$.

Definition 1.6. A Kripke model for $\mathbf{PRL}_1$ is a tuple $\mathcal{K} = \langle \mathbf{W}, <, R, \models \rangle$, where
(1) $\langle \mathbf{W}, < \rangle$ is as in the above definition,
(2) R is a binary relation on $\mathbf{W}$ satisfying
  (2.1) $\alpha R \beta$ implies $\alpha < \beta$ and
  (2.2) $\alpha < \beta R \gamma$ implies $\alpha R \gamma$
(3) $\models$ is a valuation satisfying, in addition to the usual boolean laws,
  (3.1) $\alpha \models \Box A$ iff for any $\beta \in \{\gamma \mid \alpha < \gamma\}$, $\beta \models A$ and
  (3.2) $\alpha \models \triangledown A$ iff for any $\beta \in \{\gamma \mid \alpha R \gamma\}$, $\beta \models A$.

Let $\mathcal{K} = \langle \mathbf{W}, <, X, \models \rangle$ be a Kripke model for **MOS** or $\mathbf{PRL}_1$. We say that a formula A is valid in $\mathcal{K}$ if $\alpha \models A$ for any $\alpha \in \mathbf{W}$.

Lemma 1.7([3]).
(1) $A \in \mathbf{MOS}$ *iff A is valid for any Kripke model for* **MOS**.
(2) $A \in \mathbf{PRL}_1$ *iff A is valid for any Kripke model for* $\mathbf{PRL}_1$.

2. Sequent systems

In this section, we introduce sequent systems for **MOS** and $\mathbf{PRL}_1$ and prove their soundness. We use Greek letters $\Gamma, \Delta, \Sigma, \cdots$, possibly with suffixes, for finite sets of formulas. The expressions $\Box\Gamma$ and $\triangledown\Gamma$ denote the set $\{\Box A \mid A \in \Gamma\}$ and $\{\triangledown A \mid A \in \Gamma\}$, respectively. By a sequent, we mean an expression $\Gamma \to \Delta$. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_l \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_l \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

The sequent system **GMOS** is the system obtained from the sequent system **LK** for the classical propositional logic by adding the following three inference rules.

$$\frac{\Box A, \Gamma, \Box\Gamma, \triangledown\Sigma \to A}{\Box\Gamma, \triangledown\Sigma \to \Box A}(\Box)$$

$$\frac{\triangledown A, B, \triangledown B, \Gamma, \Box\Gamma, \triangledown\Sigma \to A}{\triangledown B, \Box\Gamma, \triangledown\Sigma \to \triangledown A}(\triangledown_1)$$

$$\frac{\nabla A, \Gamma, \Box\Gamma, \nabla\Sigma \rightarrow A}{\Box\Gamma, \nabla\Sigma \rightarrow \nabla A}(\nabla_2)$$

The sequent system $\mathbf{GPRL}_1$ is the system obtained from $\mathbf{LK}$ by adding the above inference rule ($\Box$) and the following inference rule.

$$\frac{\nabla A, \Gamma, \Box\Gamma, \Sigma, \nabla\Sigma \rightarrow A}{\Box\Gamma, \nabla\Sigma \rightarrow \nabla A}(\nabla_3)$$

Theorem 2.1.

(1) *If a sequent* $A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n$ *is provable in* **GMOS**, *then* $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n)$ *is valid for any Kripke model for* **MOS**.

(2) *If a sequent* $A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n$ *is provable in* $\mathbf{GPRL}_1$, *then* $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n)$ *is valid for any Kripke model for* $\mathbf{PRL}_1$.

Proof. (1) By Lemma 1.7, it is sufficient to show that three additional inference rules hold in **MOS**.

