Two constructions of non-equivalent modal formulas[1]

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Abstract. We discuss two constructions of non-equivalent modal formulas for normal modal logics. One is basically same as the dual of the construction in Moss [Mos07] and the other is the construction in [Sas05], [Sas08] and [Sas09]. The former is useful for any normal modal logics, while as in [Sas09], the latter has much more information for modal logic **S4**. Here we define two constructions in sequent style and discuss the difference between them.

1 Preliminary

In the present section, we introduce modal formulas and normal modal logics.

Formulas are constructed from $\bot$ (contradiction) and the propositional variables $p_1, p_2, \cdots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \cdots$, possibly with suffixes, for formulas. Also we use Greek letters, $\Gamma, \Delta, \cdots$, possibly with suffixes, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. For a formula A, the *depth* $d(A)$ of A, is defined as
$d(p_i) = d(\bot) = 0$,
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
$d(\Box B) = d(B) + 1$.

Let **ENU** be an enumeration of the formulas. For a non-empty finite set Γ of formulas, the expressions

$$\bigwedge \Gamma \qquad \text{and} \qquad \bigvee \Gamma$$

denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \qquad \text{and} \qquad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = \Gamma$ and A_i occurs earlier than A_{i+1} in **ENU**. Also the expressions

$$\bigwedge \emptyset \qquad \text{and} \qquad \bigvee \emptyset$$

denote the formulas $\bot \supset \bot$ and $\bot$, respectively.

The set of propositional variables $p_1, \cdots, p_m$ $(m \geq 1)$ is denoted by $\mathbf{V}$ and the set of formulas constructed from $\mathbf{V}$ and $\bot$ is denoted by $\mathbf{F}$. Also for any $n = 0, 1, \cdots$, we define $\mathbf{F}^n$ as

$$\mathbf{F}^n = \{A \in \mathbf{F} \mid d(A) \leq n\}.$$

Below we also use n for natural numbers $0, 1, \cdots$.

By a sequent, we mean the expression $(\Gamma \to \Delta)$. For brevity's sake, we often write $\Gamma \to \Delta$ instead of $(\Gamma \to \Delta)$ and we write

$$A_1, \cdots, A_i, \Gamma_1, \cdots, \Gamma_j \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_\ell$$

instead of

$$\{A_1, \cdots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_\ell\}.$$

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We use upper case Latin letters $X, Y, Z, \cdots$, possibly with suffixes, for sequents. For a sequent $\Gamma \to \Delta$, we define $\mathbf{ant}(\Gamma \to \Delta)$ and $\mathbf{suc}(\Gamma \to \Delta)$, the *antecedent* and the *succedent* of $\Gamma \to \Delta$, respectively, as follows:

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma, \qquad \mathbf{suc}(\Gamma \to \Delta) = \Delta.$$

Also for a sequent X and for a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as follows:

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \Gamma \neq \emptyset \\ \bigvee \Delta & \text{if } \Gamma = \emptyset, \end{cases}$$

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

By **LK**, we mean the sequent system for the classical propositional logic given by Gentzen [Gen35]. Here we do not use $\neg$ as a primary connectives, so we use the additional axion $\bot \to$ instead of the inference rules $(\neg \to)$ and $(\to \neg)$.

By **K**, we mean the sequent system obtained from **LK** by adding the inference rule

$$\frac{\Gamma \to \Delta}{\Box\Gamma \to \Box\Delta}(\Box).$$

By a normal modal logic, we mean a sequent system obtained by **K** by adding sequents as axioms. We use L, possibly with suffixes, for normal modal logics.

We write $X \in L$ if X is provable in L. We write $A \equiv_L B$ instead of $\to (A \supset B) \wedge (B \supset A) \in L$. Also for $[A], [B] \in \mathbf{F}/\equiv_L$, we write $[A] \leq_L [B]$ instead of $A \to B \in L$. Then structure $\langle \mathbf{F}^n/\equiv_L, \leq_L \rangle$ expresses mutual provability of formulas.

