On sequent systems of the provability logic $\mathbf{R}^-$ for Rosser sentences [1]

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Abstract. We discuss sequent systems of the provability logic $\mathbf{R}^-$ introduced in Guaspari and Solovay [2]. Sasaki and Ohama [5] gave a sequent system with a kind of subformula property, but considering a cut-free system for the unimodal provability logic **GL**, there exists a cut that seems to be removable. Here we introduce another sequent system with a strengthened kind of subformula property. As a result, it is shown that the $\prec$-free and $\preceq$-free fragment of our system is the system for **GL**, while it has been unclear for the system in [5]. Also by our system, we can discuss what kinds of cuts are removable from the system in [5]. Moreover, in the proof of a completeness theorem for our system, we give a concrete counter model, while [5] didn't.

1 The logic $\mathbf{R}^-$

To discuss Rosser sentences, Guaspari and Solovay [2] enriched the modal language by adding, for each $\Box A$ and $\Box B$, the formulas $\Box A \prec \Box B$ and $\Box A \preceq \Box B$, with arithmetic realizations. They introduced provability logics $\mathbf{R}^-$, $\mathbf{R}$ and $\mathbf{R}^\omega$ with enriched language by extending the unimodal provability logic **GL** and proved kinds of arithmetic completeness for them. The logic $\mathbf{R}^-$ is the most preliminary one among these three logics.

In this section, we introduce the logic $\mathbf{R}^-$ and its Kripke semantics. We use $p, q, \cdots$ for propositional variables. We use logical constant $\bot$ (contradiction), and logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\Box$ (provability), $\preceq$ (witness comparison), and $\prec$ (witness comparison). Formulas are defined inductively as follows:

(1) propositional variables and $\bot$ are formulas,

(2) if A and B are formulas, then so are $(A \wedge B), (A \vee B), (A \supset B), (\Box A), (\Box A \prec \Box B)$ and $(\Box A \preceq \Box B)$.

We use upper case Latin letters $A, B, C, \cdots$, possibly with suffixes, for formulas. A formula of the form $\Box A$ is said to be a $\Box$-formula. Also a formula of the form $\Box A \preceq \Box B$ ($\Box A \prec \Box B$) is said to be a $\preceq$-formula ($\prec$-formula). By Σ, we mean the set of all $\Box$-formulas, all $\prec$-formulas and all $\preceq$-formulas.

The modal system $\mathbf{R}^-$ is defined by the following axioms and inference rules.

Axioms of $\mathbf{R}^-$

$A1$: all tautologies,
$A2 : \Box(A \supset B) \supset (\Box A \supset \Box B)$,
$A3 : \Box(\Box A \supset A) \supset \Box A$,
$A4 : A \supset \Box A$, where $A \in \Sigma$,
$A5 : (\Box A \preceq \Box B) \supset \Box A$,
$A6 : ((\Box A \preceq \Box B) \wedge (\Box B \preceq \Box C)) \supset (\Box A \preceq \Box C)$,
$A7 : (\Box A \vee \Box B) \supset ((\Box A \preceq \Box B) \vee (\Box B \prec \Box A))$,
$A8 : (\Box A \prec \Box B) \supset (\Box A \preceq \Box B)$,
$A9 : ((\Box A \preceq \Box B) \wedge (\Box B \prec \Box A)) \supset \bot$,

Inference rules of $\mathbf{R}^-$

$MP : A, A \supset B \in \mathbf{R}^-$ implies $B \in \mathbf{R}^-$,
$N : A \in \mathbf{R}^-$ implies $\Box A \in \mathbf{R}^-$.

In [2] and Smoriński [6], the following two formulas are also axioms of $\mathbf{R}^-$, but they are redundant (cf. de Jongh [3] and Voorbraak [8]).

$A10 : \Box A \supset (\Box A \preceq \Box A)$,

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$A11 : (\Box A \wedge (\Box B \supset \bot)) \supset (\Box A \prec \Box B)$.

We introduce Kripke semantics for $\mathbf{R}^-$, following [6]. For a non-empty set $\mathbf{W}$, a binary relation $<$ on $\mathbf{W}$ and an element $\alpha \in \mathbf{W}$, we put

$$\alpha{\uparrow} = \{\beta \mid \alpha < \beta\}, \qquad \alpha{\downarrow} = \{\beta \mid \beta < \alpha\} \qquad \text{and} \qquad \alpha{\downarrow\!\!\!\!-} = \alpha{\downarrow} \cup \{\alpha\}.$$

Definition 1.1 A Kripke pseudo-model for $\mathbf{R}^-$ is a triple $\langle \mathbf{W}, <, \models \rangle$ where

(1) $\mathbf{W}$ is a non-empty finite set,

(2) $<$ is an irreflexive and transitive binary relation on $\mathbf{W}$ satisfying

$$\alpha < \gamma \text{ and } \beta < \gamma \text{ imply either one of } \alpha = \beta,\ \alpha < \beta \text{ and } \beta < \alpha,$$

(3) $\models$ is a valuation satisfying, in addition to the usual boolean laws,

$$\alpha \models \Box A \text{ if and only if for any } \beta \in \alpha{\uparrow},\ \beta \models A.$$

Definition 1.2 A Kripke pseudo-model $\langle \mathbf{W}, <, \models \rangle$ for $\mathbf{R}^-$ is said to be a Kripke model for $\mathbf{R}^-$ if the following conditions hold, for any formula D,

(1) if $D \in \Sigma$ and $\alpha \models D$, then for any $\beta \in \alpha{\uparrow}, \beta \models D$,

(2) if D is either one of the axioms $A5, A6, A7, A8$ and $A9$, then $\alpha \models D$.

A formula A is said to be valid in a Kripke pseudo-model $\langle \mathbf{W}, <, \models \rangle$, if $\alpha \models A$ for any $\alpha \in \mathbf{W}$. The following lemma is proved in [2].

Lemma 1.3 *$A \in \mathbf{R}^-$ if and only if A is valid in any Kripke model for $\mathbf{R}^-$.*

2 A sequent system $\mathbf{GR}^-$

In this section we introduce a sequent system $\mathbf{GR}^-$ defined in [5]. We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of formulas. The expression $\Box\Gamma$ denotes the set $\{\Box A \mid A \in \Gamma\}$. By a sequent, we mean the expression $\Gamma \to \Delta$. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

By $\mathsf{Sub}(A)$, we mean the set of subformulas of A. Also, we put

$\mathbf{wit}(B, C) = \{\Box B \prec \Box C, \Box B \preceq \Box C, \Box C \prec \Box B, \Box C \preceq \Box B\}$,

$\mathsf{Sub}^+(A) = \mathsf{Sub}(A) \cup \{\Box B \odot \Box C \mid \mathbf{wit}(B, B') \cap \mathsf{Sub}(A) \neq \emptyset \text{ for some } B',$
$\mathbf{wit}(C, C') \cap \mathsf{Sub}(A) \neq \emptyset \text{ for some } C', \odot \in \{\prec, \preceq\}\}$,

$\mathsf{Sub}^{++}(A) = \mathsf{Sub}(A) \cup \{\Box B \odot \Box C \mid \Box B, \Box C \in \mathsf{Sub}(A), \odot \in \{\prec, \preceq\}\}$,

$\mathsf{Sub}(\Gamma \to \Delta) = \bigcup_{B \in \Gamma \cup \Delta} \mathsf{Sub}(B)$,

$\mathsf{Sub}^+(\Gamma \to \Delta) = \bigcup_{B \in \Gamma \cup \Delta} \mathsf{Sub}^+(B)$,

$\mathsf{Sub}^{++}(\Gamma \to \Delta) = \bigcup_{B \in \Gamma \cup \Delta} \mathsf{Sub}^{++}(B)$.

We note that for a $\prec$-free and $\preceq$-free sequent S, $\mathsf{Sub}(S) = \mathsf{Sub}^+(S) \subseteq \mathsf{Sub}^{++}(S)$, for instance, $\{p, \Box p\} = \mathsf{Sub}(\to \Box p) = \mathsf{Sub}^+(\to \Box p) \subseteq \mathsf{Sub}^{++}(\to \Box p) = \{p, \Box p, \Box p \prec \Box p, \Box p \preceq \Box p\}$.

By the sequent system $\mathbf{LK}$ for the classical propositional logic, we mean the system defined, in the usual way, by two axioms $A \to A$ and $\bot \to$, the usual logical inference rules, two weakening rules $(w \to)$ and $(\to w)$, and (cut) in the following form

$$\frac{\Gamma_1 \to \Delta_1, A \qquad A, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_1 \to \Delta_1, \Delta_2}(cut).$$

The system $\mathbf{GR}^-$ is the system obtained from the sequent system $\mathbf{LK}$ by adding the following axioms and inference rules in the usual way.

Additional axioms of $\mathbf{GR}^-$

$GA1$: $\Box A \preceq \Box B, \Box B \preceq \Box C \to \Box A \preceq \Box C$
$GA2$: $\Box A \to \Box A \preceq \Box B, \Box B \prec \Box A$
$GA3$: $\Box B \to \Box A \preceq \Box B, \Box B \prec \Box A$
$GA4$: $\Box A \prec \Box B \to \Box A \preceq \Box B$
$GA5$: $\Box A \preceq \Box B, \Box B \prec \Box A \to$

Additional inference rules of $\mathbf{GR}^-$

$$\frac{\Box A, \Sigma^f, \Gamma \to A}{\Sigma^f, \Box\Gamma \to \Box A}(\to \Box) \qquad \frac{\Box A, \Gamma \to \Delta}{\Box A \preceq \Box B, \Gamma \to \Delta}(\preceq\to) \qquad \frac{\Gamma \to \Delta, \Box A}{\Gamma \to \Delta, \Box A \preceq \Box A}(\to\preceq)$$

where Σ^f is a finite subset of Σ.