For ($\Box$): Suppose that

$$(\Box A \wedge C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset A \in \mathbf{MOS}.$$

Hence

$$(C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset (\Box A \supset A) \in \mathbf{MOS}.$$

Using Lemma 1.4,

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \Box(\Box A \supset A) \in \mathbf{MOS}.$$

Using L,

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \Box A \in \mathbf{MOS}.$$

For (∇_1): Suppose that

$$(\nabla A \wedge B \wedge \nabla B \wedge C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset A \in \mathbf{MOS}.$$

Hence

$$(\nabla B \wedge C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset (B \supset (\nabla A \supset A)) \in \mathbf{MOS}.$$

Using Lemma 1.4,

$$(\nabla B \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \Box(B \supset (\nabla A \supset A)) \in \mathbf{MOS}.$$

Using $A3$,

$$(\nabla B \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset (\nabla B \supset \nabla(\nabla A \supset A)) \in \mathbf{MOS}.$$

Hence

$$(\nabla B \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \nabla(\nabla A \supset A) \in \mathbf{MOS}.$$

Using Lemma 1.3(4),

$$(\nabla B \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \nabla A \in \mathbf{MOS}.$$

For (∇_2): Suppose that

$$(\nabla A \wedge C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset A \in \mathbf{MOS}.$$

Hence

$$(C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset (\nabla A \supset A) \in \mathbf{MOS}.$$

Using Lemma 1.4,

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \Box(\nabla A \supset A) \in \mathbf{MOS}.$$

Using $A2$,

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \nabla(\nabla A \supset A) \in \mathbf{MOS}.$$

Using Lemma 1.3(4),

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \nabla A \in \mathbf{MOS}.$$

(2) By (1) and Lemma 1.3(1), ($\Box$) holds in $\mathbf{PRL}_1$. So, it is sufficient to show that (∇_3) holds in $\mathbf{PRL}_1$. Suppose that

$$(\nabla A \wedge C_1 \wedge \cdots \wedge C_n \wedge \Box C_1 \wedge \cdots \wedge \Box C_n \wedge D_1 \wedge \cdots \wedge D_m \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset A \in \mathbf{PRL}_1.$$

Hence

$$C_1 \supset (\cdots (\nabla D_m \supset (\nabla A \supset A)) \cdots) \in \mathbf{PRL}_1.$$

Using necessitation and $A2$,

$$\nabla (C_1 \supset (\cdots (\nabla D_m \supset (\nabla A \supset A)) \cdots)) \in \mathbf{PRL}_1.$$

Using $A4$, possibly several times,

$$\nabla C_1 \supset (\cdots (\nabla \nabla D_m \supset \nabla(\nabla A \supset A)) \cdots) \in \mathbf{PRL}_1.$$

Using Lemma 1.3(4),

$$\nabla C_1 \supset (\cdots (\nabla \nabla D_m \supset \nabla A) \cdots) \in \mathbf{PRL}_1.$$

Hence,

$$(\nabla C_1 \wedge \cdots \wedge \nabla C_n \wedge \nabla \Box C_1 \wedge \cdots \wedge \nabla \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m \wedge \nabla \nabla D_1 \wedge \cdots \wedge \nabla \nabla D_m) \supset \nabla A \in \mathbf{PRL}_1.$$

Using $A2$,

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \Box \Box C_1 \wedge \cdots \wedge \Box \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m \wedge \Box \nabla D_1 \wedge \cdots \wedge \Box \nabla D_m) \supset \nabla A \in \mathbf{PRL}_1.$$

Using Lemma 1.3(2) and $A1$,

$$(\Box C_1 \wedge \cdots \wedge \Box C_n \wedge \nabla D_1 \wedge \cdots \wedge \nabla D_m) \supset \nabla A \in \mathbf{PRL}_1.$$

3. Cut-elimination for GMOS

Here we prove the following theorem.

Theorem 3.1. *If there exists no cut-free proof figure for $\Gamma \to \Delta$ in* **GMOS***, then there exist Kripke model $\mathcal{K} = \langle \mathbf{W}, <, F, \models \rangle$ for* **MOS** *and $\alpha \in \mathbf{W}$ such that $\alpha \models A$ for $A \in \Gamma$ and $\alpha \not\models B$ for $B \in \Delta$.*

For a proof of the above theorem, we extend the method in Avron [1], which gives a cut-free sequent system for **GL**(see also Valentini [5]).