2 The first construction

We construct non-equivalent formulas in Definition 2.1. The dual of the construction is basically same as [Mos07], which is based on Fine [Fin75]. The construction is useful to clarify the structure $\langle \mathbf{F}^n/\equiv_L, \leq_L$ since Theorem 2.2 below holds. Also Theorem 2.3 clarify the behavior of connectives. Most of the proof of two theorems here can be given by the results in [Mos07] and [Sas09]. Here we mainly prove Theorem 2.3(4), in other words, clarify the behavior of $\Box$.

Definition 2.1 The sets $\mathbf{ED}_L(n)$ of sequents and the mappings $\mathbf{Next}_L$, $\mathbf{prov}_L$, $\mathbf{next}_L$ are defined inductively as follows:

$\mathbf{ED}_L(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,

$\mathbf{Next}_L(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{ED}_L(k)), \Gamma \cap \Delta = \emptyset\}$, for $X \in \mathbf{ED}_L(k)$,

$\mathbf{prov}_L(X) = \{Y \in \mathbf{Next}_L(X) \mid Y \notin L\}$, for $X \in \mathbf{ED}_L(k)$,

$\mathbf{next}_L(X) = \mathbf{Next}(X) - \mathbf{prov}_L(X)$, for $X \in \mathbf{ED}_L(k)$,

$\mathbf{ED}_L(k+1) = \displaystyle\bigcup_{X \in \mathbf{ED}_L(k)} \mathbf{next}_L(X)$.

Theorem 2.2

(1) $\mathbf{F}^n/\equiv_L = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{ED}_L(n)\}$.

(2) *For subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}_L(n)$,*

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \textit{ if and only if } \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in \mathbf{S4}.$$

Theorem 2.3

(1) $\bot \equiv_L \bigwedge \mathbf{for}(\mathbf{ED}_L(n))$.

(2) $p_i \equiv_L \bigwedge \mathbf{for}(\{X \in \mathbf{ED}_L(n) \mid p_i \in \mathbf{suc}(X)\})$.

(3) *For any finite subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}_L(n)$,*

(3.1) $\bigwedge \mathbf{for}(\mathcal{S}_1) \wedge \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv_L \bigwedge \mathbf{for}(\mathcal{S}_1 \cup \mathcal{S}_2)$,

(3.2) $\bigwedge \mathbf{for}(\mathcal{S}_1) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv_L \bigwedge \mathbf{for}(\mathcal{S}_1 \cap \mathcal{S}_2)$,

(3.3) $\bigwedge \mathbf{for}(\mathcal{S}_1) \supset \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv_L \bigwedge \mathbf{for}((\mathbf{ED}^*_L(n) - \mathcal{S}_1) \cap \mathcal{S}_2)$.

(4) *For any finite subset* $\mathcal{S}$ *of* $\mathbf{ED}_L(n)$,
$\Box \bigwedge \mathbf{for}(\mathcal{S}) \equiv_L \bigwedge \mathbf{for}(\{Y \in \mathbf{ED}_L(k+1) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y), X \in \mathcal{S}\})$.

We prove Theorem 2.3(4), and briefly show the other parts of theorems. The following two lemmas and corollary were shown in [Sas09] in the case that L is **S4**, the normal modal logic obtained by adding two axioms $\Box A \to A$ and $\Box A \to \Box\Box A$ to **K**. The proof in [Sas09] can also show the case that L is another normal modal logic.

Lemma 2.4
(1) *None of the members in* $\mathbf{ED}_L(n)$ *is provable in* L.
(2) *For any* $X, Y \in \mathbf{ED}_L(n)$, $X \neq Y$ *implies* $\mathbf{for}(X) \vee \mathbf{for}(Y) \in L$.

Lemma 2.5 *Let* Σ, Γ, Δ *be finite sets of formulas. Then for any subset* $\Sigma' \subseteq \Sigma$,

$$\Box\Sigma', \Lambda, \Gamma \to \Delta \in L,$$

where $\Lambda = \{\mathbf{for}(\Box\Phi, \Gamma \to \Delta, \Box\Psi) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}$.

