A preliminary to no new reals

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Abstract

We look at a consistency of set theory where club guessing fails but the continuum hypothesis holds. Its proof due to S. Shelah takes many new ideas. Among others, we prepare a preliminary note to deal with a kind of properness of higher order.

Introduction

In [S1] and [S2], consistencies of statements together with the continuum hypothesis (CH) are dealt. Starting in the ground model V where we assume CH, we construct models $V[G]$ of set theory via iterated forcing. We keep the least uncountable cardinal ω_1 between V and the extensions $V[G]$. We add no new reals over V so that CH remains in $V[G]$ while forcing what we want in $V[G]$.

In this note we take a look at a model where club guessing (CG) fails and CH holds. This construction has a long history. Please see [S1] and [S2]. Recent related works include [Sa] and [M]. In [Sa], the preservation of $\neg$ CG under Cohen forcing is shown. Hence it implies a consistency of set theory where $\neg$ CG holds while 2^ω is large. [M] considers a combinatorial principle which implies both $\neg$ CG and $2^\omega = 2^{\omega_1} > \omega_1$.

We write this note based on a small fragment of [S2]. What we deal with is sets of countable elementary substructures of various H_χ and their internal structures. What appears to be left are (1): an argument with a tower of a finitely many, 5 or so, elementary substructures of [S2]. (2): a right induction hypothesis with respect to iterated forcing accommodating (1) rather than a game in [S2]. It would take a consideration to where we argue with (1). (3): exact calculations of how many layers of elementary substructures are required in advance compared to a given amount of elementary substructures to be retained. Related is formulations of clubs serving as guides to the calculations of (2) and (3) through the iteration. And of course, (4): ω^ω-bounding together with properness under countable support for this purpose as in [S1].

It would take (3) to accomplish (2). I provide no pictures at successor stages for (3). Therefore I have left a lot to a consistency proof.

§1. How to cope with losing elementary substructures

We first go through some of objects in use to set our notations.

Notation 1.1. Let C be a closed unbounded subset (club) of ω_1. The set of countable ordinals which are accumulation points of C is denoted by $\overline{C}$. Let Ω denote the set of countable limit ordinals. Thus $\overline{C} \subseteq \Omega$ and if $\delta \in \overline{C}$, then $C \cap \delta$ is cofinal below $\delta \in C$. A *ladder* A at $\delta \in \Omega$ is a cofinal subset of δ and is of order-type ω. A *ladder system* $\langle A_\delta \mid \delta \in \Omega \rangle$ is a system of ladders attatched to each $\delta \in \Omega$. *Club Guessing* (CG) means there exists a ladder system $\langle A_\delta \mid \delta \in \Omega \rangle$ such that for any club D of ω_1, there exists $\delta \in \Omega$ such that $A_\delta \setminus D$ is finite. For any set X, its size is denoted by $\mid X \mid$. Hence $\mid A_\delta \setminus D \mid < \omega$.

For a regular cardinal χ, H_χ denotes the set of sets which are hereditarily of size less than χ. An $\in$-*chain* $\langle N_i \mid i < l \rangle$ *of countable elementary substructures of* H_χ means that each $(N_i, \in)$ is a countable elementary substructure of $(H_\chi, \in)$, $\langle N_i \mid i \leq j \rangle \in N_{j+1}$ for all j with $j + 1 < l$ and $N_j = \bigcup\{N_i \mid i < j\}$ for all limit j with $j < l$. We also say $\langle M_n \mid n < \omega \rangle$ is an $\in$-*chain*, if for all $n < \omega$, we have $M_n \in M_{n+1}$ and M_n are countable elementary substructures of different H_χ's. This use of terminology is some what confusing, however there should be no real harm by considering the contexts.

We would like to get the consistency of the negation of CG (denoted by $\neg$ CG) together with the continuum hypothesis (CH). This consistency together with many others are claimed in [S2]. An account of this goes as follows; Assuming CH in the ground model, we may iterate ω_2-times with the following under countable support.

Definition 1.2. Let $\langle A_\delta \mid \delta \in \Omega \rangle$ be a ladder system. Let $p = (\alpha^p, C^p) \in P = P(\langle A_\delta \mid \delta \in \Omega \rangle)$, if

(1) $\alpha^p < \omega_1$,

(2) $C^p \subseteq \alpha^p + 1$ is closed and $\alpha^p \in C^p$,

(3) For all $\alpha \in \Omega$ with $\alpha \leq \alpha^p$, $\mid A_\alpha \cap C^p \mid < \omega$.

For $p, q \in P$, let $q \leq p$, if

(1) $\alpha^p \leq \alpha^q$,

(2) $C^p = C^q \cap (\alpha^p + 1)$.

The following is from [S2] (or may see [M]).

Lemma 1.3. Let χ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_χ with $P \in N$. Let $\delta = N \cap \omega_1$. Let $p \in N \cap P$ and F be a finite subset of δ, then there exists $q \in P$ such that q is a lower bound of some (P, N)-generic sequence and $C^q \cap F = C^p \cap F$. In particular, P is proper and σ-Baire.

We iterate with this family of notions of forcing to get $\neg$ CG via a suitable book-keeping. We denote it by $I = \langle P_\alpha \mid \alpha \leq \omega_2 \rangle$. We certainly have $\neg$ CG, since each ladder system gets killed by the generic clubs $\dot{C} = \bigcup\{C^p \mid p \in G\}$, where G denotes the P-generic filters. But to get CH, we need many new ideas as in [S2].

Since each P is σ-Baire, we add no new reals at each successor stage. However, if we are to add no new reals at the limit stages of iterated forcing, we would want some form of higher properness such as α-properness for all $\alpha < \omega_1$ and some type of completeness [S1]. But in the present context we even do not have an ω-properness. This is because ω-properness preserves CG and iterates under countable support [S1].

Motivation 1.4. (1) Let $\langle N_n \mid n \leq \omega \rangle$ be an $\in$-chain in H_χ, where χ is a sufficiently large regular cardinal. If it happens to be the case that $\langle N_n \cap \omega_1 \mid n < \omega \rangle$ coincides with the enumeration of A_{δ_ω}, where $\delta_\omega = N_\omega \cap \omega_1$, then there is no $q \in P$ which is (P, N_n)-generic for all $n \leq \omega$.

(2) Let χ_i be sufficiently large regular cardinals for $i = 0, 1, 2$. Suppose $p \in P$, $P \in H_{\chi_0} \in H_{\chi_1} \in H_{\chi_2}$. Let N be a countable elementary substructure of H_{χ_2} with $\chi_0, \chi_1, p, P \in N$. Notice that we have $H_{\chi_0}, H_{\chi_1} \in N$ and so $N \cap H_{\chi_0}$ and $N \cap H_{\chi_1}$ are elementary substructures of H_{χ_0} and H_{χ_1} respectively. Let $\langle N_{1n} \mid n \leq \omega \rangle$ be an $\in$-chain of countable elementary substructures of H_{χ_1} such that $\chi_0, p, P \in N_{10}$ and $N_{1\omega} = N \cap H_{\chi_1}$. Let $\delta_\omega = N_{1\omega} \cap \omega_1$. Let $\langle N_{0i} \mid i < \omega^2 \rangle$ be an $\in$-chain in H_{χ_0} such that $p, P \in N_{00}$, for all $n < \omega$, $N_{0\omega\cdot(n+1)} = N_{1n} \cap H_{\chi_0}$. We may choose an $\in$-subchain $\langle N_{0i_n} \mid n < \omega \rangle$ so that $\omega \cdot n < i_n < \omega \cdot (n+1)$ and $A_{\delta_\omega} \cap \{N_{0i_n} \cap \omega_1 \mid n < \omega\} = \emptyset$. Now construct $\langle q_n \mid n < \omega \rangle$ such that $q_n \in N_{0i_{n+1}} \cap P$ is (P, N_{0i_n})-generic, $C^{q_n} \cap A_{\delta_\omega} = C^p \cap A_{\delta_\omega}$ and $p \geq q_n \geq q_{n+1}$. Let $q = (\alpha^q, C^q)$, where $\alpha^q = \delta_\omega$ and $C^q = \bigcup\{C^{q_n} \mid n < \omega\} \cup \{\delta_\omega\}$. Then $C^q \cap A_{\delta_\omega} = C^p \cap A_{\delta_\omega}$. Hence $q \in P$ and q is (P, N_{0i_n})-generic for all $n < \omega$.