Definition 3.2. A sequent $\Gamma \to \Delta$ is said to be saturated if the following conditions hold:
(1) if $A \wedge B \in \Gamma$, then $A, B \in \Gamma$,
(2) if $A \vee B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$,
(3) if $A \supset B \in \Gamma$, then $A \in \Delta$ or $B \in \Gamma$,
(4) if $A \wedge B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$,
(5) if $A \vee B \in \Delta$, then $A, B \in \Delta$,
(6) if $A \supset B \in \Delta$, then $A \in \Gamma$ and $B \in \Delta$.

By $\mathsf{Sub}(A)$, we mean the set of subformulas of A. We put $\mathsf{Sub}(\Gamma) = \bigcup_{A\in\Gamma} \mathsf{Sub}(A)$ and $\mathsf{Sub}(\Gamma \to \Delta) = \mathsf{Sub}(\Gamma) \cup \mathsf{Sub}(\Delta)$.

Lemma 3.3(cf. [1]) *If $\Gamma \to \Delta$ has no cut-free proof figure in* **GMOS***, then there exists a saturated sequent $\Gamma' \to \Delta'$ having no cut-free proof figure in* **GMOS** *such that $\Gamma \subseteq \Gamma' \subseteq \mathsf{Sub}(\Gamma \to \Delta)$ and $\Delta \subseteq \Delta' \subseteq \mathsf{Sub}(\Gamma \to \Delta)$.*

In this section, we call a saturated sequent in the above lemma a saturation of $\Gamma \to \Delta$.

Definition 3.4. Let S_0 be a sequent having no cut-free proof figure in **GMOS**. We define a set $\mathbf{W}(S_0)$ as follows:
(1) $S_0 \in \mathbf{W}(S_0)$,
(2) if a sequent $\Gamma \to \Delta, \Box A$ belongs to $\mathbf{W}(S_0)$, then so does a saturation of

$$\Box A, \{D \mid \Box D \in \Gamma\}, \{\Box D \mid \Box D \in \Gamma\}, \{\triangledown D \mid \triangledown D \in \Gamma\} \to A,$$

(3) if a sequent $\Gamma \to \Delta, \triangledown A$ belongs to $\mathbf{W}(S_0)$, then so does a saturation of

$$\triangledown A, \{D \mid \Box D \in \Gamma\}, \{\Box D \mid \Box D \in \Gamma\}, \{\triangledown D \mid \triangledown D \in \Gamma\} \to A,$$

(4) if a sequent $\triangledown B, \Gamma \to \Delta, \triangledown A$ belongs to $\mathbf{W}(S_0)$, then so does a saturation of

$$\triangledown A, B, \triangledown B, \{D \mid \Box D \in \Gamma\}, \{\Box D \mid \Box D \in \Gamma\}, \{\triangledown D \mid \triangledown D \in \Gamma\} \to A.$$

Lemma 3.5. *Let S_0 be a sequent having no cut-free proof figure in* **GMOS** *and let S be a sequent in $\mathbf{W}(S_0)$. Then S has no cut-free proof figure in* **GMOS** *and contains only formulas in $\mathsf{Sub}(S_0)$.*

Proof. We use an induction on S as an element in $\mathbf{W}(S_0)$. If $S = S_0$, then the lemma is clear. Suppose that $S \neq S_0$. Then $S \in \mathbf{W}(S_0)$ is known using (2),(3) or (4) in Definition 3.4. We only show the case that (2) is used. So, there exists a sequent $\Gamma \to \Delta, \Box A \in \mathbf{W}(S)$ and S is a saturation of

$$\Box A, \{D \mid \Box D \in \Gamma\}, \{\Box D \mid \Box D \in \Gamma\}, \{\triangledown D \mid \triangledown D \in \Gamma\} \to A.$$

By the induction hypothesis, $\Gamma \to \Delta, \Box A$ has no cut-free proof figure in **GMOS** and contains only formulas in $\mathsf{Sub}(S_0)$, hence so does a sequent

$$\{\Box D \mid \Box D \in \Gamma\}, \{\triangledown D \mid \triangledown D \in \Gamma\} \to \Box A.$$

Using the inference rule $(\Box)$ and Lemma 3.3, we obtain the lemma.