Corollary 2.6 *For any* $X, Y \in \mathbf{ED}_L(n)$,
(1) $\mathbf{for}(\mathbf{next}_L(X)) \to \mathbf{for}(X) \in L$,
(2) $\bigwedge \mathbf{for}(\mathbf{next}_L(X)) \equiv_L \mathbf{for}(X)$,
(3) $\{\mathbf{for}(Z) \mid Z \in \mathbf{next}_L(X), \Box\mathbf{for}(Y) \in \mathbf{suc}(Z)\}, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y) \in L$.

Using Lemma 2.4 and Corollary 2.6(2), Theorem 2.3(1), Theorem 2.3(2) and Theorem 2.3(3) can be shown as in [Sas08] and [Sas09]. By Theorem 2.3, Theorem 2.2 can be shown as in [Sas08] and [Sas09]. By the following lemma, we obtain Theorem 2.3(4).

Lemma 2.7 *For any* $X \in \mathbf{ED}_L(n)$,

$$\mathbf{for}(\{Y \in \mathbf{ED}_L(n+1) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\}) \to \Box\mathbf{for}(X) \in L.$$

Proof. By Corollary 2.6(3), we have

$$\{\mathbf{for}(Y) \mid Y \in \mathbf{next}_L(Z), \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\} \to \mathbf{for}(Z), \Box\mathbf{for}(X) \in L.$$

for any $Z \in \mathbf{ED}_L(n)$. So,

$$\bigcup_{Z \in \mathbf{ED}_L(n)} \{\mathbf{for}(Y) \mid Y \in \mathbf{next}_L(Z), \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\} \to \bigwedge \mathbf{for}(\mathbf{ED}_L(n)), \Box\mathbf{for}(X) \in L.$$

In other words,

$$\mathbf{for}(\{Y \in \mathbf{ED}_L(n+1) \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y)\}) \to \bigwedge \mathbf{for}(\mathbf{ED}_L(n)), \Box\mathbf{for}(X) \in L.$$

Using Theorem 2.3(1), we obtain the lemma. $\dashv$

3 The second construction

We construct another non-equivalent formulas in Definition 3.1. The construction is introduced in [Sas05], [Sas08] and [Sas09] and has much more information on normal modal logics containing **S4**. [Sas09] considered **S4** and proved the corresponding results to two theorems in the previous section. Also it is also useful to clarify infinite structure $\langle \mathbf{F}/\equiv_{\mathbf{S4}}, \leq_{\mathbf{S4}} \rangle$, and useful to construct $\mathbf{prov}^*_{\mathbf{S4}}(X)$ defined below without using the provability of **S4**. So, it is natural to consider whether the corresponding theorems hold for any other normal modal logics. In the present section, using the results in the previous sections, we give another proof of theorems in [Sas09] and consider such problem.

Definition 3.1 The subsets $\mathbf{G}_L(n)$ and $\mathbf{G}^*_L(n)$ of sequents, and the mappings $\mathbf{Next}^*_L$, $\mathbf{prov}^*_L$, $\mathbf{next}^*_L$ are defined inductively as follows:

$\mathbf{G}_L(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,

$\mathbf{Next}^*_L(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}_L(k)), \Gamma \cap \Delta = \emptyset\}$, for $X \in \mathbf{G}_L(k)$,

$\mathbf{prov}^*_L(X) = \{Y \in \mathbf{Next}^*_L(X) \mid Y \notin L\}$, for $X \in \mathbf{G}_L(k)$,

$\mathbf{next}^*_L(X) = \mathbf{Next}^*(X) - \mathbf{prov}^*_L(X)$, for $X \in \mathbf{G}_L(k)$,

$\mathbf{G}_L(k+1) = \bigcup_{X \in \mathbf{G}_L(k) - \mathbf{G}^*_L(k)} \mathbf{next}^*_L(X)$,

$\mathbf{G}^*_L(k+1) = \{X \in \mathbf{G}_L(k+1) \mid (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box}$ implies $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$, for any $Y \in \mathbf{G}_L(k+1)\}$.

