On the structure corresponding to Lindenbaum algebra of Lewis logic S4

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Abstract. The structure $\langle S/\equiv, \leq \rangle$ corresponds to Lindenbaum algebra of Lewis Logic S4 if $S/\equiv$ is the quotient set of the set $S$ of all formulas modulo the provability of S4, and $\leq$ is the derivation of S4. Here we treat the structure in the case that $S$ is the set of formulas constructed from a finite set $V$ of propositional variables and whose depth of $\square$ is less than a given number $n$. It is known that this structure is Boolean (cf. Chagrov and Zakharyaschev [CZ97]). So, we have only to elucidate its generators. We give an inductive construction of concrete representatives for the generators of the Boolean. The case that $V$ has only one variable has been treated in [Sas05]. We extend it to the case that $V$ is a finite set. Also we construct representatives without the provability of S4 while the result in [Sas05] almost depends on it.

1 Preliminaries

We use lower case Latin letters $p, q, p_1, p_2, \cdots$ for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\bot$ (contradiction) by using logical connectives $\land$ (conjunction), $\lor$ (disjunction), $\top$ (implication) and $\Box$ (necessitation). We use upper case Latin letters $A, B, C, \cdots, A_0, A_1, A_2, \cdots$ for formulas. For a finite set $S$, $\#(S)$ denotes the number of elements $S$.

Let $V$ be a finite set of propositional variables. $S(V)$ denotes the set of formulas constructed from propositional variables in $V$ and $\bot$ by using $\land, \lor, \top$ and $\Box$. The depth $d(A)$ of a formula $A$ is defined inductively as follows:

1. $d(A) = 0$ if $A$ is either a propositional variable or $\bot$,
2. $d(B \land C) = d(B \lor C) = d(B \top C) = \max\{d(B), d(C)\}$,
3. $d(\Box B) = d(B) + 1$.

We define $S^n(V)$ as $S^n(V) = \{ A \in S(V) \mid d(A) \leq n \}$.

Let $ENU$ be an enumeration of the formulas. For a non-empty finite set $S$ of formulas, the expressions

$$\bigwedge S$$

and

$$\bigvee S$$

denote the formulas

$$(\cdots((A_1 \land A_2) \land A_3)\cdots \land A_n)$$

and

$$(\cdots((A_1 \lor A_2) \lor A_3)\cdots \lor A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = S$ and $A_i$ occurs earlier than $A_{i+1}$ in $ENU$. Also the expressions

$$\bigwedge \emptyset$$

and

$$\bigvee \emptyset$$

denote the formulas $\bot \top \bot$ and $\bot$, respectively.

By S4, we mean the smallest set of formulas containing all the tautologies and the axioms

$K: \Box (p \top q) \top (\Box p \top \Box q),$

$T: \Box p \top p,$

$4: \Box p \top \Box \Box p$

and closed under modus ponens, substitution and necessitation.

We introduce a sequent system for S4 following Ohnishi and Matsumoto [OM57]. We use Greek letters, $\Gamma$ and $\Delta$, possibly with suffixes, for finite sets of formulas. The expressions $\Box \Gamma$ and $\Gamma \Box$ denote

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the sets \(\{\Box A \mid A \in \Gamma\}\) and \(\{\Box A \mid \Box A \in \Gamma\}\), respectively. By a sequent, we mean the expression \(\Gamma \rightarrow \Delta\). We often write \(\Gamma \rightarrow \Delta\) instead of the expression with the parenthesis. For brevity’s sake, we write

\[
A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_t \rightarrow \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n
\]

instead of

\[
\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_t \rightarrow \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.
\]

We use upper case Latin letters \(X, Y, Z, \ldots, X_0, X_1, X_2, \ldots\) for sequents. For a sequent \(\Gamma \rightarrow \Delta\), we define \(\text{ant}(\Gamma \rightarrow \Delta)\) and \(\text{suc}(\Gamma \rightarrow \Delta)\) as follows:

\[
\text{ant}(\Gamma \rightarrow \Delta) = \Gamma, \quad \text{suc}(\Gamma \rightarrow \Delta) = \Delta.
\]

Also for a sequent \(X\) and for a set \(S\) of sequents, we define \(\text{for}(X)\) and \(\text{for}(S)\) as follows:

\[
\text{for}(X) = \begin{cases} \bigwedge \text{ant}(X) \cup \bigvee \text{suc}(X) & \text{if } \Gamma \neq \emptyset \\ \bigvee \Delta & \text{if } \Gamma = \emptyset, \end{cases}
\]

\[
\text{for}(S) = \{\text{for}(X) \mid X \in S\}.
\]

By \(\text{GS4}\), we mean the system defined by the following axioms and inference rules in the usual way.