The system $\mathbf{GR}_1^-$ is the system obtained from $\mathbf{GR}^-$ by restricting a cut to the following form:

$$\frac{\Gamma_1 \to \Delta_1, \Box A \odot \Box B \qquad \Box A \odot \Box B, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}$$

where $\odot \in \{\prec, \preceq\}$, and $\Box A$ and $\Box B$ are subformulas of a formula occurring in the lower sequent. Using Lemma 1.3, [5] proved the following lemma.

Lemma 2.1 *The following conditions are equivalent:*
(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_1^-$,
(2) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}^-$,
(3) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n \in \mathbf{R}^-$,
(4) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ *is valid in any Kripke model for* $\mathbf{R}^-$.

Corollary 2.2 *If a sequent S is provable in $\mathbf{GR}^-$, then there exists a proof figure $\mathcal{P}$ for S such that each formula occurring in $\mathcal{P}$ belongs to* $\mathsf{Sub}^{++}(S)$.

3 Another sequent system $\mathbf{GR}_2^-$

Lemma 2.1 provides a cut-elimination theorem in weakened form, and hence the decision procedure for the provability of $\mathbf{R}^-$. However, the lemma does not say that every cut in $\mathbf{GR}_1^-$ is necessary, and there seems to be a removable cut. For instance, the following cut seems to be removable if $\Gamma \to \Delta$ does not have any $\prec$-formula and $\preceq$-formula. Because we naturally conjecture that the $\prec$-free and $\preceq$-free fragment of $\mathbf{GR}^-$ is the system $\mathbf{GGL}$, the system obtained by adding $(\to \Box)$ to $\mathbf{LK}$, for the provability logic $\mathbf{GL}$, and a cut-elimination theorem of $\mathbf{GGL}$ has been proven in Valentini [7] and Avron [1].

$$\frac{\dfrac{\Gamma \to \Delta, \Box A}{\Gamma \to \Delta, \Box A \preceq \Box A}(\to\preceq) \qquad \dfrac{\Box A, \Gamma \to \Delta}{\Box A \preceq \Box A, \Gamma \to \Delta}(\preceq\to)}{\Gamma \to \Delta}(cut)$$

Here we introduce another system $\mathbf{GR}_2^-$ for $\mathbf{R}^-$ by adding only inference rules to $\mathbf{LK}$. A proof of a cut-elimination theorem of the new system will be completed in section 4. As a result, we will find that what kind of cuts are removable from $\mathbf{GR}_1^-$ and that the $\prec$-free and $\preceq$-free fragment of $\mathbf{GR}_2^-$ is $\mathbf{GGL}$.

The system $\mathbf{GR}_2^-$ is the system obtained from $\mathbf{LK}$ by adding the following inference rules in the usual way.

Additional inference rules of $\mathbf{GR}_2^-$

$(\to \Box), (\to\preceq), (\preceq\to)$ are as in $\mathbf{GR}^-$

$$\frac{\Box A \preceq \Box B, \Gamma \to \Delta, \Box B \preceq \Box A}{\Box A \prec \Box B, \Gamma \to \Delta}(\prec\to)$$

$$\frac{\Gamma \to \Delta, \Box C \preceq \Box D \quad \Gamma \to \Delta, \Box D \preceq \Box E \quad \Box C \preceq \Box E, \Gamma \to \Delta}{\Gamma \to \Delta}(tran)$$

$$\frac{\Gamma \to \Delta, \Box C, \Box D \quad \Box C \prec \Box D, \Gamma \to \Delta \quad \Box D \prec \Box C, \Gamma \to \Delta \quad \Gamma_1, \Gamma \to \Delta}{\Gamma \to \Delta}(lin)$$

where in $(tran)$, $C \neq D$, $C \neq E$, $D \neq E$ and $\{\Box C \preceq \Box D, \Box D \preceq \Box E\} \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$; in (lin), $\Gamma_1 = \{\Box C \preceq \Box D, \Box D \preceq \Box C\}$, $C \neq D$, $\Box C \prec \Box D \in \mathsf{Sub}^+(\Gamma \to \Delta)$.

The system $\mathbf{GR}_3^-$ is the system obtained from $\mathbf{GR}_2^-$ by removing cut. Here we note that for each additional inference rule $\dfrac{S_1 \quad \cdots \quad S_n}{S}$ in $\mathbf{GR}_3^-$, $\mathsf{Sub}^+(S_i) \subseteq \mathsf{Sub}^+(S)$ $(i = 1, \cdots, n)$. Hence $\mathbf{GR}_3^-$ satisfies a kind of subformula property. Also we note that if a $\prec$-free and $\preceq$-free sequent S is provable in $\mathbf{GR}_3^-$, then S is provable in **GGL** for **GL**.

Lemma 3.1 $\Gamma \to \Delta \in \mathbf{GR}^-$ *if and only if* $\Gamma \to \Delta \in \mathbf{GR}_2^-$.

Proof. First, we show "only if" part. Additional inference rules of $\mathbf{GR}^-$ are also inference rule in $\mathbf{GR}_2^-$. So, it is sufficient to show the provability of the additional axioms of $\mathbf{GR}^-$ in $\mathbf{GR}_2^-$.

The axioms $GA4(\Box A \prec \Box B \to \Box A \preceq \Box B)$ and $GA5(\Box A \preceq \Box B, \Box B \prec \Box A \to)$ are proved by $(\to w)$ and $(\prec\to)$.

The axiom $GA1(\Box A \preceq \Box B, \Box B \preceq \Box C \to \Box A \preceq \Box C)$ can be proved by $(tran)$ and weakening rules if $A \neq B$, $A \neq C$ and $B \neq C$; by weakening rules if $A = B$ or $B = C$; and by the following figure if $A = C$:

$$\dfrac{\dfrac{\dfrac{\Box A \to \Box A}{\Box A \to \Box A \preceq \Box A}(\to\preceq)}{\Box A \preceq \Box B \to \Box A \preceq \Box A}(\preceq\to)}{\Box A \preceq \Box B, \Box B \preceq \Box A \to \Box A \preceq \Box A}(w \to).$$

The axiom $GA2(\Box A \to \Box A \preceq \Box B, \Box B \prec \Box A)$ can be proved by (lin), the provability of $GA4$ and weakening rules if $A \neq B$; by $(\to\preceq)$ and weakening rules if $A = B$. Similarly, the axiom $GA3$ can be proved.

We show "if" part. It is sufficient to show that each inference rule in $\mathbf{GR}_2^-$ preserves the provability of $\mathbf{GR}^-$. We can see this in the figures in the next page. ⊣

Theorem 3.2 *The following conditions are equivalent:*

(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_3^-$,
(2) $A_1, \cdots, A_m \to B_1, \cdots, B_n \in \mathbf{GR}_2^-$,
(3) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n \in \mathbf{R}^-$,
(4) $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ *is valid in any Kripke model for* $\mathbf{R}^-$.

"(1) implies (2)" is clear. The equivalence between (2) and (3) is from Lemma 3.1, and the equivalence between (3) and (4) is from Lemma 2.1. So, it is sufficient to show "(4) implies (1)", a Kripke completeness of $\mathbf{GR}_3^-$. A proof of the completeness will be given in the next section.

Corollary 3.3 *If a sequent S is provable in $\mathbf{GR}^-$, then there exists a proof figure in $\mathbf{GR}^-$ whose cuts are of the form of cuts occurring in the next page.*[1]

Proof. Suppose that $S \in \mathbf{GR}^-$. Then by Theorem 3.2, $S \in \mathbf{GR}_3^-$. So, there exists a proof figure $\mathcal{P}$ for S in $\mathbf{GR}_3^-$. Let $\mathcal{Q}$ be the figure obtained from $\mathcal{P}$ by replacing $(\prec\to)$, $(tran)$ and (lin) with the corresponding figure in the next page. We note that $\mathcal{Q}$ is a proof figure for S in $\mathbf{GR}^-$ and each cut in $\mathcal{Q}$ is of the form of cuts occurring in the next page. ⊣

[1] We note that every cut occurring in the next page has at least one occurrence of cut formula whose ancestor occurs in an additional axiom of $\mathbf{GR}^-$.