Corollary 3.6. *Let S_0 be a sequent having no cut-free proof figure in* **GMOS***. Then $\mathbf{W}(S_0)$ is finite.*

Definition 3.7. Let S_0 be a sequent having no cut-free proof figure in **GMOS**. We define a structure $\mathcal{K}(S_0) = \langle \mathbf{W}(S_0), <, F, \models \rangle$ as follows:
(1) $\Gamma_1 \to \Delta_1 < \Gamma_2 \to \Delta_2$ iff $\{D \mid \Box D \in \Gamma_1\} \cup \{\Box D \mid \Box D \in \Gamma_1\} \cup \{\triangledown D \mid \triangledown D \in \Gamma_1\} \subseteq \Gamma_2$ and there exits a formula $B \in \Delta_1 \cap \Gamma_2$ such that $B = \Box D$ or $B = \triangledown D$ for some D,
(2) $F(\Gamma \to \Delta) = \{\{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta', D \in \Gamma'\} \mid \triangledown D \in \Gamma\} \cup \{\{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta'\}\}$.
(3) $\models$ is a valuation satisfying, in addition to the laws in Definition 1.5,

$$p \in \Gamma \text{ iff } \Gamma \to \Delta \models p$$

for any propositional variable p.

Lemma 3.8. *Let S_0 be a sequent having no cut-free proof figure in* **GMOS***. Then $\mathcal{K}(S_0)$ is a Kripke model for* **MOS***.*

Proof. By Corollary 3.6, it is sufficient to show the following five:

(1) $<$ is irreflexive,

(2) $<$ is transitive,

(3) $X \in F(\Gamma \to \Delta)$ implies $X \subseteq \{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta'\}$,

(4) $\{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta'\} \in F(\Gamma \to \Delta)$,

(5) $\Gamma_1 \to \Delta_1 < \Gamma_2 \to \Delta_2$ and $X \in F(\Gamma_1 \to \Delta_1)$ imply $X \cap \{\Gamma' \to \Delta' \mid \Gamma_2 \to \Delta_2 < \Gamma' \to \Delta'\} \in F(\Gamma_2 \to \Delta_2)$.

For (1): Suppose that $\Gamma \to \Delta < \Gamma \to \Delta$. Then there exits a formula $B \in \Delta \cap \Gamma$. So, there exists a cut-free proof figure $\Gamma \to \Delta$ in **GMOS**. This is contradictory to Lemma 3.5.

For (2): Suppose that $\Gamma_1 \to \Delta_1 < \Gamma_2 \to \Delta_2 < \Gamma_3 \to \Delta_3$. Then for any $\Box D \in \Gamma_1$, we have $\Box D \in \Gamma_2$, and so, $D, \Box D \in \Gamma_3$. Similarly, for any $\nabla D \in \Gamma_1$, we have $\nabla D \in \Gamma_2$, and so, $\nabla D \in \Gamma_3$. Hence

$$\{D \mid \Box D \in \Gamma_1\} \cup \{\Box D \mid \Box D \in \Gamma_1\} \cup \{\nabla D \mid \nabla D \in \Gamma_1\} \subseteq \Gamma_3.$$

By $\Gamma_1 \to \Delta_1 < \Gamma_2 \to \Delta_2$, there exits a formula $B \in \Delta_1 \cap \Gamma_2$ such that $B = \Box D$ or $B = \nabla D$ for some D. Using $\Gamma_2 \to \Delta_2 < \Gamma_3 \to \Delta_3$, we have $B \in \Delta_1 \cap \Gamma_3$.