**Axioms of S4:**

\[
A \rightarrow A
\]

\[
\bot \rightarrow
\]

**Inference rules of S4:**

\[
\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}^{(w \rightarrow)} 
\]

\[
\frac{\Gamma \rightarrow \Delta, A}{\Gamma, \Pi \rightarrow \Delta, A}^{(\text{cut})}
\]

\[
\frac{A_1 \land A_2, \Gamma \rightarrow \Delta}{A_1 \land A_2, \Gamma \rightarrow \Delta}^{(\land \rightarrow)}
\]

\[
\frac{A, \Gamma \rightarrow \Delta, B, \Gamma \rightarrow \Delta}{A \lor B, \Gamma \rightarrow \Delta}^{(\lor \rightarrow)}
\]

\[
\frac{\Gamma \rightarrow \Delta, A}{A \lor B, \Pi \rightarrow \Delta}^{(\lor \rightarrow)}
\]

\[
\frac{A_1 \rightarrow \Delta}{\box A, \Gamma \rightarrow \Delta}^{(\Box \rightarrow)}
\]

\[
\frac{\Gamma \rightarrow \Delta, A}{\Box A, \Gamma \rightarrow \Delta}^{(\Box \rightarrow)}
\]

\[
\frac{\Gamma \rightarrow \Delta, A}{\Box A, \Gamma \rightarrow \Delta}^{(\Box \rightarrow)}
\]

**Lemma 1.1 ([OM57])**

1. \(\Gamma \rightarrow \Delta \in \text{GS4}\) if and only if \(\text{for}(\Gamma \rightarrow \Delta) \in \text{S4}\).
2. If \(\Gamma \rightarrow \Delta \in \text{GS4}\), then there exists a cut-free proof figure for \(\Gamma \rightarrow \Delta\) in \(\text{GS4}\).

By the lemma above, we can identify \(\text{GS4}\) with \(\text{S4}\). So, if there is no confusion, we use \(\text{S4}\) as the sequent system \(\text{GS4}\).
2 Main results

Here we consider the structure \((S^n(\{p_1, \ldots, p_m\})/ \equiv, \subseteq)\), where \(A \equiv B\) if and only if \((A \supset B) \land (B \supset A) \in S_4\); and \([A] \subseteq [B]\) if and only if there exist \(A' \in [A]\) and \(B' \in [B]\) such that \(A' \supset B' \in S_4\).

Our main purpose is to give a concrete representative of each equivalence class of \((S^n(\{p_1, \ldots, p_m\})/ \equiv)\) in an inductive way and elucidate the structure. Since the structure is Boolean, we mainly construct representatives for generators. From now on, we fix the set \(\{p_1, \ldots, p_m\}\) and write \(V\).

**Definition 2.1** The sets \(G(n)\) and \(G^*(n)\) \((n = 0, 1, 2, \ldots)\) of sequents, and the mappings \(\text{next}^+, \text{prov}\), \(\text{next}\) are defined inductively as follows:

\[
G(0) = \{(V - V_1 \rightarrow V_1) \mid V_1 \subseteq V\},
\]

\[
G^*(0) = \emptyset,
\]

\[
\text{next}^+(X) = \{\langle \Gamma, \text{ant}(X) \rightarrow \text{suc}(X), \Box \Delta \rangle \mid \Gamma \cup \Delta = \text{for}(G(n)), \Gamma \cap \Delta = \emptyset, \text{for}(X) \in \Delta\}, \text{ for } X \in G(k),
\]

\[
\text{prov}(X) = \{Y \in \text{next}^+(X) \mid Y \in S_4\}, \text{ for } X \in G(k),
\]

\[
\text{next}(X) = \text{next}^+(X) - \text{prov}(X), \text{ for } X \in G(k),
\]

\[
G(k + 1) = G(k) \cup \bigcup_{X \in G(k) - G^*(k)} \text{next}(X),
\]

\[
G^*(k + 1) = \{X \in G(k + 1) \mid (\text{ant}(X))^\Box \subseteq (\text{ant}(Y))^\Box \text{ implies } (\text{ant}(X))^\Box = (\text{ant}(Y))^\Box, \text{ for any } Y \in G(k + 1)\}.
\]

Here we use the provability of \(S_4\), but in section 4, this provability will be replaced another conditions concerning the structure of sequents.

**Definition 2.2** We define \(G^n\) as follows:

\[
G^n = G(n) \cup \bigcup_{k=0}^{n-1} G^*(k).
\]

In the following theorem, it is shown that the above \(G^n\) is the set of representatives for the generators of \((S^n(\{p_1, \ldots, p_m\}), \subseteq)\).

**Theorem 2.3**

(1) \(S^n(V)/ \equiv = \{[\bigwedge \text{for}(S)] \mid S \subseteq G^n\}\).