$(\prec\to)$:

$$\dfrac{\dfrac{\Box A \prec \Box B \to \Box A \preceq \Box B \quad \Box A \preceq \Box B, \Gamma \to \Delta, \Box B \preceq \Box A}{\Box A \prec \Box B, \Gamma \to \Delta, \Box B \preceq \Box A}\,(cut) \quad \Box B \preceq \Box A, \Box A \prec \Box B \to}{\Box A \prec \Box B, \Gamma \to \Delta}\,(cut)$$

$(tran)$:

$$\dfrac{\dfrac{\Gamma \to \Delta, \Box C \preceq \Box D \quad \dfrac{\Gamma \to \Delta, \Box D \preceq \Box E \quad \Box D \preceq \Box E, \Box C \preceq \Box D \to \Box C \preceq \Box E}{\Box C \preceq \Box D, \Gamma \to \Delta, \Box C \preceq \Box E}\,(cut)}{\Gamma \to \Delta, \Box C \preceq \Box E}\,(cut) \quad \Box C \preceq \Box E, \Gamma \to \Delta}{\Gamma \to \Delta}\,(cut)$$

(lin):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, \Box C, \Box D}{\Gamma \to \Delta, \Box C, \Box D \preceq \Box D}\,(\to\preceq) \quad \mathcal{P}(D,C)}{\Gamma \to \Delta, \Box C \preceq \Box D, \Box D \preceq \Box C, \Box C}\,(cut)}{\Gamma \to \Delta, \Box C \preceq \Box D, \Box D \preceq \Box C, \Box C \preceq \Box C}\,(\to\preceq) \quad \mathcal{P}(C,D)}{\Gamma \to \Delta, \Box C \preceq \Box D, \Box D \preceq \Box C}\,(cut) \quad \mathcal{Q}(C,D)}{\Gamma \to \Delta, \Box C \preceq \Box D}\,(cut) \quad \dfrac{\mathcal{Q}(D,C) \quad \Box D \preceq \Box C, \Box C \preceq \Box D, \Gamma \to \Delta}{\Box C \preceq \Box D, \Gamma \to \Delta}\,(cut)}{\Gamma \to \Delta}\,(cut)$$

$\mathcal{P}(C,D)$:

$$\dfrac{\dfrac{\Box C \to \Box C \preceq \Box D, \Box D \prec \Box C \quad \Box D \prec \Box C \to \Box D \preceq \Box C}{\Box C \to \Box C \preceq \Box D, \Box D \preceq \Box C}\,(cut)}{\Box C \preceq \Box C \to \Box C \preceq \Box D, \Box D \preceq \Box C}\,(\preceq\to)$$

$\mathcal{Q}(C,D)$:

$$\dfrac{\dfrac{\Box D \to \Box C \preceq \Box D, \Box D \prec \Box C \quad \Box D \prec \Box C, \Gamma \to \Delta}{\Box D, \Gamma \to \Delta, \Box C \preceq \Box D}\,(cut)}{\Box D \preceq \Box C, \Gamma \to \Delta, \Box C \preceq \Box D}\,(\preceq\to)$$

Corollary 3.4 *If a sequent S is provable in $\mathbf{GR}_2^-$, then there exists a proof figure $\mathcal{P}$ for S such that each formula occurring in $\mathcal{P}$ belongs to $\mathsf{Sub}^+(S)$.*

Corollary 3.5 *If a $\prec$-free and $\preceq$-free sequent S is provable in $\mathbf{GR}_2^-$, then there exists a proof figure for S in* $\mathbf{GGL}$.

4 Completeness theorem and a concrete counter model

Here we prove the following theorem.

Theorem 4.1 *If $A_1, \cdots, A_m \to B_1, \cdots, B_n \notin \mathbf{GR}_3^-$, then there exists a Kripke model for $\mathbf{R}^-$, such that $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ is not valid.*

To prove the above theorem, we construct a concrete Kripke model for $\mathbf{R}^-$, in which $A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_n$ is not valid, while [5] only showed existence of such model in the proof of Lemma 2.1 in the following sense. To prove Lemma 2.1, [5] used the following lemma in [2], called "extension lemma",

Lemma 4.2 *Let $\mathbf{S}$ be a set of formulas satisfying $\mathsf{Sub}^{++}(A) \subseteq \mathbf{S}$ for any $A \in \mathbf{S}$ and let $\mathcal{K}^*$ be a Kripke pseudo-model for $\mathbf{R}^-$ satisfying the two conditions in Definition 1.2 for any D such that $\mathsf{Sub}(D) \cap \Sigma \subseteq \mathbf{S}$. Then there exists a Kripke model $\mathcal{K}$ for $\mathbf{R}^-$ such that for any $A \in \mathbf{S}$, A is valid in $\mathcal{K}^*$ if and only if A is valid in $\mathcal{K}$.*

[5] first gave a Kripke pseudo-model for $\mathbf{R}^-$ satisfying the conditions, in which a given formula is not valid, and using "extension lemma" several times, proved existence of counter Kripke model for $\mathbf{R}^-$. Also it may be possible to know a concrete counter model by following a proof of "extension lemma" several times, but probably it takes a long time and we do not know how to express the counter model since it would be very complicated. In a similar way, it is possible to prove Theorem 4.1(cf. [4]), but here we directly construct a concrete counter Kripke model. To do so, we need some preparations.

Definition 4.3 *A sequent $\Gamma \to \Delta$ is said to be saturated if the following conditions hold:*

(1) if $A \wedge B \in \Gamma$, then $A, B \in \Gamma$,
(2) if $A \wedge B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$,
(3) if $A \vee B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$,
(4) if $A \vee B \in \Delta$, then $A, B \in \Delta$,
(5) if $A \supset B \in \Gamma$, then $A \in \Delta$ or $B \in \Gamma$,
(6) if $A \supset B \in \Delta$, then $A \in \Gamma$ and $B \in \Delta$,
(7) if $\Box A \preceq \Box B \in \Gamma$, then $\Box A \in \Gamma$,
(8) if $\Box A \preceq \Box A \in \Delta$, then $\Box A \in \Delta$,
(9) if $\Box A \prec \Box B \in \Gamma$, then $\Box A \preceq \Box B \in \Gamma$ and $\Box B \preceq \Box A \in \Delta$,
(10) if $A \neq B$, $A \neq C$, $B \neq C$ and $\{\Box A \preceq \Box B, \Box B \preceq \Box C\} \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$, then either one of $\Box A \preceq \Box C \in \Gamma$, $\Box A \preceq \Box B \in \Delta$, and $\Box B \preceq \Box C \in \Delta$ holds,
(11) if $A \neq B$ and $\Box A \prec \Box B \in \mathsf{Sub}^+(\Gamma \to \Delta)$, then either one of $\{\Box A, \Box B\} \subseteq \Delta$, $\Box A \prec \Box B \in \Gamma$, $\Box B \prec \Box A \in \Gamma$ and $\{\Box A \preceq \Box B, \Box B \preceq \Box A\} \subseteq \Gamma$ holds.

Lemma 4.4 *If $\Gamma \to \Delta \notin \mathbf{GR}_3^-$, then there exists a saturated sequent $\Gamma' \to \Delta'$ satisfying $\Gamma' \to \Delta' \notin \mathbf{GR}_3^-$, $\Gamma \subseteq \Gamma' \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$, $\Delta \subseteq \Delta' \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$ and $\mathsf{Sub}^+(\Gamma' \to \Delta') = \mathsf{Sub}^+(\Gamma \to \Delta)$.*

Proof. Let it be that $p \notin \mathsf{Sub}(\Gamma \to \Delta)$. Since $\mathsf{Sub}^+(\Gamma \to \Delta)$ is finite, there exist formulas $A_0, A_1, \cdots, A_{n-1}$ such that

$$\{A_0, A_1, \cdots, A_{n-1}\} = \mathsf{Sub}^+(\Gamma \to \Delta) \cup \{\Box B \wedge \Box C \wedge \Box D \wedge p \mid$$
$$\{\Box B \preceq \Box C, \Box C \preceq \Box D\} \subseteq \mathsf{Sub}^+(\Gamma \to \Delta), B \neq C, B \neq D, C \neq D\}.$$

We define a sequence of sequents $(\Gamma_0 \to \Delta_0), (\Gamma_1 \to \Delta_1), \cdots$, inductively as follows.

Step 0: $(\Gamma_0 \to \Delta_0) = (\Gamma \to \Delta)$.

Step $k+1$: If $A_{k \bmod n} = \Box B \prec \Box B$, then $(\Gamma_{k+1} \to \Delta_{k+1}) = (\Gamma_k \to \Delta_k)$. If $A_{k \bmod n} = \Box B \prec \Box C$ and $B \neq C$, then

$$(\Gamma_{k+1} \to \Delta_{k+1}) = \begin{cases} S_1 & \text{if } S_1 \notin \mathbf{GR}_3^- \\ S_2^* & \text{if } S_1 \in \mathbf{GR}_3^- \text{ and } S_2 \notin \mathbf{GR}_3^- \\ S_3 & \text{if } S_1, S_2 \in \mathbf{GR}_3^- \text{ and } S_3 \notin \mathbf{GR}_3^- \\ S_4 & \text{if } S_1, S_2, S_3 \in \mathbf{GR}_3^- \text{ and } S_4 \notin \mathbf{GR}_3^- \\ (\Gamma_k \to \Delta_k) & \text{otherwise} \end{cases}$$

where
$S_1 = (\Gamma_k \to \Delta_k, \Box B, \Box C)$,
$S_2 = (\Box B \prec \Box C, \Gamma_k \to \Delta_k)$,
$S_2^* = (\Box B \preceq \Box C, \Box B \prec \Box C, \Gamma_k \to \Delta_k, \Box C \preceq \Box B)$,
$S_3 = (\Box C \prec \Box B, \Gamma_k \to \Delta_k)$ and
$S_4 = (\Box B \preceq \Box C, \Box C \preceq \Box B, \Gamma_k \to \Delta_k)$.[2]
If $A_{k \bmod n} = \Box B \preceq \Box C$, then

$$(\Gamma_{k+1} \to \Delta_{k+1}) = \begin{cases} (\Box B, \Gamma_k \to \Delta_k) & \text{if } \Box B \preceq \Box C \in \Gamma_k \\ (\Gamma_k \to \Delta_k, \Box B) & \text{if } \Box B \preceq \Box C \in \Delta_k - \Gamma_k \text{ and } B = C \\ (\Gamma_k \to \Delta_k) & \text{otherwise} \end{cases}$$