Starting with a family Y of many countable elementary substructures of various H_χ's, we may push further so that a subfamily of Y of some large size gets retained and have a common generic condition. We lose many elementary substructures out of Y but we still keep many at hand. This type of starting-many-retaining-sub-many argument takes place in the context of iterated forcing $I = \langle P_\alpha \mid \alpha \leq \omega_2 \rangle$. Therefore we need to prepare a theory to deal with many countable elementary substructures of many H_χ's in a tractable manner. The contents of subsequent sections are developed based on a small fragment of [S2] and are far from being complete at present.

§2. Many elementary substructures

Definition 2.1. Let I be an ω_2-stage iterated forcing. Let $\langle \chi_\alpha \mid \alpha < \omega_2 \rangle$ be a sequence of regular cardinals such that

(1) $I \in H_{\chi_0}$,

(2) For all $\alpha < \omega_2$, $\langle H_{\chi_\beta} \mid \beta < \alpha \rangle \in H_{\chi_\alpha}$.

Notice that if $0 < \alpha < \omega_2$, then $0 < \alpha < \omega_2 < \chi_0 < \chi_\alpha$ holds.

For $\alpha < \omega_2$, let

$$\mathcal{E}_\alpha = \{N \mid N \text{ is a countable elementary substructure of } H_{\chi_\alpha} \text{ with } I, \langle \mathcal{E}_\beta \mid \beta < \alpha \rangle \in N\} \subset H_{\chi_\alpha}.$$

$\mathcal{E}_\alpha$'s are pairwise disjoint and sort of downward closed.

Proposition 2.2. (1) If $N \in \mathcal{E}_\alpha$, then $\langle H_{\chi_\beta} \mid \beta < \alpha \rangle, \langle \chi_\beta \mid \beta < \alpha \rangle, \alpha \in N$.

(2) If $N \in \mathcal{E}_\alpha$ and $N \in \mathcal{E}_\beta$, then $\alpha = \beta$.

(3) If $N \in \mathcal{E}_\alpha$ and $\beta \in N \cap \alpha$, then $N \cap H_{\chi_\beta} \in \mathcal{E}_\beta$.

(4) If $N \in \mathcal{E}_\alpha$ and $N \cap \mathcal{E}_\beta \neq \emptyset$, then $\beta \in N$ and $\beta \leq \alpha$.

Proof. For (1): $\bigcup \mathcal{E}_\beta = H_{\chi_\beta}$ and $\chi = H_\chi \cap$ ON. Hence $\langle H_{\chi_\beta} \mid \beta < \alpha \rangle, \langle \chi_\beta \mid \beta < \alpha \rangle \in N$.

For (2): If not, then we may assume $\beta < \alpha$. Then $\langle H_{\chi_\gamma} \mid \gamma < \alpha \rangle \in N$ and $N \subset H_{\chi_\beta}$ (Actually, we have $N \in H_{\chi_\beta}$). Hence $\langle H_{\chi_\gamma} \mid \gamma < \alpha \rangle \in H_{\chi_\beta}$ and so $H_{\chi_\beta} \in H_{\chi_\beta}$. This is a contradiction.

For (3): Since N is an elementary substructure of H_{χ_α} and $H_{\chi_\beta} \in N$, we have $N \cap H_{\chi_\beta}$ is a countable elementary substructure of H_{χ_β}. We have $I \in N \cap H_{\chi_0}$. Hence $I \in N \cap H_{\chi_\beta}$. Since $\langle \mathcal{E}_\gamma \mid \gamma < \alpha \rangle \in N$ and $\beta \in N$. Hence $\langle \mathcal{E}_\gamma \mid \gamma < \beta \rangle \in N \cap H_{\chi_\beta}$.

For (4): Let $M \in N \cap \mathcal{E}_\beta$. Then $\beta \in M \subset N$ and so $\beta \in N$. Since $\langle H_{\chi_\gamma} \mid \gamma < \beta \rangle \in M \subset N$, we have $\langle H_{\chi_\gamma} \mid \gamma < \beta \rangle \in H_{\chi_\alpha}$ and so $\beta \leq \alpha$.

□

Via absoluteness considerations, we actually have the following.

Proposition 2.3. For a countable elementary substructure N of H_{χ_α} with $I \in N$, $\langle \chi_\beta \mid \beta < \alpha \rangle \in N$ if and only if $\langle H_{\chi_\beta} \mid \beta < \alpha \rangle \in N$ if and only if $\langle \mathcal{E}_\beta \mid \beta < \alpha \rangle \in N$.

□

Definition 2.4. For $N \in \mathcal{E}_\alpha$, we know $\alpha = \alpha(N)$ is uniquely determined by N. Let us denote

$$\mathrm{box}(N) = \mathrm{box}_\alpha(N) = \bigcup \{\mathcal{E}_\beta \cap N \mid \beta \in N \cap \alpha\}.$$

Let $D(N) = D_\alpha(N)$ be the set of all Y such that

- $Y \subseteq \mathrm{box}(N)$,
- For each $\beta \in N \cap \alpha$, $(Y \cap \mathcal{E}_\beta, \in)$ is a well-order. If $\langle N_{\beta i}(Y) \mid i < l_\beta(Y) \rangle$, simply denoted as, $\langle N_{\beta i} \mid i < l_\beta \rangle$ lists the elements of $Y \cap \mathcal{E}_\beta$ in the strict order $\in$, then it forms an $\in$-chain and converges to $N \cap H_{\chi_\beta}$. By this we mean
  - $\langle N_{\beta j} \mid j \leq i \rangle \in N_{\beta i+1}$ (strictly increasing),
  - If i is limit, then $N_{\beta i} = \bigcup \{N_{\beta j} \mid j < i\}$ (continuous),
  - $N \cap H_{\chi_\beta} = \bigcup \{N_{\beta i} \mid i < l_\beta\}$ ($N_{\beta i}$ converges to $N \cap H_{\chi_\beta}$).

  In particular, the order-type l_β is limit.
- For $\beta \in N \cap \alpha$, $M \in Y \cap \mathcal{E}_\beta$ and $\gamma \in M \cap \beta$, we demand $M \cap H_{\chi_\gamma} \in Y$ and $M \cap H_{\chi_\gamma} = N_{\gamma j}$ for some limit $j < l_\gamma$. In particular, we have $M \cap H_{\chi_\gamma} = \bigcup \{N_{\gamma i} \mid i < j\}$.

Proposition 2.5. (1) If $N, M \in \mathcal{E}_\alpha$ and $M \in N$, then $\mathrm{box}(M) \subset \mathrm{box}(N)$.

(2) If $N \in \mathcal{E}_0$, then $\mathrm{box}(N) = \emptyset$ and so $D_0(N) = \{\emptyset\}$.