(2) For subsets \(S_1\) and \(S_2\) of \(G^n\),

\[
\begin{align*}
&\quad \text{(2.1) } S_2 \subseteq S_1 \text{ if and only if } [\bigwedge \text{for}(S_2)] \subseteq [\bigwedge \text{for}(S_1)], \\
&\quad \text{(2.2) } S_1 = S_2 \text{ if and only if } [\bigwedge \text{for}(S_1)] = [\bigwedge \text{for}(S_2)].
\end{align*}
\]

In the next section, we prove (1) in the above theorem. In other words, we show that every equivalent class in \(S^n(V)/ \equiv\) has a representative \(\bigwedge \text{for}(S)\) for some subset \(S\) of \(G^n\). Here we prove (2) in the above theorem. To prove (2), we need some lemmas.

**Lemma 2.4**

(1) \(G(n) \subseteq S^n(V) - S^{n-1}(V)\).

(2) every member of \(G(n)\) is not provable in \(S_4\).

**Proof.** By an induction on \(n\).

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**Lemma 2.5** For any \(X, Y \in G^n\), \(X \neq Y\) implies \(\text{for}(X) \lor \text{for}(Y) \in S_4\).
Proof. We use an induction on $n$.

Basis ($n = 0$). We have $X, Y \in G^0 = \mathbb{G}(0)$. So, there exist subsets $V_1, V_2$ of $V$ such that $X = (V - V_1 \rightarrow V_1)$, $Y = (V - V_2 \rightarrow V_2)$ and $V_1 \neq V_2$. By $V_1 \neq V_2$, we have either $V_1 \cap (V - V_2) \neq \emptyset$ or $V_2 \cap (V - V_1) \neq \emptyset$. Hence either $\text{succ}(X) \cap \text{ant}(Y) \neq \emptyset$ or $\text{succ}(Y) \cap \text{ant}(X) \neq \emptyset$, and so, we obtain the lemma.

Induction step ($n \geq 1$). We divide the cases.

The case that $\{X, Y\} \subseteq G(n)$. There exist sequents $X_0, Y_0 \in G(n - 1) - G^*(n - 1)$ such that $X \in \text{next}(X_0)$ and $Y \in \text{next}(Y_0)$. So, there exist sets $\Gamma_X, \Delta_X, \Delta_Y$ of formulas such that

1. $X = (\bigotimes \Gamma_X, \text{ant}(X_0) \rightarrow \text{succ}(X_0), \Delta_X)$, $Y = (\bigotimes \Gamma_Y, \text{ant}(Y_0) \rightarrow \text{succ}(Y_0), \Delta_Y)$,
2. $\Gamma_X \cup \Delta_X = \Gamma_Y \cup \Delta_Y = \text{for}(G(n - 1))$,
3. $\Gamma_X \cap \Delta_X = \Gamma_Y \cap \Delta_Y = \emptyset$,
4. $\text{for}(X_0) \in \Delta_X, \text{for}(Y_0) \in \Delta_Y$.

If $X_0 \neq Y_0$, then by the induction hypothesis, $\text{for}(X_0) \lor \text{for}(Y_0) \in S^4$, and so, we obtain the lemma. Suppose that $X_0 = Y_0$. Then by $X \neq Y$, we have either $\Gamma_X \neq \Gamma_Y$ or $\Delta_X \neq \Delta_Y$, and using (2) and (3), we have both. Without loss of generality, we can suppose that $\Gamma_X \subseteq \Gamma_Y$. So, there exists a formula $A \in \Gamma_X - \Gamma_Y$, and using (2) and (3), $A \in \Gamma_X \cap \Delta_Y$. So, we have $\bigotimes \Gamma_X \rightarrow \Delta_Y \in S^4$. We note $\bigotimes \Gamma_X, \text{for}(X) \in S^4$ and $\Delta_Y \rightarrow \text{for}(Y) \in S^4$. Using (cut), possibly several times, we obtain $\rightarrow \text{for}(X), \text{for}(Y) \in S^4$, and hence we obtain the lemma. In the following we show how to use (cut) if each one of $\Gamma_X$ and $\Delta_Y$ has only one element:

$$
\frac{\rightarrow \bigotimes \Gamma_X, \text{for}(X) \quad \bigotimes \Gamma_X \rightarrow \Delta_Y \quad \text{(cut)} \quad \Delta_Y \rightarrow \text{for}(Y) \quad \text{(cut)}}{\rightarrow \text{for}(X), \text{for}(Y)}.
$$

The case that $\{X, Y\} \nsubseteq G(n)$. There exists $Z \in \{X, Y\} - G(n)$. Without loss of generality, we can suppose that $Z = Y \notin G(n)$, and then $Y \in \bigcup_{k=0}^{n-1} G^*(k) \subseteq G(n - 1)$. If $X \notin G(n)$, then $X \in \bigcup_{k=0}^{n-1} G^*(k) \subseteq G(n - 1)$. Using the induction hypothesis, we obtain the lemma. So, we assume that $X \in G(n)$. Then there exist $X_0 \in G(n - 1) - G^*(n - 1)$ such that $X \in \text{next}(X_0)$. By $Y \in \bigcup_{k=0}^{n-1} G^*(k)$ and Lemma 2.4(1), we have $Y \neq X_0$. By the induction hypothesis, we have $\text{for}(X_0) \lor \text{for}(Y) \in S^4$. We note that $\text{for}(X_0) \rightarrow \forall x \in X \forall y \in Y$. Using (cut), we obtain the lemma.