If $A_{k \bmod n} = \Box B \wedge \Box C \wedge \Box D \wedge p$, then

$$(\Gamma_{k+1} \to \Delta_{k+1}) = \begin{cases} S_1 & \text{if } S_1 \notin \mathbf{GR}_3^- \\ S_2 & \text{if } S_1 \in \mathbf{GR}_3^- \text{ and } S_2 \notin \mathbf{GR}_3^- \\ S_3 & \text{if } S_1, S_2 \in \mathbf{GR}_3^- \text{ and } S_3 \notin \mathbf{GR}_3^- \\ (\Gamma_k \to \Delta_k) & \text{otherwise} \end{cases}$$

where
$S_1 = (\Gamma_k \to \Delta_k, \Box B \preceq \Box C)$,
$S_2 = (\Gamma_k \to \Delta_k, \Box C \preceq \Box D)$ and
$S_3 = (\Box B \preceq \Box D, \Gamma_k \to \Delta_k)$.[3]
If $A_{k \bmod n}$ is a $\Box$-formula, then $(\Gamma_{k+1} \to \Delta_{k+1}) = (\Gamma_k \to \Delta_k)$. In the other cases, $(\Gamma_{k+1} \to \Delta_{k+1})$ is defined in the usual way.

Also in the usual way, we can prove that $\bigcup_{i=0}^{\infty} \Gamma_i \to \bigcup_{i=0}^{\infty} \Delta_i$ is a saturated sequent satisfying the conditions, where the condition $\mathsf{Sub}^+(\Gamma \to \Delta) = \mathsf{Sub}^+(\bigcup_{i=0}^{\infty} \Gamma_i \to \bigcup_{i=0}^{\infty} \Delta_i)$ is proved by using $\mathsf{Sub}(\Gamma \to \Delta) \subseteq \mathsf{Sub}(\bigcup_{i=0}^{\infty} \Gamma_i \to \bigcup_{i=0}^{\infty} \Delta_i) \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$.[4] ⊣

For a sequent $S \notin \mathbf{GR}_3^-$, there are several saturated sequents satisfying the conditions in the above lemma. In the following, however, we need fixed one, denoted $sat(S)$, in order to prove several lemmas. We will use $sat(S)$ for a sequent S that is unprovable in $\mathbf{GR}_3^-$, but its unprovability is not known at that time. So, we also define $sat(S)$ for a sequent S that is provable in $\mathbf{GR}_3^-$.

Definition 4.5 For a sequent $S \notin \mathbf{GR}_3^-$, we fix a saturated sequent satisfying the conditions in the above lemma for $sat(S)$. For $S \in \mathbf{GR}_3^-$, we put $sat(S) = S$.

Remark 4.6 *For a sequent S,*
(1) $S \in \mathbf{GR}_3^-$ *if and only if* $sat(S) \in \mathbf{GR}_3^-$,
(2) $\mathsf{Sub}^+(sat(S)) = \mathsf{Sub}^+(S)$,
(3) *the antecedent of S is a subset of antecedent of $sat(S)$.*

A sequence of formulas is defined as follows:
(1) [ ] is a sequence of formulas,

[2]By $\Gamma \to \Delta \notin \mathbf{GR}_3^-$ and (lin), we can find that "otherwise" case does not occur.
[3]By $\Gamma \to \Delta \notin \mathbf{GR}_3^-$ and $(tran)$, we can find that "otherwise" case does not occur.
[4]The proof of Lemma 3.1 provides the provability of additional axioms in $\mathbf{GR}^-$ in $\mathbf{GR}_3^-$. This makes it easier to check that the sequent is saturated.

(2) if $[A_1, \cdots, A_n]$ is a sequence of formulas, then so is $[A_1, \cdots, A_n, B]$.
A binary operator $\circ$ is defined by

$$[A_1, \cdots, A_m] \circ [B_1, \cdots, B_n] = [A_1, \cdots, A_m, B_1, \cdots, B_n].$$

We use τ and σ, possibly with suffixes, for sequences of formulas.

From now on, we fix a sequent S_0, as a sequent that is not provable in $\mathbf{GR}_3^-$.

Definition 4.7 We define the set $\mathbf{W}(S_0)$ of pairs of a sequent and a sequence of formulas, inductively as follows:
(1) $(sat(S_0); [\,]) \in \mathbf{W}(S_0)$,
(2) if a pair $(\Gamma \to \Delta, \Box A; \tau)$ belongs to $\mathbf{W}(S_0)$, then so does the pair

$$(sat(\Box A, \{D \mid \Box D \in \Gamma\}, \Gamma \cap \Sigma \to A); \tau \circ [\Box A]).$$

We note that S is saturated if $(S; \tau) \in \mathbf{W}(S_0)$.

The set $\mathbf{W}(S_0)$ will be the set of possible worlds in the final model. A world $(\Gamma_1 \to \Delta_1; \sigma) \in \mathbf{W}(S_0)$ will validate the formulas in Γ_1 and refute the formulas in Δ_1. So, $w = (\Gamma \to \Delta, \Box A; \tau) \in \mathbf{W}(S_0)$ will refute $\Box A$. Also the pair

$$w_1 = (sat(\Box A, \{D \mid \Box D \in \Gamma\}, \Gamma \cap \Sigma \to A); \tau \circ [\Box A]),$$

which we find in Definition 4.7(2), will be a world refuting A and satisfying $w < w_1$.

Lemma 4.8 *If $(S_1; \tau), (S_2; \tau) \in \mathbf{W}(S_0)$, then $S_1 = S_2$.*

Proof. We use an induction on τ. If $\tau = [\,]$, then we have $S_1 = S_2 = sat(S_0)$. Suppose that $\tau = \sigma \circ [\Box A]$. Then there exist $(\Gamma_1 \to \Delta_1, \Box A; \sigma)$ and $(\Gamma_2 \to \Delta_2, \Box A; \sigma) \in \mathbf{W}(S_0)$ such that for $i = 1, 2$,

$$S_i = sat(\Box A, \{D \mid \Box D \in \Gamma_i\}, \Gamma_i \cap \Sigma \to A).$$

By the induction hypothesis, we have $(\Gamma_1 \to \Delta_1, \Box A) = (\Gamma_2 \to \Delta_2, \Box A)$, and so, $\Gamma_1 = \Gamma_2$. Hence we obtain $S_1 = S_2$. ⊣

Lemma 4.9 *Let $(\Gamma_1 \to \Delta_1; \tau)$ and $(\Gamma_2 \to \Delta_2; \tau \circ [\Box A])$ be in $\mathbf{W}(S_0)$. Then*
(1) $\Box A \in \mathsf{Sub}(\Gamma_1 \to \Delta_1)$,
(2) $\Gamma_1 \to \Delta_1 \notin \mathbf{GR}_3^-$ *implies* $\Gamma_2 \to \Delta_2 \notin \mathbf{GR}_3^-$,
(3) $\mathsf{Sub}^+(\Gamma_1 \to \Delta_1) \supseteq \mathsf{Sub}^+(\Gamma_2 \to \Delta_2)$,
(4) $\Gamma_1 \cap \Sigma \subseteq \Gamma_2 \cap \Sigma$.

Proof. First, we note that $\Gamma_i \to \Delta_i$ $(i = 1, 2)$ are saturated, and so, every condition in Definition 4.3 for these sequents holds. By $(\Gamma_2 \to \Delta_2; \tau \circ [\Box A]) \in \mathbf{W}(S_0)$, there exists a pair $(\Gamma_3 \to \Delta_3, \Box A; \tau) \in \mathbf{W}(S_0)$ such that $(\Gamma_2 \to \Delta_2) = sat(S)$, where $S = (\Box A, \{D \mid \Box D \in \Gamma_3\}, \Gamma_3 \cap \Sigma \to A)$. On the other hand, by Lemma 4.8, $(\Gamma_3 \to \Delta_3, \Box A) = (\Gamma_1 \to \Delta_1)$. So, (1) is clear. By Remark 4.6(1) and $(\to \Box)$, we have that $S \in \mathbf{GR}_3^-$ if and only if $sat(S) = (\Gamma_2 \to \Delta_2) \in \mathbf{GR}_3^-$ and that $S \in \mathbf{GR}_3^-$ implies $(\Gamma_3 \to \Delta_3, \Box A) = (\Gamma_1 \to \Delta_1) \in \mathbf{GR}_3^-$, respectively, and hence we obtain (2). (3) and (4) are shown using Remark 4.6(2) and Remark 4.6(3), respectively. ⊣

Lemma 4.10 *Let $(\Gamma_1 \to \Delta_1; \tau)$ and $(\Gamma_2 \to \Delta_2; \tau \circ \sigma)$ be in $\mathbf{W}(S_0)$. Then*
(1) *σ consists of only $\Box$-formulas in* $\mathsf{Sub}(\Gamma_1 \to \Delta_1)$,
(2) $\Gamma_1 \to \Delta_1 \notin \mathbf{GR}_3^-$ *implies* $\Gamma_2 \to \Delta_2 \notin \mathbf{GR}_3^-$,
(3) $\mathsf{Sub}^+(\Gamma_1 \to \Delta_1) \supseteq \mathsf{Sub}^+(\Gamma_2 \to \Delta_2)$,
(4) $\Gamma_1 \cap \Sigma \subseteq \Gamma_2 \cap \Sigma$.