For (3): From Definition 3.7(2).

For (4): From Definition 3.7(2).

For (5): Suppose that $\Gamma_1 \to \Delta_1 < \Gamma_2 \to \Delta_2$ and $X \in F(\Gamma_1 \to \Delta_1)$. If $X = \{\Gamma' \to \Delta' \mid \Gamma_1 \to \Delta_1 < \Gamma' \to \Delta'\}$, then by (2), $X \cap \{\Gamma' \to \Delta' \mid \Gamma_2 \to \Delta_2 < \Gamma' \to \Delta'\} = \{\Gamma' \to \Delta' \mid \Gamma_2 \to \Delta_2 < \Gamma' \to \Delta'\} \in F(\Gamma_2 \to \Delta_2)$. So, we assume that $X = \{\Gamma' \to \Delta' \mid \Gamma_1 \to \Delta_1 < \Gamma' \to \Delta', D \in \Gamma'\}$ for some $\nabla D \in \Gamma_1$. By $\Gamma_1 \to \Delta_1 < \Gamma_2 \to \Delta_2$, we have $\nabla D \in \Gamma_2$. Using (2), $X \cap \{\Gamma' \to \Delta' \mid \Gamma_2 \to \Delta_2 < \Gamma' \to \Delta'\} = \{\Gamma' \to \Delta' \mid \Gamma_2 \to \Delta_2 < \Gamma' \to \Delta', D \in \Gamma'\} \in F(\Gamma_2 \to \Delta_2)$.

Lemma 3.9. *Let S_0 be a sequent having no cut-free proof figure in* **GMOS**. *Then in $\mathcal{K}(S_0)$,*

(1) $A \in \Gamma$ *implies* $\Gamma \to \Delta \models A$ *and*

(2) $A \in \Delta$ *implies* $\Gamma \to \Delta \not\models A$.

Proof. We use an induction on A.

If A is a propositional variable or $\bot$, then we obtain (1) and (2) by the definition of $\models$ and Lemma 3.5.

Suppose that A is not variable or $\bot$ and the lemma holds for any proper subformula of A. We only show the following two cases.

The case that $A = \Box B$: Suppose that $\Box B \in \Gamma$. Then for any $\Gamma' \to \Delta' \in \{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta'\}$, we have $B \in \Gamma'$, and by the induction hypothesis, $\Gamma' \to \Delta' \models B$. Hence $\Gamma \to \Delta \models \Box B$.

Suppose that $\Box B \in \Delta$. Then a saturation S of a sequent $\Box B, \{C \mid \Box C \in \Gamma\}, \{\Box C \mid \Box C \in \Gamma\}, \{\nabla C \mid \nabla C \in \Gamma\} \to B$ belongs to $\mathbf{W}(S_0)$. We note that B belongs to the succeedent of S and $\Gamma \to \Delta < S$. So, by the induction hypothesis, $S \not\models B$, and hence, $\Gamma \to \Delta \not\models \Box B$.

The case that $A = \nabla B$: Suppose that $\nabla B \in \Gamma$. Then $\{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta', B \in \Gamma'\} \in F(\Gamma \to \Delta)$. By the induction hypothesis, $S \models B$ for any $S \in \{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta', B \in \Gamma'\}$. Hence $\Gamma \to \Delta \models \nabla B$.

Suppose that $\nabla B \in \Delta$. We will show that for any $X \in F(\Gamma \to \Delta)$, there exists $S \in X$ such that $S \not\models B$. If $X = \{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta'\}$, then a saturation S_1 of a sequent $\nabla B, \{C \mid \Box C \in \Gamma\}, \{\Box C \mid \Box C \in \Gamma\}, \{\nabla C \mid \nabla C \in \Gamma\} \to B$ belongs to X, and by the induction hypothesis, $S_1 \not\models B$. So, we assume that $X = \{\Gamma' \to \Delta' \mid \Gamma \to \Delta < \Gamma' \to \Delta', D \in \Gamma'\}$ for some $\nabla D \in \Gamma$. Then a saturation S_2 of a sequent $D, \nabla B, \{C \mid \Box C \in \Gamma\}, \{\Box C \mid \Box C \in \Gamma\}, \{\nabla C \mid \nabla C \in \Gamma\} \to B$ belongs to X, and by the induction hypothesis, $S_2 \not\models B$.