Proof of Theorem 2.5(2). (2.1) is clear. The “if part” of (2.2) is also clear. We show the “only if” part of (2.2). Suppose that $S_1 \nsubseteq S_2$. Then there exists a sequent $X \in S_1 - S_2$. By Lemma 2.5, we have $\text{for}(X) \lor \bigwedge \text{for}(S_2) \in S^4$. By Lemma 2.4(2), we have $\text{for}(X) \notin S^4$. Also we have $\bigwedge \text{for}(S_1) \rightarrow \text{for}(X) \in S^4$. Hence considering the figure

$$
\frac{\rightarrow \text{for}(X) \lor \bigwedge \text{for}(S_2) \quad \bigwedge \text{for}(S_2) \rightarrow \text{for}(X) \quad \text{(cut)}}{\rightarrow \text{for}(X) \lor \bigwedge \text{for}(S_2) \rightarrow \text{for}(X) \quad \text{for}(X) \lor \bigwedge \text{for}(S_2) \rightarrow \text{for}(X) \quad \text{(cut)},}
$$

we obtain $\bigwedge \text{for}(S_2) \rightarrow \bigwedge \text{for}(S_1) \notin S^4$. Similarly, we can show that $S_2 \nsubseteq S_1$ implies $\bigwedge \text{for}(S_1) \rightarrow \bigwedge \text{for}(S_2) \notin S^4$.

3 Representatives of the equivalent classes in $S^n(V)/\equiv$

Here we prove the following theorem.
Theorem 3.1 For any $A \in S^n(V)$, there exists a subset $S$ of $G^n$ such that $A \equiv \bigwedge \text{for}(S)$.

From the above theorem, we obtain Theorem 3.1(1), and that every equivalent class in $S^n(V)/\equiv$ has a representative $\bigwedge \text{for}(S)$ for some subset $S$ of $G^n$. To prove Theorem 3.1, we need some lemmas.

Lemma 3.2 For any subsets $S_1$ and $S_2$ of $G^n$,

1. $\bigwedge S_1 \wedge \bigwedge S_2 \equiv \bigwedge (S_1 \cup S_2)$,
2. $\bigwedge S_1 \vee \bigwedge S_2 \equiv \bigwedge (S_1 \cap S_2)$.

Proof. (1) is clear. We show (2). Let $A$ be in $S_1$. Then by Lemma 2.5, we have $A \vee B \in S_4$ for any $B \in S_2 - \{A\}$. So, if $A \in S_2$, then

$$A \vee \bigwedge S_2 \equiv (A \vee A) \wedge (A \vee \bigwedge (S_2 - \{A\}))$$

$$\equiv A \wedge (A \vee \bigwedge (S_2 - \{A\}))$$

$$\equiv A;$$

if not,

$$A \vee \bigwedge S_2 \equiv \bigwedge \{A \vee B \mid B \in S_2\}$$

$$\equiv p \supset p.$$ 

Hence

$$\bigwedge S_1 \vee \bigwedge S_2 \equiv \bigwedge \{A \vee \bigwedge S_2 \mid A \in S_1\}$$

$$\equiv \bigwedge \{A \vee \bigwedge S_2 \mid A \in S_1 - S_2\} \wedge \bigwedge \{A \vee \bigwedge S_2 \mid A \in S_1 \cap S_2\}$$

$$\equiv (p \supset p) \wedge \bigwedge \{A \mid A \in S_1 \cap S_2\}$$

$$\equiv \bigwedge (S_1 \cap S_2).$$

Lemma 3.3 Let $\Sigma, \Gamma, \Gamma_1, \Delta, \Delta_1$ be finite sets of formulas. Then for any subset $\Sigma' \subseteq \Sigma$,

$$\Box \Sigma', \{\text{for}(\Box \Gamma, \Box \Phi, \Gamma_1 \rightarrow \Delta_1, \Box \Psi, \Box \Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}, \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4.$$ 

Proof. We define $S$ as follows:

$$S = \{\text{for}(\Box \Gamma, \Box \Phi, \Gamma_1 \rightarrow \Delta_1, \Box \Psi, \Box \Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\},$$

and prove

$$\Box \Sigma', S, \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4.$$ 

We use an induction on $\#(\Sigma - \Sigma')$.

Basis($\Sigma' = \Sigma$). We note that

$$\text{for}(\Box \Gamma, \Box \Sigma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta) \in S$$

and

$$\Box \Sigma, \text{for}(\Box \Gamma, \Box \Sigma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta), \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4.$$

Using weakening rule, we obtain the lemma.