(3) Let $N \in \mathcal{E}_\alpha$ and $Y \in D_\alpha(N)$. Then no $N \cap H_{\chi_\beta}$ belongs to Y for any $\beta \in N \cap \alpha$.

Proof. For (3): This is because if $N \cap H_{\chi_\beta} \in Y$, then $N \cap H_{\chi_\beta} \in N$ and so $N \cap H_{\chi_\beta} \in N \cap H_{\chi_\beta}$. This would be a contradiction.

□

Pictorially, we may draw Y as a sort of a triangle inside a rectangular shape box(N). We see Y consists of $\in$-chains $\langle N_{\beta i} \mid i < l_\beta \rangle$ converging to $N \cap H_{\chi_\beta}$ at all levels H_{χ_β} for all $\beta \in N \cap \alpha$.

Example 2.6. We pay attention to typical elements of $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$.

(1) Let N be a countable elementary substructure of H_{χ_0} with $I \in N$. Then this is equivalent to $N \in \mathcal{E}_0$.

(2) Let N be a countable elementary substructure of H_{χ_1} such that $I, \chi_0 \in N$. Then this is equivalent to $N \in \mathcal{E}_1$. Let $\langle N_n \mid n < \omega \rangle$ be an $\in$-chain in $\mathcal{E}_0$ such that $\bigcup\{N_n \mid n < \omega\} = N \cap H_{\chi_0}$. Then $\{N_n \mid n < \omega\} \in D_1(N)$.

(3) Let N be a countable elementary substructure of H_{χ_2} with $I, \chi_0, \chi_1 \in N$. Then this is equivalent to $N \in \mathcal{E}_2$. Let $\langle N_{1n} \mid n < \omega \rangle$ be an $\in$-chain in $\mathcal{E}_1$ such that $N \cap H_{\chi_1} = \bigcup\{N_{1n} \mid n < \omega\}$. Let $\langle N_{0i} \mid i < \omega^2 \rangle$ be an $\in$-chain in H_{χ_0} with $I \in N_{00}$. For all $n < \omega$, we demand $N_{1n} \cap H_{\chi_0} = N_{0\omega\cdot(n+1)} = \bigcup\{N_{0i} \mid i < \omega\cdot(n+1)\}$. Then $\{N_{0i} \mid i < \omega \cdot (n+1)\} \in D_1(N_{1n})$ for all $n < \omega$ and $\{N_{0i} \mid i < \omega^2\} \cup \{N_{1n} \mid n < \omega\} \in D_2(N)$.

Though we draw $Y \in D_\alpha(N)$ as a triangle in box(N), Y's structure would be much more complex. In particular, we see no easy comparisions of l_β's for $\beta \in N \cap \alpha$.

Proposition 2.7. Let $N \in \mathcal{E}_\alpha$ and $Y^N = \{N_{\beta i} \mid \beta \in N \cap \alpha, i < l_\beta\} \in D_\alpha(N)$. Let $\gamma < \beta < \alpha$ and $\gamma, \beta \in N$. Let i_0 be the least $i < l_\beta$ such that $\gamma \in N_{\beta i}$. Let $f : [i_0, l_\beta) \to l_\gamma$ be a function such that $f(i) = j$, where $i_0 \le i < l_\beta$ and $N_{\beta i} \cap H_{\chi_\gamma} = N_{\gamma j}$. We know $f(i)$ is a limit ordinal and so

$$f(i_0) + \omega \cdot (i - i_0) \le f(i).$$

Hence $\omega \cdot \big(1 + (i - i_0)\big) \le f(i)$ and so $\omega \cdot \big(1 + (l_\beta - i_0)\big) \le l_\gamma$.

□

For a simpler situation with $i_0 = 0$, we have

Proposition 2.8. Let $N \in \mathcal{E}_\alpha$ and $Y^N = \{N_{\beta i} \mid \beta \in N \cap \alpha, i < l_\beta\} \in D_\alpha(N)$. Let $\beta \in N \cap \alpha$ and $\gamma \in N_{\beta 0} \cap \beta$. Then

(1) For all $i < l_\beta$, if $N_{\beta i} \cap H_{\chi_\gamma} = N_{\gamma j}$, then we have $N_{\gamma j} \cap \omega_1 < N \cap \omega_1$ and so

$$\omega \cdot (1 + i) \le j < N \cap \omega_1.$$

(2) In particular, we have $l_\beta \le l_\gamma \le l_0 \le N \cap \omega_1$.

□

By example 2.6, we see that l_β may not be indecomposable.

§3. Internal strutures of $D_\alpha(N)$

We consider two natural projections.

Proposition 3.1. Let $N \in \mathcal{E}_\alpha$ and $Y = \{N_{\beta i} \mid \beta \in N \cap \alpha, i < l_\beta\} \in D_\alpha(N)$. Let $\beta \in N \cap \alpha$ and $M \in Y \cap \mathcal{E}_\beta$.

(1) $Y \cap$ box$(M) \in D_\beta(M)$. We may call $\big(M, Y \cap \text{box}(M)\big)$ the projection of (N, Y) down to M.

(2) For each $\gamma \in M \cap \beta$, $(Y\cap \text{ box}(M)) \cap \mathcal{E}_\gamma$ listed as $\langle N_{\gamma i} \mid i < f_\gamma(M)\rangle$, where $N_{\gamma f_\gamma(M)} = M \cap H_{\chi_\gamma}$ and $f_\gamma(M)$ is a limit ordinal.

(3) $Y\cap \text{ box}(N \cap H_{\chi_\beta}) \in D_\beta(N \cap H_{\chi_\beta})$. We may call $\big(N \cap H_{\chi_\beta}, Y\cap \text{ box}(N \cap H_{\chi_\beta}\big)$ the projection of (N, Y) down to β.

(4) For each $\gamma \in (N \cap H_{\chi_\beta}) \cap \beta = N \cap \beta$, $\big(Y\cap \text{ box}(N \cap H_{\chi_\beta})\big) \cap \mathcal{E}_\gamma = Y \cap \mathcal{E}_\gamma$ listed as $\langle N_{\gamma i} \mid i < l_\gamma\rangle$.

Proof. For (1) and (2): We have three conditions to check. First $Y \cap \text{box}(M) \subseteq \text{box}(M)$.

Next, for $\gamma \in M \cap \beta$,

$$(Y \cap \text{box}(M)) \cap \mathcal{E}_\gamma = Y \cap (\mathcal{E}_\gamma \cap M) = (Y \cap \mathcal{E}_\gamma) \cap (M \cap H_{\chi_\gamma}).$$

Hence $(Y \cap \text{box}(M)) \cap \mathcal{E}_\gamma = \{N_{\gamma i} \mid i < j\}$, where $N_{\gamma j} = M \cap H_{\chi_\gamma}$. And so it is well-ordered by $\in$. It forms an initial segment of the $\in$-chain $\langle N_{\gamma i} \mid i < l_\gamma\rangle$ which lists the elemennts of $Y \cap \mathcal{E}_\gamma$ in the strict order $\in$. We have $\bigcup\{N_{\gamma i} \mid i < j\} = N_{\gamma j} = M \cap H_{\chi_\gamma}$.

Lastly, for any $\gamma_1 \in M \cap \beta$, $M_1 \in (Y \cap \text{box}(M)) \cap \mathcal{E}_{\gamma_1}$ and $\gamma_2 \in M_1 \cap \gamma_1 \subset M \cap \beta$, we have

$$M_1 \cap H_{\chi_{\gamma_2}} \in Y \cap \mathcal{E}_{\gamma_2} \subseteq Y \cap \text{box}(M),$$

and if $M_1 \cap H_{\chi_{\gamma_2}} = N_{\gamma_2 j}$, then j is limit. We know $\langle N_{\gamma_2 i} \mid i \leq j\rangle$ is an initial segment of $Y\cap \text{ box}(M) \cap \mathcal{E}_{\gamma_2}$.