Proof. We use an induction on σ. From Lemma 4.8, we have Basis($\sigma = [\,]$). By Lemma 4.9, we also have Induction Step($\sigma \neq [\,]$). ⊣

Corollary 4.11 *Let* $(\Gamma \to \Delta; \tau)$ *be in* $\mathbf{W}(S_0)$*. Then*

(1) τ *consists of only* $\Box$*-formulas in* $\mathsf{Sub}(sat(S_0))$*,*

(2) $\Gamma \to \Delta \notin \mathbf{GR}_3^-$*.*

Lemma 4.12 $\mathbf{W}(S_0)$ *is finite.*

Proof. By Lemma 4.8 and Corollary 4.11(1), it is sufficient to show $(S; \tau_1 \circ [\Box A] \circ \tau_2 \circ [\Box A] \circ \tau_3) \notin \mathbf{W}(S_0)$, for any A, S, τ_1, τ_2 and τ_3. Suppose that the above pair belongs to $\mathbf{W}(S_0)$. Since we can show, by the induction on σ,

$$(S_2; \tau \circ \sigma) \in \mathbf{W}(S_0) \text{ implies } (S_1; \tau) \in \mathbf{W}(S_0) \text{ for some } S_1,$$

we have $(\Gamma_1 \to \Delta_1; \tau_1 \circ [\Box A])$, $(\Gamma_2 \to \Delta_2; \tau_1 \circ [\Box A] \circ \tau_2)$, $(\Gamma_3 \to \Delta_3; \tau_1 \circ [\Box A] \circ \tau_2 \circ [\Box A]) \in \mathbf{W}(S_0)$ for some $\Gamma_i \to \Delta_i$ $(i = 1, 2, 3)$. Using the definition of $\mathbf{W}(S_0)$, we have $\Box A \in \Gamma_1$, and using Lemma 4.10(4), $\Box A \in \Gamma_2$. On the other hand, by $(\Gamma_3 \to \Delta_3; \tau_1 \circ [\Box A] \circ \tau_2 \circ [\Box A]) \in \mathbf{W}(S_0)$ and the definition of $\mathbf{W}(S_0)$, there exists $(\Gamma_4 \to \Delta_4; \tau_1 \circ [\Box A] \circ \tau_2) \in \mathbf{W}(S_0)$ such that $\Box A \in \Delta_4$. Using Lemma 4.8, we have $\Box A \in \Delta_4 = \Delta_2$. This is in contradiction with $\Box A \in \Gamma_2$ and Corollary 4.11(2). ⊣

Definition 4.13 We define a structure $\mathcal{K}(S_0) = \langle \mathbf{W}(S_0), <, \models \rangle$ as follows:

(1) $(S_1; \tau_1) < (S_2; \tau_2)$ if and only if $\tau_2 = \tau_1 \circ \sigma$ for some $\sigma \neq [\,]$,

(2) $\models$ is a valuation satisfying, in addition to the conditions in Definition 1.1(3), for any $\gamma (= (\Gamma \to \Delta; \tau)) \in \mathbf{W}(S_0)$,

(2.1) $\gamma \models p$ if and only if $p \in \Gamma$,

(2.2) $\gamma \models \Box A \prec \Box B$ if and only if either one of the following three holds:

(2.2.1) $\Box A \prec \Box B \in \Gamma$,

(2.2.2) there exists $\alpha \in \gamma\!\downarrow\!\!\!\!-$ such that $\alpha \models \Box A$ and $\alpha \not\models \Box B$,

(2.2.3) $\Box D \prec \Box B \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any D and there exist a formula C and $\gamma' (= (\Gamma' \to \Delta'; \tau')) \in \gamma\!\downarrow\!\!\!\!-$ such that $\Box A \prec \Box C \in \Gamma'$, $\gamma' \models \Box A$, $\gamma' \models \Box B$, $\gamma' \models \Box C$ and for any $\beta \in \gamma'\!\downarrow$, $\beta \not\models \Box A$, $\beta \not\models \Box B$ and $\beta \not\models \Box C$,

(2.3) $\gamma \models \Box A \preceq \Box B$ if and only if $\gamma \models \Box A$ and $\gamma \not\models \Box B \prec \Box A$.

Here we note that for any $(\Gamma_1 \to \Delta_1; \tau_1)$, $(\Gamma_2 \to \Delta_2; \tau_2) \in \mathbf{W}(S_0)$, every condition in Lemma 4.10 holds if $(\Gamma_1 \to \Delta_1; \tau_1) < (\Gamma_2 \to \Delta_2; \tau_2)$.

Lemma 4.14 $\mathcal{K}^*(S_0)$ *is a Kripke pseudo-model for* $\mathbf{R}^-$*.*

Proof. By Lemma 4.12, $\mathbf{W}(S_0)$ is finite. The irreflexivity and the transitivity of $<$ can be shown easily. We show the remaining condition of $<$. Suppose that $(S_1; \tau_1) < (S_3; \tau_3)$ and $(S_2; \tau_2) < (S_3; \tau_3)$. Then $\tau_3 = \tau_1 \circ \sigma_1 = \tau_2 \circ \sigma_2$ for some non-empty sequences σ_1 and σ_2. Hence either $\tau_1 = \tau_2 \circ \sigma_2'$ or $\tau_1 \circ \sigma_1' = \tau_2$ holds. Using Lemma 4.8, we have either one of $(S_1; \tau_1) = (S_2; \tau_2)$, $(S_1; \tau_1) < (S_2; \tau_2)$ and $(S_2; \tau_2) < (S_1; \tau_1)$. ⊣

Lemma 4.15 *The axioms* $A5$*,* $A7$ *and* $A9$ *are valid in* $\mathcal{K}(S_0)$*.*

Proof. From Definition 4.13(2.3), the validity of $A5$ and $A9$ are clear. We show the validity of $A7$. Suppose that $\gamma \in \mathbf{W}(S_0)$ and $\gamma \models \Box A \vee \Box B$. If $\gamma \models \Box A$, then by Definition 4.13(2.3), we have that $\gamma \not\models \Box B \prec \Box A$ implies $\gamma \models \Box A \preceq \Box B$, and hence $\gamma \models (\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$. If $\gamma \not\models \Box A$, then by $\gamma \models \Box A \vee \Box B$, we have $\gamma \models \Box B$. So, we have $\alpha (= \gamma) \in \gamma\!\downarrow\!\!\!\!-$ such that $\alpha \not\models \Box A$ and $\alpha \models \Box B$, and using Definition 4.13(2.2), $\gamma \models \Box B \prec \Box A$. Hence we obtain $\gamma \models (\Box A \preceq \Box B) \vee (\Box B \prec \Box A)$. ⊣

Lemma 4.16 *Let* $(\Gamma \to \Delta; \tau)$ *be in* $\mathbf{W}(S_0)$*. Then for any formula* D*,*

(1) $D \in \Gamma$ *implies* $(\Gamma \to \Delta; \tau) \models D$*,*

(2) $D \in \Delta$ *implies* $(\Gamma \to \Delta; \tau) \not\models D$*.*

Proof. First, we note that $\Gamma \to \Delta$ is saturated, and so, every condition in Definition 4.3 holds. To prove the lemma, we use an induction on D. Basis(D is a propositional variable) is from Definition 4.13(2.1) and Corollary 4.11(2). For Induction Step, we only show the case that D is a $\prec$-formula and the case that D is a $\preceq$-formula.

We show the case that $D = \Box A \prec \Box B$. From Definition 4.13(2.2), (1) is clear. We show (2). Suppose that $\Box A \prec \Box B \in \Delta$ and $(\Gamma \to \Delta; \tau) \models \Box A \prec \Box B$. Then either one of the conditions (2.2.1), (2.2.2) and (2.2.3) in Definition 4.13 holds. However, by $\Box A \prec \Box B \in \Delta$ and Corollary 4.11(2), we have $\Box A \prec \Box B \in \mathsf{Sub}^+(\Gamma \to \Delta)$ and $\Box A \prec \Box B \notin \Gamma$, and so, none of (2.2.1) and (2.2.3) holds. Hence (2.2.2) holds, in other words, there exists $(\Gamma_1 \to \Delta_1; \sigma) \in (\Gamma \to \Delta; \tau)\updownarrow$ such that $(\Gamma_1 \to \Delta_1; \sigma) \models \Box A$ and $(\Gamma_1 \to \Delta_1; \sigma) \not\models \Box B$. Immediately, we have $A \neq B$. Also by the induction hypothesis, we have $\Box B \notin \Gamma_1$ and $\Box A \notin \Delta_1$. Here we also note that $\Gamma_1 \to \Delta_1$ is saturated, and so, every condition in Definition 4.3 for $\Gamma_1 \to \Delta_1$ holds. So, by $\Box B \notin \Gamma_1$, Definition 4.3(7) and Definition 4.3(9), we have $\Box B \preceq \Box A \notin \Gamma_1$ and $\Box B \prec \Box A \notin \Gamma_1$. Also by $\Box A \prec \Box B \in \Delta \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$ and Lemma 4.10(3), $\Box A \prec \Box B \in \mathsf{Sub}^+(\Gamma_1 \to \Delta_1)$. Using $A \neq B$, $\Box A \notin \Delta_1$ and Definition 4.3(11), we have $\Box A \prec \Box B \in \Gamma_1$. Using Lemma 4.10(4), we have $\Box A \prec \Box B \in \Gamma$. This is in contradiction with $\Box A \prec \Box B \notin \Gamma$.