Induction step($\Sigma' \neq \Sigma$). By the induction hypothesis, for any $A \in \Sigma - \Sigma'$,

$$\Box (\Sigma' \cup \{A\}), S, \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4.$$ 

Using ($\vee \rightarrow$), possibly several times,

$$\Box \Sigma', \bigvee (\Box (\Sigma - \Sigma'))), S, \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4.$$
Hence using (⊣), possibly several times,
\[ \Box \Sigma', \bigvee (\Delta_1 \cup \Box \Delta \cup \Box (\Sigma - \Sigma')), \Sigma, \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4. \]

Using (⊢→), possibly several times,
\[ \Box \Sigma', \text{for} (\Box \Gamma, \Gamma_1, \Box \Sigma' \rightarrow \Delta_1, \Box \Delta, \Box (\Sigma - \Sigma')), \Sigma, \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4. \]

We note that
\[ \text{for} (\Box \Gamma, \Gamma_1, \Box \Sigma' \rightarrow \Delta_1, \Box \Delta, \Box (\Sigma - \Sigma')) \in \mathcal{S}, \]
and so,
\[ \Box \Sigma', \Sigma, \Box \Gamma, \Gamma_1 \rightarrow \Delta_1, \Box \Delta \in S_4. \]

**Corollary 3.4** Let \( X \) be a sequent in \( G(n) \) and let \( Y \) be a sequent in \( G_\ell \). Then

1. \( \text{for}(\text{next}(X)) \rightarrow \text{for}(X) \in S_4 \),
2. \( \bigwedge \text{for}(\text{next}(X)) \equiv \text{for}(X) \),
3. \( \{ \text{for}(Z) \mid Z \in \text{next}(X), \Box \text{for}(Y) \in \text{suc}(Z) \} \rightarrow \text{for}(X), \Box \text{for}(Y) \in S_4 \).

**Proof.** Considering the case that \( \Gamma = \emptyset, \Gamma_1 = \text{ant}(X), \Delta = \{ \text{for}(X) \}, \Delta_1 = \text{suc}(X), \Sigma = \text{for}(G(n)) - \{ \text{for}(X) \} \) and \( \Sigma' = \emptyset \) in the above lemma, we have \( \text{for}(\text{next}^+(X)) \rightarrow \text{for}(X), \Box \text{for}(X) \in S_4 \). Using (cut), possibly several times, we obtain (1). (2) follows from (1). Also considering the case that \( \Gamma = \emptyset, \Gamma_1 = \text{ant}(X), \Delta = \{ \text{for}(X), \text{for}(Y) \}, \Delta_1 = \text{suc}(X), \Sigma = \text{for}(G(n)) - \{ \text{for}(X), \text{for}(Y) \} \) and \( \Sigma' = \emptyset \) in the above lemma, we obtain (3) similarly to (1).

**Definition 3.5** We define \( B G_\ell \) as follows:
\[ B G_\ell = V \cup \bigcup_{i=0}^{\ell-1} \Box \text{for}(G(i)). \]

**Lemma 3.6** Let \( X \) be a sequent in \( G(n) \). Then

1. \( \text{ant}(X) \cup \text{suc}(X) = B G_n \),
2. \( \text{ant}(X) \cap \text{suc}(X) = \emptyset \).

**Proof.** By Lemma 2.4(1) and an induction on \( n \).

**Lemma 3.7** Let \( X \) and \( Y \) be sequents in \( G(n) \). Then
\[ (\text{ant}(X))^\Box \not\subseteq (\text{ant}(Y))^\Box \text{ implies } (\rightarrow \text{for}(X), \Box \text{for}(Y)) \in S_4. \]

**Proof.** By \( (\text{ant}(X))^\Box \not\subseteq (\text{ant}(Y))^\Box \), there exists a formula \( \Box A \in (\text{ant}(X))^\Box - (\text{ant}(Y))^\Box \). Using Lemma 3.6, we have \( \Box A \in (\text{ant}(X))^\Box \cap (\text{suc}(Y))^\Box \). So,
\[ \Box A \rightarrow \text{suc}(Y) \in S_4. \]
Hence
\[ \Box A \rightarrow \text{for}(Y) \in S_4. \]
Using \( (\rightarrow \Box) \),
\[ \Box A \rightarrow \Box \text{for}(Y) \in S_4. \]
Using weakening rule,
\[ \text{ant}(X) \rightarrow \text{suc}(X), \Box \text{for}(Y) \in S_4. \]
Hence we obtain the lemma.
Lemma 3.8 Let $X$ be a sequent in $G^*(n)$ and let $Y$ be a sequent in $G(n)$ satisfying $(\ant(X))^\mathsf{D} = (\ant(Y))^\mathsf{D}$. Then

$$\Box \for(Y) \rightarrow \for(X) \in S4.$$ Proof. If $n = 0$, then the lemma is clear from $G^*(0) = \emptyset$. Also, if $X = Y$, then the lemma is clear. So, we assume $n > 0$ and $X \neq Y$. By $X \in G^*(n)$ and $Y \in G(n)$, we have

- $X = s\Gamma_X, \ant(X_0) \rightarrow \suc(X_0, \Box \Delta_X), Y = s\Gamma_Y, \ant(Y_0) \rightarrow \suc(Y_0, \Box \Delta_Y)$,
- $\Gamma_X \cup \Delta_X = \Gamma_Y \cup \Delta_Y = \for(G(n))$,
- $\Gamma_X \cap \Delta_X = \Gamma_Y \cap \Delta_Y = \emptyset$,
- $\for(X_0) \in \Delta_X, \for(Y_0) \in \Delta_Y$.