For (3) and (4): Similar.

$\square$

Motivation 3.2. Let $M \in \mathcal{E}_{\alpha^M}$ and $N \in \mathcal{E}_{\alpha^N}$. We are interested in their possible configurations.

(1) Let $M \in N$. Then we have $\alpha^M \leq \alpha^N$, $M \cap \alpha^M \subseteq N \cap \alpha^N$ and for all $\beta \in M \cap \alpha^M$, we have $H_{\chi_\beta} \in M$ and so $M \cap H_{\chi_\beta} \in N$ holds. In particular, we have $\text{box}(M) \subseteq \text{box}(N)$.

(2) In turn, let $\text{box}(M) \subseteq \text{box}(N)$ and for all $\beta \in M \cap \alpha^M$, $M \cap H_{\chi_\beta} \in N$. Suppose $N^* \in \mathcal{E}_{\alpha^*}$ and $M, N \in Y^* \in D_{\alpha^*}(N^*)$. Let us denote $Y^M = Y^* \cap \text{box}(M)$ and $Y^N = Y^* \cap \text{box}(N)$. Then we have seen that $Y^M \in D_{\alpha^M}(M)$ and $Y^N \in D_{\alpha^N}(N)$. We have $Y^M = Y^N \cap \text{box}(M)$. For any $\beta \in M \cap \alpha^M$, we have $M \cap H_{\chi_\beta} \in Y^*$ and so $M \cap H_{\chi_\beta} \in Y^*\cap \text{ box}(N) = Y^N$.

Definition 3.3. Let $N \in \mathcal{E}_{\alpha^N}$, $Y^N \in D_{\alpha^N}(N)$, $M \in \mathcal{E}_{\alpha^M}$ and $Y^M \in D_{\alpha^M}(M)$.

We denote $(M, Y^M) <_0 (N, Y^N)$, if

- $Y^M = Y^N \cap \text{box}(M)$,
- For all $\beta \in M \cap \alpha^M$, $M \cap H_{\chi_\beta} \in Y^N$.

We similarly denote $(M, Y^M) <_{\text{ho}} (N, Y^N)$, if

- $\alpha^M = \alpha^N$,
- $M \in N$,
- $(M, Y^M) <_0 (N, Y^N)$.

Lastly, $(M, Y^M) <_{\text{up}} (N, Y^N)$, if

- $M \in Y^N$,
- $(M, Y^M) <_0 (N, Y^N)$.

Notice that $(M, Y^M) <_{\text{ho}} (N, Y^N)$ does not imply $\sup(M \cap \alpha^M) = \sup(N \cap \alpha^N)$. It is possible that $\sup(M \cap \alpha^M) < \sup(N \cap \alpha^N)$. For example, consider $\alpha^N = \alpha^M = \omega_1$. On the other hand if $(M, Y^M) <_{\text{up}} (N, Y^N)$ and α^N is limit, then $\sup(M \cap \alpha^M) < \sup(N \cap \alpha^N)$ holds. Since $M, \alpha^M \in N$ and $\alpha^M < \alpha^N$, we have $\sup(M \cap \alpha^M) \in N$ and $\sup(M \cap \alpha^M) \leq \alpha^M < \alpha^M + 1 < \alpha^N$.

Proposition 3.4. Let $N \in \mathcal{E}_{\alpha^N}$, $Y^N \in D_{\alpha^N}(N)$, $M \in \mathcal{E}_{\alpha^M}$, $Y^M \in D_{\alpha^M}(M)$, $N^* \in \mathcal{E}_{\alpha^{N^*}}$ and $Y^{N^*} \in D_{\alpha^{N^*}}(N^*)$.

(1) $(M, Y^M) <_0 (M, Y^M)$ iff $Y^M = \emptyset$ iff $\alpha^M = 0$.

(2) If $(M, Y^M) <_0 (N, Y^N) <_0 (N^*, Y^{N^*})$, then $(M, Y^M) <_0 (N^*, Y^{N^*})$ (transitive).

(3) If $\beta \in N \cap \alpha^N$ and $M \in Y^N \cap \mathcal{E}_\beta$, then $(M, Y^N \cap \mathrm{box}(M)) <_{\mathrm{up}} (N, Y^N)$.

Let $(M, Y^M), (N, Y^N) <_0 (N^*, Y^{N^*})$. Then

(4) If $M \in N$, then $\alpha^M \leq \alpha^N$ and for all $\beta \in M \cap \alpha^M$, $M \cap H_{\chi_\beta} \in Y^N$.

(5) If for all $\beta \in M \cap \alpha^M$, $M \cap H_{\chi_\beta} \in Y^N$, then $(M, Y^M) <_0 (N, Y^N)$.

(6) If $\alpha^M = \alpha^N$ and $M \in N$, then $(M, Y^M) <_{\mathrm{ho}} (N, Y^N)$.

(7) If $M \in Y^N$, then $\alpha^M < \alpha^N$ and $(M, Y^M) <_{\mathrm{up}} (N, Y^N)$.

Proof. For (4): Let $\beta \in M \cap \alpha^M$. Then $\beta \in N \cap \alpha^N$. Hence $H_{\chi_\beta} \in N$ and so $M \cap H_{\chi_\beta} \in N \cap H_{\chi_\beta} \in Y^*$. We conclude $M \cap H_{\chi_\beta} \in Y^N$.

For (5): Want $Y^M = Y^N \cap \mathrm{box}(M)$. For $\beta \in M \cap \alpha^M$, we have $M \cap H_{\chi_\beta} \in \mathcal{E}_\beta \cap Y^N$ and so $\beta \in N \cap \alpha^N$. Hence $M \cap \alpha^M \subseteq N \cap \alpha^N$. For $\beta \in M \cap \alpha^M$, $Y^M \cap \mathcal{E}_\beta$ is the initial segment of $Y^* \cap \mathcal{E}_\beta$ below $M \cap H_{\chi_\beta} \in Y^* \cap \mathcal{E}_\beta$. This initial segment can be viewed as the initial segment of $Y^N \cap \mathcal{E}_\beta$ below $M \cap H_{\chi_\beta} \in Y^N \cap \mathcal{E}_\beta$. Hence $Y^M = \bigcup\{Y^M \cap \mathcal{E}_\beta \mid \beta \in \alpha^M \cap M\} = \bigcup\{Y^N \cap \mathcal{E}_\beta \cap (M \cap H_{\chi_\beta}) \mid \beta \in \alpha^M \cap M\} = Y^N \cap \bigcup\{\mathcal{E}_\beta \cap M \mid \beta \in \alpha^M \cap M\} = Y^N \cap \mathrm{box}(M)$.

□

By the following two propositions, we see that $D_\alpha(N)$ has a recursive construction.

Proposition 3.5. (Successor) Let $N \in \mathcal{E}_{\alpha+1}$. Then the following are equivalent.

(1) $Y \in D_{\alpha+1}(N)$.

(2) There exists an $\in$-chain $\langle N_i \mid i < l\rangle$ in $\mathcal{E}_\alpha$ such that l is limit and $\bigcup\{N_i \mid i < l\} = N \cap H_{\chi_\alpha}$ and for all $i < l$, there exist $Y_i \in D_\alpha(N_i)$ such that $\langle (N_i, Y_i) \mid i < l\rangle$ is a $<_{\mathrm{ho}}$-increasing sequence and $Y = \bigcup\{Y_i \cup \{N_i\} \mid i < l\}$.