We show the case that $D = \Box A \preceq \Box B$. First we show (1). Suppose that $\Box A \preceq \Box B \in \Gamma$ and $(\Gamma \to \Delta; \tau) \not\models \Box A \preceq \Box B$. Then by Definition 4.3(7) $\Box A \in \Gamma$, and using the induction hypothesis, we have $(\Gamma \to \Delta; \tau) \models \Box A$. Using $(\Gamma \to \Delta; \tau) \not\models \Box A \preceq \Box B$ and Definition 4.13(2.3), we have $(\Gamma \to \Delta; \tau) \models \Box B \prec \Box A$, and using Definition 4.13(2.2), either one of the following three holds:

(4) $\Box B \prec \Box A \in \Gamma$,

(5) there exists $(\Gamma_1 \to \Delta_1; \tau_1) \in (\Gamma \to \Delta; \tau)\updownarrow$ such that $(\Gamma_1 \to \Delta_1; \tau_1) \models \Box B$ and $(\Gamma_1 \to \Delta_1; \tau_1) \not\models \Box A$,

(6) $\Box C \prec \Box A \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any C.

However, we have $\Box B \prec \Box A \in \mathsf{Sub}^+(\Gamma \to \Delta)$ from $\Box A \preceq \Box B \in \Gamma \subseteq \mathsf{Sub}(\Gamma \to \Delta)$, and so, (6) does not hold. Also by $\Box A \preceq \Box B \in \Gamma$ and Corollary 4.11(2), we have $\Box A \preceq \Box B \notin \Delta$, and using Definition 4.3(9), $\Box B \prec \Box A \notin \Gamma$, and hence (4) does not hold. So, (5) holds. Immediately, we have $A \neq B$. Also by the induction hypothesis, we have $\Box B \notin \Delta_1$ and $\Box A \notin \Gamma_1$. Here we also note that $\Gamma_1 \to \Delta_1$ is saturated, and so, every condition in Definition 4.3 for $\Gamma_1 \to \Delta_1$ holds. So, by $\Box A \notin \Gamma_1$, Definition 4.3(7) and Definition 4.3(9), we have $\Box A \preceq \Box B \notin \Gamma_1$ and $\Box A \prec \Box B \notin \Gamma_1$. On the other hand, by Lemma 4.10(3), we have $\Box B \prec \Box A \in \mathsf{Sub}^+(\Gamma \to \Delta) \subseteq \mathsf{Sub}^+(\Gamma_1 \to \Delta_1)$. Using $A \neq B$ and Definition 4.3(11), we have $\Box B \prec \Box A \in \Gamma_1$, and using Lemma 4.10(4), $\Box B \prec \Box A \in \Gamma$, which is in contradiction with $\Box B \prec \Box A \notin \Gamma$ shown above as the negation of (4).

We show (2). Suppose that $\Box A \preceq \Box B \in \Delta$. If $A = B$, then by using Definition 4.3(8), we have $\Box A \in \Delta$, and using the induction hypothesis, $(\Gamma \to \Delta; \tau) \not\models \Box A$, moreover using Definition 4.13(2.3), $(\Gamma \to \Delta; \tau) \not\models \Box A \preceq \Box B$. So, we assume that $A \neq B$. By $\Box A \preceq \Box B \in \Delta$, Corollary 4.11(2) and Definition 4.3(9), we have $\Box A \prec \Box B \in \mathsf{Sub}^+(\Gamma \to \Delta)$, $\Box A \preceq \Box B \notin \Gamma$ and $\Box A \prec \Box B \notin \Gamma$. Using Definition 4.3(11), we have either $\Box A \in \Delta$ or $\Box B \prec \Box A \in \Gamma$. Using the induction hypothesis, $(\Gamma \to \Delta; \tau) \not\models \Box A$ or $(\Gamma \to \Delta; \tau) \models \Box B \prec \Box A$, and using Definition 4.13(2.3), $(\Gamma \to \Delta; \tau) \not\models \Box A \preceq \Box B$. ⊣

Corollary 4.17 *Let A be a formula in the antecedent of S_0 and let B be a formula in the succeedent of S_0. Then in $\mathcal{K}(S_0)$, $(sat(S_0); [\,]) \models A$ and $(sat(S_0); [\,]) \not\models B$.*

Lemma 4.18 $(\Gamma \to \Delta; \tau) \in \mathbf{W}(S_0)$ *implies* $\Box A \prec \Box A \notin \Gamma$.

Proof. First, we note that $\Gamma \to \Delta$ is saturated, and so, every condition in Definition 4.3 holds. Suppose that $\Box A \prec \Box A \in \Gamma$. Then by Definition 4.3(9), $\Box A \preceq \Box A$ belongs to Γ and Δ. This is in contradiction with Corollary 4.11(2). ⊣

Lemma 4.19 *Let $\gamma = (\Gamma \to \Delta; \tau)$ and $\gamma_2 = (\Gamma_2 \to \Delta_2; \tau_2)$ be in $\mathbf{W}(S_0)$ and let be that $\gamma < \gamma_2$. Then for any formula D,*

(1) $\Box D \notin \Delta$ *and* $\Box E_2 \prec \Box D \notin \mathsf{Sub}^+(\Gamma_2 \to \Delta_2)$ *for any* E_2 *imply* $\Box E \prec \Box D \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ *for any* E,

(2) $D \in \Sigma$ *and* $\gamma \models D$ *imply* $\gamma_2 \models D$.

Proof. First, we note that $\Gamma \to \Delta$ and $\Gamma_2 \to \Delta_2$ are saturated, and so, every condition in Definition 4.3 for these sequents holds.

We show (1). Suppose that

(1.1) $\Box D \notin \Delta$,

(1.2) $\Box E_2 \prec \Box D \notin \mathsf{Sub}^+(\Gamma_2 \to \Delta_2)$ for any E_2,

(1.3) $\Box E \prec \Box D \in \mathsf{Sub}^+(\Gamma \to \Delta)$ for some E.

Then by (1.3), we have $\mathbf{wit}(D, D') \cap \mathsf{Sub}(\Gamma \to \Delta) \neq \emptyset$ for some D'. So, either $\mathbf{wit}(D, D') \cap \mathsf{Sub}(\Gamma \to) \neq \emptyset$

or $\mathbf{wit}(D, D') \cap \mathsf{Sub}(\to \Delta) \neq \emptyset$ holds. If the latter holds, then using (1.1) and Definition 4.3(11), we have the former. So, the former holds in both cases. Using Lemma 4.10(4) and $\gamma < \gamma_2$, we have $\mathbf{wit}(D, D') \cap \mathsf{Sub}(\Gamma_2 \to) \neq \emptyset$, and so, $\Box D' \prec \Box D \in \mathsf{Sub}^+(\Gamma_2 \to \Delta_2)$, which is in contradiction with (1.2).

We show (2). If D is a $\Box$-formula, then by the definition of $\models$, we have the lemma.

Suppose that $D = \Box A \prec \Box B$ and $\gamma \models \Box A \prec \Box B$. Then by Definition 4.13(2.2), either one of the conditions (2.2.1), (2.2.2) and (2.2.3) in Definition 4.13 holds. If (2.2.1) holds, then by Lemma 4.10(4), $\Box A \prec \Box B \in \Gamma_2$, and using Lemma 4.16, $\gamma_2 \models \Box A \prec \Box B$. If (2.2.2) holds, then by $\gamma < \gamma_2$, the condition obtained from (2.2.2) by replacing γ with γ_2 holds, and hence $\gamma_2 \models \Box A \prec \Box B$. If (2.2.3) holds, then by Lemma 4.10(3) and $\gamma < \gamma_2$, the condition obtained from (2.2.3) by replacing $\Gamma \to \Delta$ and γ with $\Gamma_2 \to \Delta_2$ and γ_2, respectively, holds, and hence $\gamma_2 \models \Box A \prec \Box B$.

Suppose that $D = \Box A \preceq \Box B$ and $\gamma \models \Box A \preceq \Box B$. Then by Definition 4.13(2.3), we have

(2.1) $\gamma \models \Box A$ and $\gamma \not\models \Box B \prec \Box A$.

From the definition of $\models$, we have $\gamma_2 \models \Box A$. So, it is sufficient to show $\gamma_2 \not\models \Box B \prec \Box A$. Suppose that $\gamma_2 \models \Box B \prec \Box A$. Then by Definition 4.13(2.2), either one of the following three holds:

(2.2) $\Box B \prec \Box A \in \Gamma_2$,

(2.3) there exists $\alpha \in \gamma_2 \downarrow\!\!\!\!\!\downarrow$ such that $\alpha \models \Box B$ and $\alpha \not\models \Box A$,

(2.4) $\Box E \prec \Box A \notin \mathsf{Sub}^+(\Gamma_2 \to \Delta_2)$ for any E and there exist a formula C and $\gamma_2' (= (\Gamma_2' \to \Delta_2'; \tau_2')) \in \gamma_2 \downarrow\!\!\!\!\!\downarrow$ such that $\Box B \prec \Box C \in \Gamma_2'$, $\gamma_2' \models \Box A$, $\gamma_2' \models \Box B$, $\gamma_2' \models \Box C$ and for any $\beta \in \gamma_2' \downarrow$, $\beta \not\models \Box A$, $\beta \not\models \Box B$ and $\beta \not\models \Box C$.