Definition 3.2 We define $\mathbf{ED}^*_L(n)$ as follows:

$$\mathbf{ED}^*_L(n) = \mathbf{G}_L(n) \cup \bigcup_{k=0}^{n-1} \mathbf{G}^*_L(k).$$

By sketching the proof in [Sas09], the following two theorems hold.

Theorem 3.3 *Let L be a normal modal logic containing* **S4**.

(1) $\mathbf{F}^n/\equiv_L = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{ED}^*_L(n)\}$.

(2) *For subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}^*_L(n)$,*

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \textit{ if and only if } \bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in L.$$

Theorem 3.4 *Let L be a normal modal logic containing* **S4**.

(1) $\perp \equiv_L \bigwedge \mathbf{for}(\mathbf{ED}^*_L(n))$.

(2) $p_i \equiv_L \bigwedge \mathbf{for}(\{X \in \mathbf{ED}^*_L(n) \mid p_i \in \mathbf{suc}(X)\})$.

(3) *For any finite subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}^*_L(n)$,*

(3.1) $\bigwedge \mathbf{for}(\mathcal{S}_1) \wedge \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv_L \bigwedge \mathbf{for}(\mathcal{S}_1 \cup \mathcal{S}_2)$,

(3.2) $\bigwedge \mathbf{for}(\mathcal{S}_1) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv_L \bigwedge \mathbf{for}(\mathcal{S}_1 \cap \mathcal{S}_2)$,

(3.3) $\bigwedge \mathbf{for}(\mathcal{S}_1) \supset \bigwedge \mathbf{for}(\mathcal{S}_2) \equiv_L \bigwedge \mathbf{for}((\mathbf{ED}^*_L(n) - \mathcal{S}_1) \cap \mathcal{S}_2)$.

(4) *For any finite subset $\mathcal{S}$ of $\mathbf{ED}^k$, $\Box \bigwedge \mathbf{for}(\mathcal{S}) \equiv_L \bigwedge \mathbf{for}(\mathcal{S}_1 \cup \mathcal{S}_2)$, where*
$\mathcal{S}_1 = \{Y \in \mathbf{G}^*(i) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}, X \in \mathcal{S} \cap \mathbf{G}^*(i), 0 \leq i \leq k\}$,
$\mathcal{S}_2 = \{Y \in \mathbf{ED}^{k+1} \mid \Box\mathbf{for}(X) \in \mathbf{suc}(Y), X \in \mathcal{S}\}$.

Also we can show the two theorems using the results in the previous section.

Definition 3.5 For any $X \in \mathbf{G}_L(n)$, we define $\mathbf{C}(X)$ and $\mathbf{n}_L(X)$ as follows:

$\mathbf{C}(X) = \{Y \in \mathbf{G}_L(n) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}\}$,

$\mathbf{n}_L(X) = (\Box\mathbf{for}(\mathbf{G}_L(n) - \mathbf{C}(X)), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\mathbf{C}(X)))$.

Definition 3.6 We define $\mathbf{BG}_L(n)$ as follows:

$$\mathbf{BG}_L(n) = \mathbf{V} \cup \bigcup_{i=0}^{n-1} \Box\mathbf{for}(\mathbf{G}_L(i)).$$

The following two lemmas were shown in [Sas09] in the case that $L = \mathbf{S4}$, and we find that the same proof can show the two lemmas.

Lemma 3.7 *For any $X \in \mathbf{G}_L(n)$, $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{BG}_L(n)$ and $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.*

Lemma 3.8 *Let L be a normal modal logic containing $\mathbf{S4}$ and let X and Y be sequents in $\mathbf{G}_L(n)$ satisfying $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Then*
(1) *$X \in \mathbf{G}^*_L(n)$ if and only if $Y \in \mathbf{G}^*_L(n)$,*
(2) *$X \in \mathbf{G}^*_L(n)$ implies $\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in L$.*

Lemma 3.9 (Ohnishi and Matsumoto [OM57]) *Let L be a normal modal logic containing $\mathbf{S4}$. The the following two inference rule hold:*