If (2) holds, then for all $i < l$, we have $(N_i, Y_i) <_{\mathrm{up}} (N, Y)$.

Proof. (1) implies (2): $\alpha \in N$, as $\langle H_{\chi_\beta} \mid \beta < \alpha + 1\rangle \in N$. Let $\langle N_i \mid i < l\rangle$ list the elements of $(Y \cap \mathcal{E}_\alpha, \in)$ increasingly. Then $\langle N_i \mid i < l\rangle$ is an $\in$-chain with $\bigcup\{N_i \mid i < l\} = N \cap H_{\chi_\alpha}$. Let $Y_i = Y \cap \mathrm{box}(N_i)$. Then we have $(N_i, Y_i) <_{\mathrm{up}} (N, Y)$. Therefore we may conclude $\langle (N_i, Y_i) \mid i < l\rangle$ is $<_{\mathrm{ho}}$-increasing such that $Y = \bigcup\{Y_i \cup \{N_i\} \mid i < l\}$.

(2) implies (1): We have three conditions to check. First $Y_i \subseteq \mathrm{box}(N_i) \subset \mathrm{box}(N)$ and $N_i \in \mathrm{box}(N)$. Hence $Y \subseteq \mathrm{box}(N)$. Next let $\beta \in N \cap (\alpha+1)$. We want $Y \cap \mathcal{E}_\beta$ is an $\in$-chain. If $\beta < \alpha$, then $\beta \in N_{i^*}$ for some $i^* < l$. Then $Y \cap \mathcal{E}_\beta = \bigcup\{Y_i \cap \mathcal{E}_\beta \mid i \geq i^*\}$. Since (N_i, Y_i) forms a $<_{\mathrm{ho}}$-increasing sequence, $\bigcup\{Y_i \cap \mathcal{E}_\beta \mid i \geq i^*\}$ is an $\in$-chain converging to $N \cap H_{\chi_\beta}$. If $\beta = \alpha$, then $Y \cap \mathcal{E}_\beta = \{N_i \mid i < l\}$ is an $\in$-chain converging to $N \cap H_{\chi_\beta}$. Lastly, let $\gamma_1 \in N \cap (\alpha+1)$, $M \in Y \cap \mathcal{E}_{\gamma_1}$ and $\gamma_2 \in M \cap \gamma_1$. We want $M \cap H_{\chi_{\gamma_2}} \in Y \cap \mathcal{E}_{\gamma_2}$ and if $M \cap H_{\chi_{\gamma_2}} = N_{\gamma_2 j}$, then j is limit. But since $\langle (N_i, Y_i) \mid i < l\rangle$ is $<_{\mathrm{ho}}$-increasing, we are done.

□

Proposition 3.6. (Limit) Let $N \in \mathcal{E}_\alpha$ and α be limit. Then the following are equivalent.

(1) $Y \in D_\alpha(N)$.

(2) There exists $\langle (\alpha_n, N_n, Y_n) \mid n < \omega\rangle$ such that $\langle \alpha_n \mid n < \omega\rangle$ is a strictly increasing cofinal sequence in $N \cap \alpha$, $N_n \in \mathcal{E}_{\alpha_n} \cap N$, $Y_n \in D_{\alpha_n}(N_n)$ and $(N_n, Y_n) <_{\mathrm{up}} (N_{n+1}, Y_{n+1})$, N_n converges to $\bigcup\{N \cap H_{\chi_\beta} \mid \beta \in N \cap \alpha\}$ and $Y = \bigcup\{Y_n \mid n < \omega\} = \bigcup\{Y_n \cup \{N_n\} \mid n < \omega\}$.

If (2) holds, then for all $n < \omega$, $(N_n, Y_n) <_{\mathrm{up}} (N, Y)$.

Proof. (1) implies (2): Take $\langle \alpha_n \mid n < \omega\rangle$ such that $\alpha_n \in N \cap \alpha$ are strictly increasing and cofinal in $N \cap \alpha$. Let us then take an $\in$-chain $\langle N_n \mid n < \omega\rangle$ such that $N_n \in \mathcal{E}_{\alpha_n} \cap Y$ and $\bigcup\{N_n \mid n < \omega\} = \bigcup\{N \cap H_{\chi_\beta} \mid \beta \in N \cap \alpha\}$.

Now set

$$Y_n = Y \cap \mathrm{box}(N_n).$$

Since $(N_n, Y_n), (N_{n+1}, Y_{n+1}) <_{\mathrm{up}} (N, Y)$ and $N_n \in Y_{n+1}$, we have

$$(N_n, Y_n) <_{\mathrm{up}} (N_{n+1}, Y_{n+1}).$$

Since $\bigcup\{N \cap \mathcal{E}_\beta \mid \beta \in N \cap \alpha\} = \bigcup\{N_n \cap \mathcal{E}_\beta \mid n < \omega, \beta \in N_n \cap \alpha_n\}$, we have

$$\mathrm{box}(N) = \bigcup\{\mathrm{box}(N_n) \mid n < \omega\}$$

and so

$$Y = \bigcup\{Y_n \mid n < \omega\}.$$

(2) implies (1): We have three conditions to check. First we want $Y \subseteq \mathrm{box}(N)$. This holds, because $Y = \bigcup\{Y_n \mid n < \omega\}$ and $Y_n \subseteq \mathrm{box}(N_n) \subseteq \mathrm{box}(N)$.

Next let $\beta \in N \cap \alpha$. Then $Y \cap \mathcal{E}_\beta$ is well-ordered by $\in$ and if $\langle N_{\beta i}(Y) \mid i < l\rangle$ lists the elements of $Y \cap \mathcal{E}_\beta$ in the strict order $\in$, then it is an $\in$-chain in H_{χ_β} converging to $N \cap H_{\chi_\beta}$. This holds, because α_n are strictly increasing cofinally in $N \cap \alpha$, $(N_n, Y_n) <_{\mathrm{up}} (N_{n+1}, Y_{n+1})$ and N_n converges to $\bigcup\{N \cap H_{\chi_\beta} \mid \beta \in N \cap \alpha\}$.

Lastly, let $\beta \in N \cap \alpha$ and $M \in Y \cap \mathcal{E}_\beta$. Then $\beta \in N_n \cap \alpha_n$ and $M \in Y_n \cap \mathcal{E}_\beta$ for some n. Let $\gamma \in M \cap \beta$. Then $M \cap H_{\chi_\gamma} \in Y_n$. Hence $M \cap H_{\chi_\gamma} \in Y$. If $M \cap H_{\chi_\gamma} = N_{\gamma j}(Y_n)$, then $M \cap H_{\chi_\gamma} = N_{\gamma j}(Y_n) = N_{\gamma j}(Y)$ and j is limit.

□

We extract constructions above.

Lemma 3.7. ($<_0$-Limit) We have two typical constructions.

(1) (successor step) Let $N \in \mathcal{E}_{\alpha+1}$. Let $\langle N_i \mid i < l\rangle$ be an $\in$-chain in $\mathcal{E}_\alpha$ such that l is limit and N_i converge to $N \cap H_{\chi_\alpha}$. Let $\langle Y_i \mid i < l\rangle$ satisfy for all $i < l$, $Y_i \in D_\alpha(N_i)$ and for all $i < j$, $(N_i, Y_i) <_{\mathrm{ho}} (N_j, Y_j)$. Let $Y = \bigcup\{Y_i \cup \{N_i\} \mid i < l\}$. Then $Y \in D_{\alpha+1}(N)$ such that for all $i < l$, $(N_i, Y_i) <_{\mathrm{up}} (N, Y)$.