If (2.2) holds, then by Lemma 4.10(3), we have $\Box B \prec \Box A \in \mathsf{Sub}^+(\Gamma \to \Delta)$. Also by (2.2) and Lemma 4.18, we have $A \neq B$. On the other hand, by Lemma 4.16 and (2.1), we have $\Box A \notin \Delta$ and $\Box B \prec \Box A \notin \Gamma$. So, using Definition 4.3(11), we have either $\Box A \preceq \Box B$ or $\Box A \prec \Box B$ belongs to Γ, and using Definition 4.3(9), we have $\Box A \preceq \Box B \in \Gamma$ in both cases. Using Lemma 4.10(4) and Corollary 4.11(2), we have $\Box A \preceq \Box B \in \Gamma_2$ and $\Box A \preceq \Box B \notin \Delta_2$. This is in contradiction with (2.2) and Definition 4.3(9).

If (2.3) holds, then by Lemma 4.14, either one of $\alpha < \gamma$, $\alpha = \gamma$ and $\gamma < \alpha$ holds. Here by (2.3) and (2.1), we have $\alpha \not\models \Box A$ and $\gamma \models \Box A$. So, using the definition of $\models$, we have $\alpha < \gamma$. So, the condition obtained from (2.3) by replacing γ_2 with γ holds, and hence $\gamma \models \Box B \prec \Box A$. This is in contradiction with (2.1).

If (2.4) holds, then similarly to the case that (2.3) holds, we have $\gamma_2' < \gamma$. Using $\gamma_2' \models \Box A$ in (2.4), we have $\gamma \models \Box A$, and using Lemma 4.16, $\Box A \notin \Delta$. By (2.4), $\Box E \prec \Box A \notin \mathsf{Sub}^+(\Gamma_2 \to \Delta_2)$ for any E, and using (1), we have $\Box E \prec \Box A \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any E. So, the condition obtained from (2.4) by replacing $\Gamma_2 \to \Delta_2$ and γ_2 with $\Gamma \to \Delta$ and γ, respectively, holds. Hence we have $\gamma \models \Box B \prec \Box A$. This is in contradiction with (2.1). ⊣

Lemma 4.20 *Let γ be in $\mathbf{W}(S_0)$ and let be that either $\gamma \models \Box A \preceq \Box B$ or $\gamma \models \Box A \prec \Box B$. Then for any $\gamma_1 \in \gamma \downarrow\!\!\!\!\!\downarrow$, $\gamma_1 \models \Box B$ implies $\gamma_1 \models \Box A$.*

Proof. We put $\gamma = (\Gamma \to \Delta; \tau)$ and $\gamma_1 = (\Gamma_1 \to \Delta_1; \tau_1)$. We note that $\Gamma \to \Delta$ and $\Gamma_1 \to \Delta_1$ are saturated, and so, every condition in Definition 4.3 for these two sequents holds. By Definition 4.13(2.2) and Definition 4.13(2.3), either one of the following four holds:

(1) $\gamma \not\models \Box B \prec \Box A$,

(2) $\Box A \prec \Box B \in \Gamma$,

(3) there exists $\alpha \in \gamma \downarrow\!\!\!\!\!\downarrow$ such that $\alpha \models \Box A$ and $\alpha \not\models \Box B$,

(4) there exists $\alpha \in \gamma \downarrow\!\!\!\!\!\downarrow$ such that $\alpha \models \Box A$ and for any $\beta \in \alpha \downarrow$, $\beta \not\models \Box B$,

If (1) holds, then by Definition 4.13(2.2), the following does not hold:

"there exists $\alpha \in \gamma \downarrow\!\!\!\!\!\downarrow$ such that $\alpha \models \Box B$ and $\alpha \not\models \Box A$",

and the negation of the above condition is the goal of the lemma.

If (2) holds, then by Definition 4.3(9), we have $\Box B \preceq \Box A \in \Delta$. Using Corollary 4.11(2) and Definition 4.3(9), we have $\Box B \preceq \Box A \notin \Gamma$ and $\Box B \prec \Box A \notin \Gamma$, and using Lemma 4.10(4), $\Box B \prec \Box A \notin \Gamma_1$. By (2) and Lemma 4.10(3), we have $\Box A \prec \Box B \in \mathsf{Sub}^+(\Gamma_1 \to \Delta_1)$. Also by (2) and Lemma 4.18, we have $A \neq B$. So, using Definition 4.3(11), we have either one of $\Box B \in \Delta_1$, $\Box A \prec \Box B \in \Gamma_1$ and $\Box A \preceq \Box B \in \Gamma_1$. Using Definition 4.3(9) and Definition 4.3(7), we have either $\Box B \in \Delta_1$ or $\Box A \in \Gamma_1$. Using Lemma 4.16, we have either $\gamma_1 \not\models \Box B$ or $\gamma_1 \models \Box A$.

If (3) or (4) holds, then by Lemma 4.19(2), $\gamma_1 \models \Box A$ if $\gamma_1 \in \alpha\uparrow \cup\{\alpha\}$; and $\gamma_1 \not\models \Box B$ if $\gamma_1 \in \alpha\downarrow$. By $\gamma_1 \in \gamma\downarrow$, $\alpha \in \gamma\downarrow$ and Lemma 4.14, we have either $\gamma_1 \in \alpha\uparrow \cup\{\alpha\}$ or $\gamma_1 \in \alpha\downarrow$. Hence we have either $\gamma_1 \not\models \Box B$ or $\gamma_1 \models \Box A$. ⊣

Lemma 4.21 *The axiom A8 is valid in $\mathcal{K}(S_0)$.*

Proof. Let $\gamma = (\Gamma \to \Delta; \tau)$ be in $\mathbf{W}(S_0)$. We note that $\Gamma \to \Delta$ is saturated, and so, every condition in Definition 4.3 holds. Suppose that

(1) $\gamma \models \Box A \prec \Box B$.

Then either one of the conditions (2.2.1), (2.2.2) and (2.2.3) in Definition 4.13 holds. If (2.2.1) holds, then Definition 4.3(9) and Definition 4.3(7), we have $\Box A \in \Gamma$, and using Lemma 4.16, $\gamma \models \Box A$. If either (2.2.2) or (2.2.3) holds, then there exists $\alpha \in \gamma\downarrow$ such that $\alpha \models \Box A$, and using Lemma 4.19(2), $\gamma \models \Box A$. Hence in any case, we have $\gamma \models \Box A$. Hence by Definition 4.13(2.3), it is sufficient to show $\gamma \not\models \Box B \prec \Box A$. Suppose that

(2) $\gamma \models \Box B \prec \Box A$.

Then by Definition 4.13(2.2), either one of the following three holds:

(2a) $\Box B \prec \Box A \in \Gamma$,

(2b) there exists $\alpha \in \gamma\downarrow$ such that $\alpha \models \Box B$ and $\alpha \not\models \Box A$,

(2c) $\Box D \prec \Box A \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any D.

Here we note that (2b) is in contradiction with (1) and Lemma 4.20. Hence we have either (2a) or (2c). On the other hand, by (1) and Definition 4.13(2.2), either one of the following three holds:

(1a) $\Box A \prec \Box B \in \Gamma$,

(1b) there exists $\alpha \in \gamma\downarrow$ such that $\alpha \models \Box A$ and $\alpha \not\models \Box B$,

(1c) $\Box D \prec \Box B \notin \mathsf{Sub}(\Gamma \to \Delta)$ for any D and there exist C and $(\Gamma_1 \to \Delta_1; \tau_1) \in \gamma\downarrow$ such that $\Box A \prec \Box C \in \Gamma_1$.

We note that (1a) is in contradiction with (2a), Definition 4.3(9) and Corollary 4.11(2) and that (1a) is in contradiction with (2c). Also we note that (1b) is in contradiction with (2) and Lemma 4.20. Moreover, we note that (1c) is in contradiction with (2a) and that $\Box A \prec \Box C \in \Gamma_1$ of (1c) is in contradiction with (2c) and Lemma 4.10(4). ⊣

Lemma 4.22 *The axiom A6 is valid in $\mathcal{K}(S_0)$.*

Proof. Let $\gamma = (\Gamma \to \Delta; \tau)$ be in $\mathbf{W}(S_0)$. We note that $\Gamma \to \Delta$ is saturated, and so, every condition in Definition 4.3 holds. Suppose that

(1) $\gamma \models \Box A \preceq \Box B$,

(2) $\gamma \models \Box B \preceq \Box C$

(3) $\gamma \not\models \Box A \preceq \Box C$.

If $A = B$, then (2) is in contradiction with (3). If $B = C$, then (1) is in contradiction with (3). So, we assume that $A \neq B$ and $B \neq C$.