By Lemma 3.8 and Lemma 3.9, we obtain Theorem 3.1.

Corollary 3.10. *The following four conditions are equivalent:*

(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in$ **GMOS**,

(2) $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n)$ *is valid for any Kripke model for* **MOS**,

(3) *there exists a cut-free proof figure for* $A_1, \cdots, A_m \to B_1, \cdots, B_n$ *in* **GMOS**,

(4) $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n) \in \mathbf{MOS}$.

Proof. Theorem 2.1 shows that (1) implies (2). By Theorem 3.1, we have that (2) implies (3). It is clear that (3) implies (1). The equivalence between (2) and (4) is shown in Lemma 1.7.

4. Cut-elimination for $\mathbf{GPRL}_1$

Here we prove the following theorem.

Theorem 4.1. *If there exists no cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GPRL}_1$, then there exist Kripke model $\mathcal{K} = \langle \mathbf{W}, <, R, \models \rangle$ for $\mathbf{PRL}_1$ and $\alpha \in \mathbf{W}$ such that $\alpha \models A$ for $A \in \Gamma$ and $\alpha \not\models B$ for $B \in \Delta$.*

To prove the theorem above, we show some lemmas.

Lemma 4.2(cf. [1]). *If $\Gamma \to \Delta$ has no cut-free proof figure in $\mathbf{GPRL}_1$, then there exists a saturated sequent $\Gamma' \to \Delta'$ having no cut-free proof figure in $\mathbf{GPRL}_1$ such that $\Gamma \subseteq \Gamma' \subseteq \mathsf{Sub}(\Gamma \to \Delta)$ and $\Delta \subseteq \Delta' \subseteq \mathsf{Sub}(\Gamma \to \Delta)$.*

In this section, we call a saturated sequent in the above lemma a saturation of $\Gamma \to \Delta$.

Definition 4.3. Let S_0 be a sequent having no cut-free proof figure in $\mathbf{GPRL}_1$. We define a set $\mathbf{W}^*(S_0)$ as follows:
(1) $S_0 \in \mathbf{W}^*(S_0)$,
(2) if a sequent $\Gamma \to \Delta, \Box A$ belongs to $\mathbf{W}^*(S_0)$, then so does a saturation of

$$\Box A, \{D \mid \Box D \in \Gamma\}, \{\Box D \mid \Box D \in \Gamma\}, \{\triangledown D \mid \triangledown D \in \Gamma\} \to A,$$

(3) if a sequent $\Gamma \to \Delta, \triangledown A$ belongs to $\mathbf{W}^*(S_0)$, then so does a saturation of

$$\triangledown A, \{D \mid \Box D \in \Gamma\}, \{\Box D \mid \Box D \in \Gamma\}, \{D \mid \triangledown D \in \Gamma\}\{\triangledown D \mid \triangledown D \in \Gamma\} \to A.$$

Lemma 4.4. *Let S_0 be a sequent having no cut-free proof figure in $\mathbf{GPRL}_1$ and let S be a sequent in $\mathbf{W}^*(S_0)$. Then S has no cut-free proof figure in $\mathbf{GPRL}_1$ and contains only formulas in $\mathsf{Sub}(S_0)$.*

Proof. Similarly to Lemma 3.5.