(2) (limit step) Let α be limit. Let $N \in \mathcal{E}_\alpha$. Let $\langle \alpha_n \mid n < \omega\rangle$ be strictly increasing and cofinal in $N \cap \alpha$. Let $\langle N_n \mid n < \omega\rangle$ be an $\in$-chain such that $N_n \in N \cap \mathcal{E}_{\alpha_n}$ and N_n converge to $\bigcup\{N \cap H_{\chi_\beta} \mid \beta \in N \cap \alpha\}$. Let $\langle Y_n \mid n < \omega\rangle$ satisfy for all $n < \omega$, $Y_n \in D_{\alpha_n}(N_n)$ and $(N_n, Y_n) <_{\mathrm{up}} (N_{n+1}, Y_{n+1})$. Let $Y = \bigcup\{Y_n \mid n < \omega\}$. Then $Y \in D_\alpha(N)$ and for all $n < \omega$, $(N_n, Y_n) <_{\mathrm{up}} (N, Y)$.

Proof. Routine. For (1): Let $\beta \in N \cap \alpha$. Then

$$\langle N_{\beta j}(Y) \mid j < l_\beta(Y)\rangle = \bigcup\{\langle N_{\beta j}(Y_i) \mid j < l_\beta(Y_i)\rangle \mid i < l, \beta \in N_i \cap \alpha\}.$$

For (2): Let $\beta \in N \cap \alpha$. Then

$$\langle N_{\beta j}(Y) \mid j < l_\beta(Y)\rangle = \bigcup\{\langle N_{\beta j}(Y_n) \mid j < l_\beta(Y_n)\rangle \mid n < \omega, \beta \in N_n \cap \alpha\}.$$

□

§4. **Extension Properties**

We define 4 extension types.

Definition 4.1. Let $N_i \in \mathcal{E}_{\alpha_i}$ for $i = 1,2,3$ and let $Y_i \in D_{\alpha_i}(N_i)$ for $i = 1,2$ such that $N_1, N_2 \in N_3$, $\alpha_2 \in N_1 \cap \alpha_1$ and

$$(N_1 \cap H_{\chi_{\alpha_2}}, Y_1 \cap \mathrm{box}(N_1 \cap H_{\chi_{\alpha_2}})) <_{\mathrm{ho}} (N_2, Y_2).$$

We want to find $Y_3 \in D_{\alpha_3}(N_3)$ such that

$$(N_1, Y_1), (N_2, Y_2) <_0 (N_3, Y_3).$$

Depending on the configurations, we have 2 types.

(**Type 1**) $\alpha_2 < \alpha_1 = \alpha_3$: $(N_1, Y_1) <_{\mathrm{ho}} (N_3, Y_3)$ and $(N_2, Y_2) <_{\mathrm{up}} (N_3, Y_3)$.

(**Type 2**) $\alpha_2 < \alpha_1 < \alpha_3$: $(N_1, Y_1) <_{\mathrm{up}} (N_3, Y_3)$ and $(N_2, Y_2) <_{\mathrm{up}} (N_3, Y_3)$.

Let $N_i \in \mathcal{E}_{\alpha_i}$ for $i = 1,2$ such that $N_1 \in N_2$. Let $Y_1 \in D_{\alpha_1}(N_1)$. We want to find $Y_2 \in D_{\alpha_2}(N_2)$ such that

$$(N_1, Y_1) <_0 (N_2, Y_2).$$

Depending on the configurations, we have two types.

(**Type 3**) $\alpha_1 = \alpha_2$: $(N_1, Y_1) <_{\mathrm{ho}} (N_2, Y_2)$.

(**Type 4**) $\alpha_1 < \alpha_2$: $(N_1, Y_1) <_{\mathrm{up}} (N_2, Y_2)$.

Proposition 4.2. All of the 4 extension types hold.

Proof. We prove all of the 4 extension types simultaneously. Let $\alpha < \omega_2$ be the greatest ordinal in each type. We prove by induction on α.

Successor $\underline{\alpha \to \alpha + 1}$:

(**Type 1**) We have two cases.

Case 1. $\alpha_2 < \alpha$: Look at $N_1 \cap H_{\chi_\alpha}, N_2$ and $N_3 \cap H_{\chi_\alpha}$. Let $\langle M_n \mid n < \omega \rangle$ be an $\in$-chain in $\mathcal{E}_\alpha$ such that $N_1 \cap H_{\chi_\alpha}, N_2 \in M_0$ and M_n converges to $N_3 \cap H_{\chi_\alpha}$. Then apply (Type 1) at α to $\big(N_1 \cap H_{\chi_\alpha}, Y_1 \cap \mathrm{box}(N_1 \cap H_{\chi_\alpha})\big), (N_2, Y_2)$ and M_0. We have $Y^{M_0} \in D_\alpha(M_0)$ such that $\big(N_1 \cap H_{\chi_\alpha}, Y_1 \cap \mathrm{box}(N_1 \cap H_{\chi_\alpha})\big) <_{\mathrm{ho}} (M_0, Y^{M_0})$ and $(N_2, Y_2) <_{\mathrm{up}} (M_0, Y^{M_0})$. We then apply (Type 3) repeatedly to get $Y^{M_{n+1}} \in D_\alpha(M_{n+1})$ such that $(M_n, Y^{M_n}) <_{\mathrm{ho}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_3 = (Y_1 \cap \mathcal{E}_\alpha) \cup \{N_1 \cap H_{\chi_\alpha}\} \cup \bigcup \{Y^{M_n} \cup \{M_n\} \mid n < \omega\}$. Then this Y_3 works.

Case 2. $\alpha_2 = \alpha$: Take an $\in$-chain $\langle M_n \mid n < \omega \rangle$ in $\mathcal{E}_\alpha$ such that $N_2 \in M_0$ and M_n converges to $N_3 \cap H_{\chi_\alpha}$. The rest is the same as case 1 except no use of (Type 1) at α made. We just repeatedly apply (Type 3) at α. Let $Y_3 = (Y_1 \cap \mathcal{E}_\alpha) \cup \{N_1 \cap H_{\chi_\alpha}, N_2\} \cup \bigcup \{Y^{M_n} \cup \{M_n\} \mid n < \omega\}$. Then this Y_3 works.

(**Type 2**) We have two cases

Case 1. $\alpha_1 < \alpha$: Let us take an $\in$-chain $\langle M_n \mid n < \omega \rangle$ in $\mathcal{E}_\alpha$ such that $N_1, N_2 \in M_0$ and M_n converges to $N_3 \cap H_{\chi_\alpha}$. Then apply (Type 2) at α to get $Y^{M_0} \in D_\alpha(M_0)$ such that $(N_1, Y_1), (N_2, Y_2) <_{\mathrm{up}} (M_0, Y^{M_0})$. Then repeatedly apply (Type 3) at α to get $Y^{M_{n+1}} \in D_\alpha(M_{n+1})$ such that $(M_n, Y^{M_n}) <_{\mathrm{ho}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_3 = \bigcup \{Y^{M_n} \cup \{M_n\} \mid n < \omega\}$. Then this Y_3 works.

Case 2. $\alpha_1 = \alpha$: Take an $\in$-chain $\langle M_n \mid n < \omega \rangle$ in $\mathcal{E}_\alpha$ such that $N_1, N_2 \in M_0$ and M_n converges to $N_3 \cap H_{\chi_\alpha}$. The rest is the same as case 1 except one use (Type 1) at α. We then repeatedly apply (Type 3) at α. Let $Y_3 = \{N_1\} \cup \bigcup \{Y^{M_n} \cup \{M_n\} \mid n < \omega\}$. Then this Y_3 works.