By (1) and Definition 4.13(2.3), we have $\gamma \models \Box A$. So, using (3) and Definition 4.13(2.3), we have $\gamma \models \Box C \prec \Box A$, and using Definition 4.13(2.2), either one of the following three holds:

(3a) $\Box C \prec \Box A \in \Gamma$,

(3b) there exists $\alpha \in \gamma\downarrow$ such that $\alpha \models \Box C$ and $\alpha \not\models \Box A$,

(3c) $\Box E \prec \Box A \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any E and there exist a formula D and $\gamma'(= (\Gamma' \to \Delta'; \tau')) \in \gamma\downarrow$ such that $\Box C \prec \Box D \in \Gamma'$, $\gamma' \models \Box A$, $\gamma' \models \Box C$, $\gamma' \models \Box D$ and for any $\beta \in \gamma'\downarrow$, $\beta \not\models \Box A$, $\beta \not\models \Box C$ and $\beta \not\models \Box D$.

By (1), (2) and Lemma 4.20, we have that $\alpha \models \Box C$ implies $\alpha \models \Box A$ for any $\alpha \in \gamma\downarrow$. So, (3b) does not hold. We have either (3a) or (3c).

Suppose that (3a) holds. Then by Lemma 4.18, we have $A \neq C$.

If $\Box E \prec \Box B \in \mathsf{Sub}^+(\Gamma \to \Delta)$ for some E, then by (3a), we have $\{\Box A \preceq \Box B, \Box B \preceq \Box C\} \subseteq \mathsf{Sub}^+(\Gamma \to \Delta)$. Using Definition 4.3(10), either one of $\Box A \preceq \Box B \in \Delta$, $\Box B \preceq \Box C \in \Delta$ and $\Box A \preceq \Box C \in \Gamma$ holds, but this is in contradiction with (1), (2), (3) and Lemma 4.16.

We assume that $\Box E \prec \Box B \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any E. By (2) and Definition 4.13(2.3), we have $\gamma \not\models \Box C \prec \Box B$. So, it is sufficient to show $\gamma \models \Box C \prec \Box B$. To show this, we will show the condition corresponding to Definition 4.13(2.2.3).

By (3a), Definition 4.3(9) and Corollary 4.11(2), $\Box A \preceq \Box C$ belongs to Δ, but to Γ, moreover, $\Box A \prec \Box C \notin \Gamma$. Let $\gamma_1(= (\Gamma_1 \to \Delta_1; \tau_1))$ be in $\gamma\downarrow\!\!\!\!\!\!\downarrow$. Here we note that $\Gamma_1 \to \Delta_1$ is saturated, and so, every condition in Definition 4.3 for $\Gamma_1 \to \Delta_1$ holds. By Lemma 4.10(4), we have $\Box A \preceq \Box C \notin \Gamma_1$ and $\Box A \prec \Box C \notin \Gamma_1$. Also by Lemma 4.10(3), we have $\Box A \prec \Box C \in \mathsf{Sub}^+(\Gamma_1 \to \Delta_1)$. Using $A \neq C$ and Definition 4.3(11), we have

(4) either $\Box C \prec \Box A \in \Gamma_1$ or $\Box A \in \Delta_1$.

Using Definition 4.3(9) and Definition 4.3(7), we have either $\Box C \in \Gamma_1$ or $\Box A \in \Delta_1$. Using Lemma 4.16, we have either $\gamma_1 \models \Box C$ or $\gamma_1 \not\models \Box A$. Using (1),(2) and Lemma 4.20, we have

(5) all of $\gamma_1 \models \Box A$, $\gamma_1 \models \Box B$ and $\gamma_1 \models \Box C$ hold or none of them holds.

We put $\mathbf{W}' = \{\delta \in \gamma\downarrow\!\!\!\!\!\!\downarrow \mid \delta \models \Box A\}$. By $\gamma \models \Box A$, we have $\mathbf{W}' \neq \emptyset$. Also by Lemma 4.14, $\langle \mathbf{W}', < \rangle$ is linear(i.e., either one of $\delta < \zeta$, $\delta = \zeta$ and $\zeta < \delta$ holds for any $\delta, \zeta \in \mathbf{W}'$) and finite. So, there exists the minimum element $\gamma_2(= (\Gamma_2 \to \Delta_2; \tau_2))$ of $\mathbf{W}'$. By Lemma 4.16, we have $\Box A \notin \Delta_2$, and using (4), $\Box C \prec \Box A \in \Gamma_2$. Also by (5), we have $\gamma_2 \models \Box A$, $\gamma_2 \models \Box B$ and $\gamma_2 \models \Box C$. Since γ_2 is minimum, by (5), for any $\delta \in \gamma_2\downarrow$, $\delta \not\models \Box A$, $\delta \not\models \Box B$ and $\delta \not\models \Box C$. Using Definition 4.13(2.2), we obtain $\gamma \models \Box C \prec \Box B$.

Suppose that (3c) holds. We note that $\Gamma' \to \Delta'$ is saturated, and so, every condition in Definition 4.3 for $\Gamma' \to \Delta'$ holds. By (1), (2) and Definition 4.13(2.3), we have $\gamma \not\models \Box B \prec \Box A$ and $\gamma \not\models \Box C \prec \Box B$. So, it is sufficient to show one of the following two:

(6) $\gamma \models \Box B \prec \Box A$,

(7) $\gamma \models \Box C \prec \Box B$.

We show (7) if $\Box E \prec \Box B \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any E; and (6) if not.

Suppose that $\Box E \prec \Box B \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any E. To show (7), we show the condition corresponding to Definition 4.13(2.2.3), namely, the condition obtained from (3c) by replacing all the occurrences of A with B. There are three occurrences of A in (3c). By $\Box E \prec \Box B \notin \mathsf{Sub}^+(\Gamma \to \Delta)$ for any E, the condition obtained from (3c) by replacing the first occurrence holds. Using (2) and Lemma 4.20, the condition obtained from (3c) by replacing the first two occurrences holds. Using (1) and Lemma 4.20, we obtain the condition that we wanted to show.

Suppose that $\Box E \prec \Box B \in \mathsf{Sub}^+(\Gamma \to \Delta)$ for some E. To show (6), we show the condition corresponding to Definition 4.13(2.2.3), namely, the condition obtained from (3c) by replacing all the occurrences of C with B. There are three occurrences of C in (3c). Similarly to the above case, by (1), (2) and Lemma 4.20, the condition obtained from (3c) by replacing the last two occurrences holds.

So, we have only to show $\Box B \prec \Box D \in \Gamma'$. By (3c), we have

(8) $\Box C \prec \Box D \in \Gamma'$.

So, $\Box C \prec \Box D \in \mathsf{Sub}^+(\Gamma' \to \Delta')$. Also, by Lemma 4.10(3), we have $\Box E \prec \Box B \in \mathsf{Sub}^+(\Gamma \to \Delta) \subseteq \mathsf{Sub}^+(\Gamma' \to \Delta')$. Hence we have

(9) $\{\Box B \prec \Box C, \Box B \preceq \Box C, \Box C \preceq \Box D, \Box D \preceq \Box B, \Box B \prec \Box D\} \subseteq \mathsf{Sub}^+(\Gamma' \to \Delta')$.

By (2) and Lemma 4.16, we have $\Box B \preceq \Box C \notin \Delta$, and using Definition 4.3(7) and Lemma 4.10(4), $\Box C \prec \Box B \notin \Gamma$ and $\Box C \prec \Box B \notin \Gamma'$. By (3c), we have $\gamma' \models \Box C$, and using Lemma 4.16, we have $\Box C \notin \Delta'$. Using (9) and Definition 4.3(11), either $\Box B \prec \Box C$ or $\Box B \preceq \Box C$ belongs to Γ', and using Definition 4.3(7), we have

(10) $\Box B \preceq \Box C \in \Gamma'$

in both cases. By (8) and Lemma 4.18, we have $C \neq D$. Also by (8) and $\Box C \prec \Box B \notin \Gamma'$, we have $B \neq D$. Using (9) and Definition 4.3(10), either one of $\Box B \preceq \Box C \in \Delta'$, $\Box C \preceq \Box D \in \Delta'$ and $\Box B \preceq \Box D \in \Gamma'$ holds. However, the first one is in contradiction with (10) and Corollary 4.11(2). Also the second one is in contradiction with (8), Definition 4.3(9) and Corollary 4.11(2). Hence we have the third one, $\Box B \preceq \Box D \in \Gamma'$. Using Corollary 4.11(2) and Definition 4.3(7), we have $\Box B \preceq \Box D \notin \Delta'$ and $\Box D \prec \Box B \notin \Gamma'$. By $\gamma' \models \Box C$ of (3c), (2) and Lemma 4.20, we have $\gamma' \models \Box B$, and using Lemma 4.16, $\Box B \notin \Delta'$. Using (9) and Definition 4.3(11), we have either $\Box B \prec \Box D \in \Gamma'$ or $\Box D \preceq \Box B \in \Gamma'$. If $\Box D \preceq \Box B \in \Gamma'$, then by (10), (9), Corollary 4.11(2) and Definition 4.3(10), we have $\Box D \preceq \Box C \in \Gamma'$, and using Corollary 4.11(2), $\Box D \preceq \Box C \notin \Delta'$. This is in contradiction with (8) and Definition 4.3(7). Hence we obtain $\Box B \prec \Box D \in \Gamma'$. ⊣

By Lemma 4.14, Lemma 4.15, Lemma 4.19(2), Lemma 4.21 and Lemma 4.22, we obtain

Corollary 4.23 *$\mathcal{K}(S_0)$ is a Kripke model for* $\mathbf{R}^-$.

From Corollary 4.17 and Corollary 4.24, we obtain Theorem 4.1.

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