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad\qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box).$$

Lemma 3.10 *Let L be a normal modal logic containing $\mathbf{S4}$ and X be a sequent in $\mathbf{G}^*_L(n)$.*
(1) *$(\mathbf{ant}(X))^\Box \to \Box\mathbf{for}(Y) \in L$, for any $Y \in (\bigcup_{k=n}^{\infty} \mathbf{G}_L(k)) - \mathbf{C}(X)$,*
(2) *$\mathbf{for}(X) \equiv_L \mathbf{for}(\mathbf{n}_L(X))$,*
(3) *$\mathbf{next}^*_L(X) = \{\mathbf{n}_L(X)\}$.*

Proof. For (1). Let Y be a sequent in $\mathbf{G}_L(n) - \mathbf{C}(X)$. Then we have $(\mathbf{ant}(X))^\Box \neq (\mathbf{ant}(Y))^\Box$. Using $X \in \mathbf{G}^*_L(n)$, we have $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$. So, there exists a formula $\Box A \in (\mathbf{ant}(X))^\Box - (\mathbf{ant}(Y))^\Box$. Using Lemma 3.7, we have $\Box A \in (\mathbf{ant}(X))^\Box \cap (\mathbf{suc}(Y))^\Box$. Hence

$$(\mathbf{ant}(X))^\Box \to \mathbf{for}(Y) \in L.$$

Using Lemma 3.9, we have

$$(\mathbf{ant}(X))^\Box \to \Box\mathbf{for}(Y) \in L. \tag{1.1}$$

So, (1.1) holds for any $Y \in \mathbf{G}_L(n) - \mathbf{C}(X)$. Using Lemma 3.8(1), (1.1) holds for any $Y \in \mathbf{G}_L(n) - \mathbf{G}^*_L(n)$. Since $\Box\mathbf{for}(Z) \to \Box\mathbf{for}(Z_\oplus) \in L$, for any $Z_\oplus \in \mathbf{next}^*_L(Z)$, using an induction on $i (> 0)$, (1.1) holds for any $Y \in \mathbf{G}_L(n+i)$. Hence we obtain (1).

For (2). It is easily seen that

$$\mathbf{for}(X) \to \mathbf{for}(\mathbf{n}_L(X)) \in L.$$

So, we have only to show

$$\mathbf{for}(\mathbf{n}_L(X)) \to \mathbf{for}(X) \in L. \tag{2.1}$$

By (1),

$$\mathbf{ant}(X) \to B \in L \quad \text{for any } B \in \mathbf{ant}(\mathbf{n}_L(X)). \tag{2.2}$$

On the other hand, by Lemma 3.8, we have

$$B \to \mathbf{for}(X) \in L \quad \text{for any } B \in \mathbf{suc}(\mathbf{n}_L(X)). \tag{2.3}$$

By (2.2) and (2.3), we have

$$\mathbf{ant}(X), \mathbf{for}(\mathbf{n}_L(X)) \to \mathbf{for}(X) \in L.$$

Hence we obtain (2.1).

For (3). By (2), we have $\mathbf{n}_L(X) \notin L$, and so,

$$\mathbf{next}^*_L(X) \supseteq \{\mathbf{n}_L(X)\}.$$

We show

$$\mathbf{next}^*_L(X) \subseteq \{\mathbf{n}_L(X)\}.$$

Suppose that

$$\Box\mathbf{for}(\mathbf{G}_L(n) - \Delta), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\Delta) \in \mathbf{next}^*_L(X) - \{\mathbf{n}_L(X)\}. \tag{2.1}$$

We note that $\Delta \subseteq \mathbf{G}_L(n)$ and $\Delta \neq \mathbf{C}(X)$.

We divide the cases.

The case that $\Delta \not\subseteq \mathbf{C}(X)$. There exists a sequent $Z \in \Delta - \mathbf{C}(X) \subseteq \mathbf{G}_L(n) - \mathbf{C}(X)$. So, using (1),

$$(\mathbf{ant}(X))^{\Box} \to \Box\mathbf{for}(Z) \in L.$$

Hence

$$\Box\mathbf{for}(\mathbf{G}_L(n) - \Delta), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\Delta) \in L,$$

which is in contradiction with (2.1).