(Type 3) Take an $\in$-chain $\langle M_n \mid n < \omega \rangle$ in $\mathcal{E}_\alpha$ such that $N_1 \cap H_{\chi_\alpha} \in M_0$ and M_n converges to $N_2 \cap H_{\chi_\alpha}$. Then we repeatedly apply (Type 3) at α to get $Y^{M_n} \in D_\alpha(M_n)$ such that $\big(N_1 \cap H_{\chi_\alpha}, Y_1 \cap \mathrm{box}(N_1 \cap H_{\chi_\alpha})\big) <_{\mathrm{ho}} (M_0, Y^{M_0})$ and $(M_n, Y^{M_n}) <_{\mathrm{ho}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_2 = (Y_1 \cap \mathcal{E}_\alpha) \cup \{N_1 \cap H_{\chi_\alpha}\} \cup \bigcup \{Y^{M_n} \cup \{M_n\} \mid n < \omega\}$. Then this Y_2 works.

(Type 4) We have two cases.

Case 1. $\alpha_1 < \alpha$: Take an $\in$-chain $\langle M_n \mid n < \omega\rangle$ in $\mathcal{E}_\alpha$ such that $N_1 \in M_0$ and M_n converges $N_2 \cap H_{\chi_\alpha}$. Take $Y^{M_0} \in D_\alpha(M_0)$ such that $(N_1, Y_1) <_{\text{up}} (M_0, Y^{M_0})$ by (Type 4) at α. Then repeatedly apply (Type 3) at α to get $Y^{M_{n+1}} \in D_\alpha(M_{n+1})$ such that $(M_n, Y^{M_n}) <_{\text{ho}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_2 = \bigcup\{Y^{M_n} \cup \{M_n\} \mid n < \omega\}$. Then this Y_2 works.

Case 2. $\alpha_1 = \alpha$: Take an $\in$-chain $\langle M_n \mid n < \omega\rangle$ in $\mathcal{E}_\alpha$ such that $N_1 \in M_0$ and M_n converges to $N_2 \cap H_{\chi_\alpha}$. The rest is the same as case 1 except no use of (Type 4) at α made. Let $Y_2 = \{N_1\} \cup \bigcup\{Y^{M_n} \cup \{M_n\} \mid n < \omega\}$. Then this Y_2 works.

Limit <u>α is limit</u>: We first prepare a lemma.

Lemma. (White-hole Lemma) Let $\alpha < \omega_2$ be limit. Let $N \in \mathcal{E}_\alpha$ and $Y^N \in D_\alpha(N)$. Let $\alpha^* = \sup(N \cap \alpha) \leq \alpha$ and let $M \in \mathcal{E}_{\alpha^*}$ with $N \cap H_{\chi_{\alpha^*}} \in M$.

(1) If $\alpha^* < \alpha$, then $N \cap H_{\chi_{\alpha^*}} \notin \mathcal{E}_{\alpha^*}$ (White-hole).

(2) There exists $Y^M \in D_{\alpha^*}(M)$ such that $(N, Y^N) <_0 (M, Y^M)$.

Proof. For (1): By contradiction. Suppose $N \cap H_{\chi_{\alpha^*}} \in \mathcal{E}_{\alpha^*}$. Then $\langle \mathcal{E}_\beta \mid \beta < \alpha^*\rangle \in N \cap H_{\chi_{\alpha^*}}$ and so $\alpha^* \in N$. Since we assume that $\alpha^* < \alpha$ and that α is limit, we have $\alpha^* + 1 \in N \cap \alpha$. This contradicts $\sup(N \cap \alpha) = \alpha^*$.

For (2): Fix a sequence of strictly increasing ordinals $\langle \beta_n \mid n < \omega\rangle$ cofinally in $\alpha \cap N$. Hence we have $\sup\{\beta_n \mid n < \omega\} = \alpha^*$. Let $\langle M_n \mid n < \omega\rangle$ be an $\in$-chain such that $N \cap H_{\chi_{\beta_n}} \in M_n$ and $M_n \in \mathcal{E}_{\beta_n} \cap M$ converge to $\bigcup\{H_{\chi_\beta} \cap M \mid \beta \in M \cap \alpha^*\}$. Form $Y^N \cap \text{box}(N \cap H_{\chi_{\beta_n}}) \in D_{\beta_n}(N \cap H_{\chi_{\beta_n}})$ for each $n < \omega$. First get $Y^{M_0} \in D_{\beta_0}(M_0)$ such that $\big(N \cap H_{\chi_{\beta_0}}, Y^N \cap \text{box}(N \cap H_{\chi_{\beta_0}})\big) <_0 (M_0, Y^{M_0})$. This is possible by (Type 3) at β_0. We then construct $Y^{M_{n+1}} \in D_{\beta_{n+1}}(M_{n+1})$ by repeatedly applying (Type 1) at β_{n+1} so that $\big(N \cap H_{\chi_{\beta_{n+1}}}, Y^N \cap \text{box}(N \cap H_{\chi_{\beta_{n+1}}})\big) <_{\text{ho}} (M_{n+1}, Y^{M_{n+1}})$ and $(M_n, Y^{M_n}) <_{\text{up}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y^M = \bigcup\{Y^{M_n} \mid n < \omega\}$. Then this Y^M works.

□

(Type 1) We have two cases.

Case 1. $\sup(N_1 \cap \alpha) = \sup(N_3 \cap \alpha)$: In this case we have $\sup(N_1 \cap \alpha) = \sup(N_3 \cap \alpha) = \alpha$. This is because $\alpha, N_1 \in N_3$ and so $\sup(N_1 \cap \alpha) \in N_3$. If $\sup(N_1 \cap \alpha) < \alpha$, then we would have $\sup(N_1 \cap \alpha) + 1 \in N_3 \cap \alpha$. This contradicts $\sup(N_1 \cap \alpha) = \sup(N_3 \cap \alpha)$.

Take a strictly increasing sequence $\langle \beta_n \mid n < \omega\rangle$ of ordinals which are cofinal in $N_1 \cap \alpha$ with $\alpha_2 < \beta_0$. Then take an $\in$-chain $\langle M_n \mid n < \omega\rangle$ such that $N \cap H_{\chi_{\beta_0}}, N_2 \in M_0 \in N_3 \cap \mathcal{E}_{\beta_0}$ and $N \cap H_{\chi_{\beta_n}} \in M_n \in N_3 \cap \mathcal{E}_{\beta_n}$ and that M_n converge to $\bigcup\{N_3 \cap H_{\chi_\gamma} \mid \gamma \in N_3 \cap \alpha\}$. We repeatedly apply (Type 1) to get $Y^{M_n} \in D_{\gamma_n}(M_n)$ such that

$$(N_1 \cap H_{\chi_{\beta_0}}, Y_1 \cap \text{box}(N_1 \cap H_{\chi_{\beta_0}})) <_{\text{ho}} (M_0, Y^{M_0}),\ (N_2, Y_2) <_{\text{up}} (M_0, Y^{M_0}),$$

$$(N_1 \cap H_{\chi_{\beta_{n+1}}}, Y_1 \cap \text{box}(N_1 \cap H_{\chi_{\beta_{n+1}}})) <_{\text{ho}} (M_{n+1}, Y^{M_{n+1}}),\ (M_n, Y^{M_n}) <_{\text{up}} (M_{n+1}, Y^{M_{n+1}}).$$

Let $Y_3 = \bigcup\{Y^{M_n} \mid n < \omega\}$. Then this Y_3 works.