Also we have

- $X \notin S4$, $Y \notin S4$.

By $Y \in G(n)$ and Corollary 3.4(1),

$$\for(\next(X_0)) \rightarrow \for(Y_0) \in S4.$$ Using $\Box \rightarrow \Box$ and $\Box \rightarrow \Box$,

$$\Box \for(\next(X_0)) \rightarrow \Box \for(Y_0) \in S4.$$ By $\ant(X)^\mathsf{D} = \ant(Y)^\mathsf{D}$, (1) and Lemma 2.4(1), we have $\Gamma_X = \Gamma_Y$. Using (2),(3) and (4), we have

$$\Box \for(Y_0) \in \Box \Delta_Y = \Box \Delta_X, \text{ and so, } \Box \for(Y_0) \rightarrow \for(X) \in S4.$$ Using (cut),

$$\Box \for(\next(X_0)) \rightarrow \for(X) \in S4,$$

that is,

$$\Box \for((Z \in \next(Y_0) \mid (\ant(X))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D} \text{ or } (\ant(X))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D})) \rightarrow \for(X) \in S4.$$ By $X \in G^*(n)$, we have that $(\ant(X))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D}$ if and only if $(\ant(X))^\mathsf{D} = (\ant(Z))^\mathsf{D}$, and so,

$$\Box \for((Z \in \next(Y_0) \mid (\ant(X))^\mathsf{D} = (\ant(Z))^\mathsf{D} \text{ or } (\ant(X))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D})) \rightarrow \for(X) \in S4.$$ By $\ant(X)^\mathsf{D} = \ant(Y)^\mathsf{D}$, (1) and Lemma 3.6, we have

$$(Z \in \next(Y_0) \mid (\ant(X))^\mathsf{D} = (\ant(Z))^\mathsf{D}) = \{Y\},$$

and so,

$$\Box \for(Y), \Box \for((Z \in \next(Y_0) \mid (\ant(X))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D})) \rightarrow \for(X) \in S4.$$ Using $(\lor \rightarrow \lor)$, possibly several times,

$$\Box \for(Y), \{\for(X) \lor \Box \for(Z) \mid Z \in \next(Y_0), (\ant(X))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D}\} \rightarrow \for(X) \in S4.$$ Using Lemma 3.7, and (cut), possibly several times, we obtain the lemma. $\dashv$

Lemma 3.9 Let $X$ and $Y$ be sequents in $G(n)$ satisfying $(\ant(X))^\mathsf{D} = (\ant(Y))^\mathsf{D}$. Then $X \in G^*(n)$ if and only if $Y \in G^*(n)$.

Proof. From the definition of $G^*(n)$,

- $X \in G^*(n)$ if and only if $(\ant(X))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D}$ implies $(\ant(X))^\mathsf{D} = (\ant(Z))^\mathsf{D}$, for any $Z \in G(n)$,
- $Y \in G^*(n)$ if and only if $(\ant(Y))^\mathsf{D} \subseteq (\ant(Z))^\mathsf{D}$ implies $(\ant(Y))^\mathsf{D} = (\ant(Z))^\mathsf{D}$, for any $Z \in G(n)$.

Using $(\ant(X))^\mathsf{D} = (\ant(Y))^\mathsf{D}$, we obtain the lemma. $\dashv$

Definition 3.10 We define a mapping $\text{cf}$ as follows:

$$\text{cf}(X) = \begin{cases} \bigwedge \for((Y \in G(n) \mid (\ant(X))^\mathsf{D} = (\ant(Y))^\mathsf{D})) & \text{if } X \in G^*(n) \\ \perp \cup \perp & \text{if } X \in G(n) - G^*(n) \end{cases}$$
Lemma 3.11 Let \( X \) be a sequent in \( G(n) \) and let \( \Sigma \) be a subset of \((\text{ant}(X))^\circ\). Then

\[
\Sigma, \text{cf}(X), \Phi \rightarrow \Box \text{for}(X) \in S4.
\]

where \( \Phi = \{\text{for}(Y) \mid Y \in G(n) - G^*(n), (\text{ant}(Y))^\circ \subseteq (\text{ant}(X))^\circ\} \).

Proof. We use an induction on \( \omega n + \#((\text{ant}(X))^\circ - \Sigma) \).