The case that $\Delta \not\supseteq \mathbf{C}(X)$. There exists a sequent $Z \in \mathbf{C}(X) - \Delta \subseteq \mathbf{C}(X)$. So, using Lemma 3.8, we have $\Box\mathbf{for}(Z) \to \mathbf{for}(X) \in L$. Hence

$$\Box\mathbf{for}(\mathbf{G}_L(n) - \Delta), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\Delta) \in L,$$

which is in contradiction with (2.1). $\dashv$

For sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of sequents, we write $\mathcal{S}_1 \cong_L \mathcal{S}_2$ if there exists a one-to-one mapping f from $\mathcal{S}_1$ onto $\mathcal{S}_2$ satisfying $\mathbf{for}(X) \equiv_L \mathbf{for}(f(X))$ for any $X \in \mathcal{S}_1$.

Lemma 3.11 *Let L be a normal modal logic containing* **S4**. *Then*

$$\mathbf{ED}_L(n) \cong_L \mathbf{ED}^*_L(n).$$

Proof. We use an induction on n.

Basis($n = 0$) is clear from $\mathbf{ED}_L(0) = \mathbf{ED}^*_L(0)$.

Induction step($n > 0$). By the induction hypothesis, there exists a one-to-one mapping f from $\mathbf{ED}_L(n-1)$ onto $\mathbf{ED}^*_L(0)$ satisfying

$$\mathbf{for}(X) \equiv_L \mathbf{for}(f(X)) \text{ for any } X \in \mathbf{ED}_L(n-1). \tag{1}$$

We define a mapping f' from $\mathbf{ED}_L(n)$ to the set of sequents as follows:

$$f'(X_\oplus) = (\Box\mathbf{for}(\{f(Y) \mid Y \in \mathcal{S}_1\}), \mathbf{ant}(f(X)) \to \mathbf{suc}(f(X)), \Box\mathbf{for}(\{f(Y) \mid Y \in \mathcal{S}_2\})),$$

where

$$X_\oplus = (\Box\mathbf{for}(\mathcal{S}_1), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\mathcal{S}_2)) \in \mathbf{ED}_L(n), \tag{2}$$

for some $X \in \mathbf{ED}_L(n-1)$ and subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{ED}_L(n-1)$. We note

$$f(X) \in \mathbf{ED}_L(n-1) - \bigcup_{i=0}^{n-1} \mathbf{G}^*_L(i) \quad \text{implies} \quad f'(X_\oplus) \in \mathbf{ED}_L(n)$$

and by (1) and Lemma 2.4(1), we have

$$\mathbf{for}(X_\oplus) \equiv_L \mathbf{for}(f'(X_\oplus)) \tag{3}$$

and

$$f'(X_\oplus) \notin L \tag{4}$$

So, we can define a mapping g from $\mathbf{ED}_L(n)$ to $\mathbf{ED}^*_L(n)$ as follows:

$$g(X_\oplus) = \begin{cases} f(X) & \text{if } f(X) \in \bigcup_{i=0}^{n-1} \mathbf{G}^*_L(i) \\ f'(X_\oplus) & \text{otherwise,} \end{cases}$$

where $X_\oplus$ is a sequent as in (2).

We show

$$\mathbf{for}(X_\oplus) \equiv_L \mathbf{for}(g(X_\oplus)) \text{ for any } X_\oplus \in \mathbf{ED}_1. \tag{5}$$

We use $X_\oplus$ as in (2). If $f(X) \notin \bigcup_{i=0}^{n-1} \mathbf{G}^*_L(i)$, then by (3), we have

$$\mathbf{for}(X_\oplus) \equiv_L \mathbf{for}(f'(X_\oplus)) = \mathbf{for}(g(X_\oplus)).$$