Corollary 4.5. *Let S_0 be a sequent having no cut-free proof figure in $\mathbf{GPRL}_1$. Then $\mathbf{W}^*(S_0)$ is finite.*

Definition 4.6. Let S_0 be a sequent having no cut-free proof figure in $\mathbf{GPRL}_1$. We define a structure $\mathcal{K}^*(S_0) = \langle \mathbf{W}^*(S_0), <, R, \models \rangle$ as follows:
(1) $<$ is as in Definition 3.5,
(2) $\Gamma_1 \to \Delta_1 R \Gamma_2 \to \Delta_2$ iff $\{D \mid \Box D \in \Gamma_1\} \cup \{\Box D \mid \Box D \in \Gamma_1\} \cup \{D \mid \triangledown D \in \Gamma_1\} \cup \{\triangledown D \mid \triangledown D \in \Gamma_1\} \subseteq \Gamma_2$ and there exits a formula $B \in \Delta_1 \cap \Gamma_2$ such that $B = \Box D$ or $B = \triangledown D$ for some D.
(3) $\models$ is a valuation satisfying, in addition to the laws in Definition 1.6,

$$p \in \Gamma \text{ iff } \Gamma \to \Delta \models p$$

for any propositional variable p.

Lemma 4.7. *Let S_0 be a sequent having no cut-free proof figure in $\mathbf{GPRL}_1$. Then $\mathcal{K}^*(S_0)$ is a Kripke model for $\mathbf{PRL}_1$.*

Proof. The proof of Lemma 3.6 shows the irreflexivity and transitivity of $<$. We only show the condition for R. By the definition, we have that $S_1 R S_2$ implies $S_1 < S_2$. Suppose that $\Gamma_1 \to \Delta_1 < \Gamma_2 \to \Delta_2 R \Gamma_3 \to \Delta_3$. Then for $\triangledown D \in \Gamma_1$, we have $\triangledown D \in \Gamma_2$, and so, $D \in \Gamma_3$. On the other hand, by the transitivity of $<$, we have $\Gamma_1 \to \Delta_1 < \Gamma_3 \to \Delta_3$. Hence we obtain $\Gamma_1 \to \Delta_1 R \Gamma_3 \to \Delta_3$.

Lemma 4.8. *Let S_0 be a sequent having no cut-free proof figure in* $\mathbf{GPRL}_1$. *Then in* $\mathcal{K}(S_0)$,
(1) $A \in \Gamma$ *implies* $\Gamma \to \Delta \models A$ *and*
(2) $A \in \Delta$ *implies* $\Gamma \to \Delta \not\models A$.

Proof. We use an induction on A. We show only the case that $A = \triangledown B$. The other cases can be shown similarly to Lemma 3.9.

Suppose that $\triangledown B \in \Gamma$. Then for any $\Gamma' \to \Delta' \in \{\Gamma' \to \Delta' \mid \Gamma \to \Delta R \Gamma' \to \Delta'\}$, we have $B \in \Gamma'$, and by the induction hypothesis. $\Gamma' \to \Delta' \models B$. Hence $\Gamma \to \Delta \models \triangledown B$.

Suppose that $\triangledown B \in \Delta$. Then a saturation $\Gamma' \to \Delta'$ of a sequent $\triangledown B, \{C \mid \Box C \in \Gamma\}, \{\Box C \mid \Box C \in \Gamma\}, \{C \mid \triangledown C \in \Gamma\}, \{\triangledown C \mid \triangledown C \in \Gamma\} \to B$ belongs to $\mathbf{W}(S_0)$. We note that $B \in \Delta'$ and $\Gamma \to \Delta R \Gamma' \to \Delta'$. Using the induction hypothesis, $\Gamma' \to \Delta' \not\models B$, and hence, $\Gamma \to \Delta \not\models \triangledown B$.

By Lemma 4.7 and Lemma 4.8, we obtain Theorem 4.1.

Corollary 4.9. *The following four conditions are equivalent:*
(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GPRL}_1$,
(2) $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n)$ *is valid for any Kripke model for* $\mathbf{PRL}_1$,
(3) *there exists a cut-free proof figure for* $A_1, \cdots, A_m \to B_1, \cdots, B_n$ *in* $\mathbf{GPRL}_1$,
(4) $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n) \in \mathbf{PRL}_1$.

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