Basis (\( n = 0 \)). We note that \((\text{ant}(X))^\circ = \emptyset\) and for any \( Y \in G(0) - G^*(0) = G^*(0) \), \((\text{ant}(Y))^\circ = \emptyset\). Hence \( \Phi = G(0) \). So, it is not hard to see that \( \Phi \rightarrow \in S4 \). Hence we obtain the lemma.

Induction step (\( n > 0 \)). By \( n > 0 \), there exists a sequent \( X_0 \in G(n - 1) - G^*(n - 1) \) such that \( X \in \text{next}(X_0) \). By the induction hypothesis,

\[
\perp \supset \perp, \{\text{for}(Y_0) \mid Y_0 \in G(n - 1) - G^*(n - 1), (\text{ant}(Y_0))^\circ \subseteq (\text{ant}(X_0))^\circ\} \rightarrow \Box \text{for}(X_0) \in S4.
\]

Since \((\Box \text{for}(X_0) \rightarrow \Box \text{for}(X)), (\perp \rightarrow \perp) \in S4\), using (cut), twice,

\[
\{\text{for}(Y_0) \mid Y_0 \in G(n - 1) - G^*(n - 1), (\text{ant}(Y_0))^\circ \subseteq (\text{ant}(X_0))^\circ\} \rightarrow \Box \text{for}(X) \in S4.
\]

Using weakening rule,

\[
\Sigma, \Phi, \{\text{for}(Y_0) \mid Y_0 \in G(n - 1) - G^*(n - 1)\} \rightarrow \Box \text{for}(X) \in S4. \tag{1}
\]

On the other hand, by the induction hypothesis,

\[
\Sigma, \text{cf}(X), \Phi, A \rightarrow \Box \text{for}(X) \in S4, \tag{2}
\]

for any formula \( A \in (\text{ant}(X))^\circ - \Sigma \). (2) also holds for any \( A \in (\text{suc}(X))^\circ \), and so, for any \( A \in G(n - 1) - \Sigma \). Let \( Y \) be a sequent in \( G(n) \) such that \((\text{ant}(Y))^\circ = \Sigma\). Then (2) holds for any \( A \in G(n - 1) - (\text{ant}(Y))^\circ = (\text{suc}(Y))^\circ \). We note that \(\text{suc}(Y) = \{\text{for}(Y_0)\} \cup (\text{suc}(Y))^\circ\) if \( Y \in \text{next}(Y_0) \), so using (1) and (\( \lor \rightarrow \)), possibly several times,

\[
\Sigma, \text{cf}(X), \Phi, \{\bigvee\text{suc}(Y) \mid Y \in \bigcup_{Y_0 \in G(n - 1) - G^*(n - 1)} \text{next}(Y_0), (\text{ant}(Y))^\circ = \Sigma\} \rightarrow \Box \text{for}(X) \in S4.
\]

Also we have that \((\text{ant}(Y))^\circ = \Sigma\) implies \( \Sigma \rightarrow \forall \text{ant}(Y) \in S4 \), for any \( Y \in G(n) \); so using (\( \supset \rightarrow \)),

\[
\Sigma, \text{cf}(X), \Phi, \{\text{for}(Y) \mid Y \in G(n), (\text{ant}(Y))^\circ = \Sigma\} \rightarrow \Box \text{for}(X) \in S4.
\]

Using (\( w \rightarrow \)), possibly several times,

\[
\Sigma, \text{cf}(X), \Phi, \{\text{for}(Y) \mid Y \in G(n), (\text{ant}(Y))^\circ \subseteq (\text{ant}(X))^\circ\} \rightarrow \Box \text{for}(X) \in S4.
\]

Using the definition of \( \Phi\),

\[
\Sigma, \text{cf}(X), \Phi, \{\text{for}(Y) \mid Y \in G^*(n), (\text{ant}(Y))^\circ \subseteq (\text{ant}(X))^\circ\} \rightarrow \Box \text{for}(X) \in S4.
\]

Using the definition of \( G^*(n)\),

\[
\Sigma, \text{cf}(X), \Phi, \{\text{for}(Y) \mid Y \in G^*(n), (\text{ant}(Y))^\circ = (\text{ant}(X))^\circ\} \rightarrow \Box \text{for}(X) \in S4. \tag{3}
\]

If \( X \not\in G^*(n) \), then by Lemma 3.9, \((\text{ant}(Y))^\circ = (\text{ant}(X))^\circ\) implies \( Y \not\in G^*(n) \), and so,

\[
\{\text{for}(Y) \mid Y \in G^*(n), (\text{ant}(Y))^\circ = (\text{ant}(X))^\circ\} = \emptyset \subseteq \text{cf}(X).
\]

If \( X \in G^*(n) \), then from Definition 3.10, we also have

\[
\{\text{for}(Y) \mid Y \in G^*(n), (\text{ant}(Y))^\circ = (\text{ant}(X))^\circ\} \subseteq \text{cf}(X).
\]