If $f(X) \in \mathbf{G}^*_L(n-1)$, then by (3), (4), Lemma 3.10(3) and Lemma 3.10(2), we have

$$\mathbf{for}(X_\oplus) \equiv_L \mathbf{for}(f'(X_\oplus)) = \mathbf{for}(\mathbf{n}(f(X))) \equiv_L \mathbf{for}(f(X)) = \mathbf{for}(g(X_\oplus)).$$

So, we assume that $f(X) \in \bigcup_{i=0}^{n-2} \mathbf{G}^*_L(i)$. By (4) and Lemma 3.10(1), we have

$$f'(X_\oplus) = (\Box\mathbf{for}(\mathbf{ED}^*_L(n-1)), \mathbf{ant}(f(X)) \to \mathbf{suc}(f(X))).$$

Using Lemma 3.10(1), again,

$$\mathbf{for}(f'(X_\oplus)) \equiv_L \mathbf{for}(f(X)).$$

So, using (3),

$$\mathbf{for}(X_\oplus) \equiv_L \mathbf{for}(f'(X_\oplus)) \equiv_L \mathbf{for}(f(X)) = \mathbf{for}(g(X_\oplus)).$$

Hence we obtain (5).

By Theorem 2.2(3) and the above (5), we have that g is one-to-one.

We show that g is onto. Let $Z_\oplus$ be a sequent in $\mathbf{ED}^*_L(n)$. If $Z_\oplus \in \mathbf{G}_L(n)$, then there exists a sequent $Z \in \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)$ such that $Z_\oplus \in \mathbf{next}^*_L(Z)$, and so, $Z'_\oplus$ defined as

$$\begin{aligned} Z'_\oplus = (&\Box\mathbf{for}(\{f^{-1}(Y) \mid \Box\mathbf{for}(Y) \in \mathbf{ant}(Z_\oplus) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)\})), \mathbf{ant}(f^{-1}(Z)) \\ &\to \mathbf{suc}(f^{-1}(Z)), \Box\mathbf{for}(\{f^{-1}(Y) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(Z_\oplus) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1))\})) \end{aligned}$$

satisfies

$$g(Z'_\oplus) = Z_\oplus \text{ and } Z'_\oplus \in \mathbf{ED}_L(n),$$

using the properties of f. If $Z_\oplus \in \mathbf{G}^*_L(n-1)$, then $Z'_\oplus$ defined as

$$\begin{aligned} Z'_\oplus = (&\Box\mathbf{for}(\{f^{-1}(Y) \mid \Box\mathbf{for}(Y) \in \mathbf{ant}(\mathbf{n}(Z_\oplus)) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1)\})), \mathbf{ant}(f^{-1}(Z_\oplus)) \\ &\to \mathbf{suc}(f^{-1}(Z_\oplus)), \Box\mathbf{for}(\{f^{-1}(Y) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(\mathbf{n}(Z_\oplus)) \cap \Box\mathbf{for}(\mathbf{G}_L(n-1))\})) \end{aligned}$$

satisfies

$$g(Z'_\oplus) = Z_\oplus \text{ and } Z'_\oplus \in \mathbf{ED}_L(n),$$

using the properties of f and Lemma 3.10(2). If $Z_\oplus \in \bigcup_{i=0}^{n-2} \mathbf{G}^*_L(i)$, then $Z'_\oplus$ defined as

$$Z'_\oplus = (\Box\mathbf{for}(\mathbf{ED}_L(n-1)), \mathbf{ant}(f^{-1}(Z_\oplus)) \to \mathbf{suc}(f^{-1}(Z_\oplus)))$$

satisfies

$$g(Z'_\oplus) = Z_\oplus \text{ and } Z'_\oplus \in \mathbf{ED}_L(n),$$

using the properties of f and Lemma 3.10(1). $\dashv$

By Theorem 2.2 and Lemma 3.11, we obtain Theorem 3.2.

We note that property of **S4** is used only in Lemma 3.8(2) and Lemma 3.10(1). So, considering these two lemmas, there is a possibility to obtain the corresponding results to [Sas09] for a normal modal logic without containing **S4**.

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