Case 2. $\sup(N_1 \cap \alpha) < \sup(N_3 \cap \alpha)$:

Let $\alpha^* = \sup(N_1 \cap \alpha)$. Then $\alpha^* \in N_3 \cap \alpha$. We know $N_1 \cap H_{\chi_{\alpha^*}} \notin \mathcal{E}_{\alpha^*}$ (white-hole). Then take a strictly increasing sequence $\langle \gamma_n \mid n < \omega\rangle$ of ordinals which are cofinal in $N_3 \cap \alpha$ and $\alpha^* = \gamma_0$. Then take an $\in$-chain $\langle M_n \mid n < \omega\rangle$ such that $M_0 \in N_3 \cap \mathcal{E}_{\alpha^*}$ with $N_1 \cap H_{\chi_{\alpha^*}}, N_2 \in M_0$ and $M_{n+1} \in N_3 \cap \mathcal{E}_{\gamma_{n+1}}$ and M_n converges to $\bigcup\{N_3 \cap H_{\chi_\gamma} \mid \gamma \in N_3 \cap \alpha\}$. As in case 1 and white-hole lemma, get $Y^{M_0} \in D_{\alpha^*}(M_0)$ such that $(N_2, Y_2) <_{\text{up}} (M_0, Y^{M_0})$ and $(N_1, Y_1) <_0 (M_0, Y^{M_0})$. Namely, $(N_1 \cap H_{\chi_\beta}, Y_1 \cap \text{box}(N_1 \cap H_{\chi_\beta})) <_{\text{up}} (M_0, Y^{M_0})$ for all $\beta \in N_1 \cap \alpha$.

We repeatedly apply (Type 4) to get $Y^{M_{n+1}} \in D_{\gamma_{n+1}}(M_{n+1})$ such that $(M_n, Y^{M_n}) <_{\text{up}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_3 = \bigcup\{Y^{M_n} \mid n < \omega\}$. Then this Y_3 works.

(Type 2) Take a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of ordinals cofinally in $N_3 \cap \alpha$ with $\alpha_1 < \gamma_0$. Take $M_0 \in N_3 \cap \mathcal{E}_{\gamma_0}$ such that $(N_1, Y_1) <_{\mathrm{up}} (M_0, Y^{M_0})$ and $(N_2, Y_2) <_{\mathrm{up}} (M_0, Y^{M_0})$ by (Type 2) at γ_0. Then take an $\in$-chain $\langle M_{n+1} \mid n < \omega \rangle$ such that $M_{n+1} \in N_3 \cap \mathcal{E}_{\gamma_{n+1}}$ and M_{n+1} converges to $\bigcup\{N_3 \cap H_{\chi_\gamma} \mid \gamma \in N_3 \cap \alpha\}$. Then repeatedly appy (Type 4) at γ_{n+1} to get $Y^{M_{n+1}} \in D_{\gamma_{n+1}}(M_{n+1})$ such that $(M_n, Y^{M_n}) <_{\mathrm{up}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_3 = \bigcup\{Y^{M_n} \mid n < \omega\}$. Then this Y_3 works.

(Type 3) We have two cases. Similar to (Type 1).

Case 1. $\sup(N_1 \cap \alpha) = \sup(N_2 \cap \alpha)$: In this case we have $\sup(N_1 \cap \alpha) = \sup(N_2 \cap \alpha) = \alpha$.

Take a strictly increasing sequence $\langle \beta_n \mid n < \omega \rangle$ of ordinals cofinally in $N_1 \cap \alpha$. Take an $\in$-chain $\langle M_n \mid n < \omega \rangle$ such that $N_1 \cap H_{\chi_{\beta_n}} \in M_n$ and $M_n \in N_2 \cap \mathcal{E}_{\beta_n}$ and that M_n converge to $\bigcup\{N_2 \cap H_{\chi_\gamma} \mid \gamma \in N_2 \cap \alpha\}$. First appy (Type 3) at β_0 to get $Y^{M_0} \in D_{\beta_0}(M_0)$ such that $\big(N_1 \cap H_{\chi_{\beta_0}}, Y_1 \cap \mathrm{box}(N_1 \cap H_{\chi_{\beta_0}})\big) <_{\mathrm{ho}} (M_0, Y^{M_0})$. Then repeatedly apply (Type 1) to get $Y^{M_{n+1}} \in D_{\beta_{n+1}}(M_{n+1})$ such that $\big(N_1 \cap H_{\chi_{\beta_{n+1}}}, Y_1 \cap \mathrm{box}(N_1 \cap H_{\chi_{\beta_{n+1}}})\big) <_{\mathrm{ho}} (M_{n+1}, Y^{M_{n+1}})$ and $(M_n, Y^{M_n}) <_{\mathrm{up}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_2 = \bigcup\{Y^{M_n} \mid n < \omega\}$. Then this Y_2 works.

Case 2. $\sup(N_1 \cap \alpha) <\sup(N_2 \cap \alpha)$:

Let $\alpha^* = \sup(N_1 \cap \alpha)$. Then $\alpha^* \in N_2 \cap \alpha$. We know $N_1 \cap H_{\chi_{\alpha^*}} \notin \mathcal{E}_{\alpha^*}$ (white-hole). Then take a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of ordinals cofinally in $N_2 \cap \alpha$ with $\alpha^* = \gamma_0$. Take an $\in$-chain $\langle M_n \mid n < \omega \rangle$ such that $M_0 \in N_2 \cap \mathcal{E}_{\gamma_0}$ such that $N_1 \cap H_{\chi_{\alpha^*}} \in M_0$ and $M_{n+1} \in N_2 \cap \mathcal{E}_{\gamma_{n+1}}$ and M_n converges to $\bigcup\{N_2 \cap H_{\chi_\gamma} \mid \gamma \in N_2 \cap \alpha\}$. Then get $Y^{M_0} \in D_{\alpha^*}(M_0)$ such that $(N_1, Y_1) <_0 (M_0, Y^{M_0})$ by white-hole lemma. Namely, $\big(N_1 \cap H_{\chi_\beta}, Y_1 \cap \mathrm{box}(N_1 \cap H_{\chi_\beta})\big) <_{\mathrm{up}} (M_0, Y^{M_0})$ for all $\beta \in N_1 \cap \alpha$. Then repeatedly apply (Type 4) at γ_{n+1} to get $Y^{M_{n+1}} \in D_{\gamma_{n+1}}(M_{n+1})$ such that $(M_n, Y^{M_n}) <_{\mathrm{up}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_2 = \bigcup\{Y^{M_n} \mid n < \omega\}$. Then this Y_2 works.

(Type 4) Take a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of ordinals cofinally in $N_2 \cap \alpha$ with $\alpha_1 < \gamma_0$. Take an $\in$-chain $\langle M_n \mid n < \omega \rangle$ such that $N_1 \in M_0$, $M_n \in N_2 \cap \mathcal{E}_{\gamma_n}$ and M_n converges to $\bigcup\{N_2 \cap H_{\chi_\gamma} \mid \gamma \in N_2 \cap \alpha\}$. Then repeatedly apply (Type 4) to get $Y^{M_n} \in D_{\gamma_n}(M_n)$ such that $(N_1, Y_1) <_{\mathrm{up}}< (M_n, Y^{M_n}) <_{\mathrm{up}} (M_{n+1}, Y^{M_{n+1}})$. Let $Y_2 = \bigcup\{Y^{M_n} \mid n < \omega\}$. Then this Y_2 works.

$\square$

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