So, the above condition also holds in any case. Using (3), we obtain the lemma.
Lemma 3.12 Let $X$ be a sequent in $G(n)$. Then

\[ \Box \text{for}(X) \equiv \text{cf}(X) \land \bigwedge \{ \text{for}(X_1) \mid X_1 \in G(n + 1), \Box \text{for}(X) \in \text{suc}(X_1) \} . \]

Proof. By Lemma 3.8 and $\rightarrow \land$, possibly several times,

\[ \Box \text{for}(X) \rightarrow \text{cf}(X) \in S4. \]

Also we note that

\[ \Box \text{for}(X) \rightarrow \bigwedge \{ \text{for}(X_1) \mid X_1 \in G(n + 1), \Box \text{for}(X) \in \text{suc}(X_1) \} \in S4. \]

Using $\rightarrow \land$,

\[ \Box \text{for}(X) \rightarrow \text{cf}(X) \land \bigwedge \{ \text{for}(X_1) \mid X_1 \in G(n + 1), \Box \text{for}(X) \in \text{suc}(X_1) \} \in S4. \]

We show the converse. By Corollary 3.4(3), for any $Y \in G(n) - G^*(n)$,

\[ \{ \text{for}(Y_1) \mid Y_1 \in \text{next}(Y), \Box \text{for}(X) \in \text{suc}(Y_1) \} \rightarrow \text{for}(Y), \Box \text{for}(X) \in S4. \]

Using $\rightarrow \land$, possibly several times,

\[ \{ \text{for}(Y_1) \mid Y_1 \in \bigcup_{Y \in G(n) - G^*(n)} \text{next}(Y), \Box \text{for}(X) \in \text{suc}(Y_1) \} \rightarrow \bigwedge \text{for}(G(n) - G^*(n)), \Box \text{for}(X) \in S4. \]

On the other hand, by Lemma 3.11,

\[ \text{cf}(X), \bigwedge \text{for}(G(n) - G^*(n)) \rightarrow \Box \text{for}(X) \in S4. \]

Using (cut),

\[ \text{cf}(X), \{ \text{for}(Y_1) \mid Y_1 \in \bigcup_{Y \in G(n) - G^*(n)} \text{next}(Y), \Box \text{for}(X) \in \text{suc}(Y_1) \} \rightarrow \Box \text{for}(X) \in S4. \]

Hence we obtain the lemma.

\[ \dashv \]

\[ \text{Lemma 3.13} \]

\[ \bot \equiv \bigwedge \text{for}(G^n). \]

Proof. We use an induction on $n$.

Basis ($n = 0$). It is not hard to see

\[ \bot \equiv \bigwedge \text{for}(G^0). \]

Induction step ($n > 0$). By the induction hypothesis,

\[ \bot \equiv \bigwedge \text{for}(G^{n-1}). \]

So,

\[ \bot \equiv \bigwedge \bigcup_{k=0}^{n-1} \text{for}(G^*(n-1)) \land \bigwedge \text{for}(G(n-1) - G^*(n-1)). \]

Using Corollary 3.4(2),

\[ \bot \equiv \bigwedge \bigcup_{k=0}^{n-1} \text{for}(G^*(n-1)) \land \bigwedge \text{for}(X \in G(n-1) - G^*(n-1)). \]

Hence

\[ \bot \equiv \bigwedge \bigcup_{k=0}^{n-1} \text{for}(G^*(n-1)) \land \bigwedge \text{for}(G(n)) \equiv \bigwedge \text{for}(G^n). \]

\[ \dashv \]
Lemma 3.14 For a subset $S$ of $G^n$

$$\bigwedge \text{for}(S) \cup \bot \equiv \bigwedge \text{for}(G^n - S).$$

**Proof.** By Lemma 3.13,

$$\bigwedge \text{for}(G_n - S) \rightarrow \bigwedge \text{for}(S) \cup \bot \in S4.$$

By Lemma 2.5,

$$\rightarrow \bigwedge \text{for}(S), \bigwedge \text{for}(G^n - S) \in S4.$$

Using ($\cup \rightarrow$),

$$\bigwedge \text{for}(S) \cup \bot \rightarrow \bigwedge \text{for}(G^n - S) \in S4.$$

Proof of Theorem 3.1. We use an induction on $n$.

Basis ($n = 0$). The theorem follows from the results in Classical propositional logic.

Induction step ($n > 0$). We use an induction on $A$.

If $A = \bot$, then from Lemma 3.13, we obtain the lemma.

If $A$ is a propositional variable $p_i$, then by the induction hypothesis, there exists a subset $S \subseteq G^{n-1}$ such that $p_i \equiv \bigwedge \text{for}(S)$. So,

$$p_i \equiv \bigwedge \text{for}((S \cap (G(n - 1) - G^*(n - 1))) \cup (S \cap (\bigcup_{k=0}^{n-1} G^*(k)))).$$

Using Corollary 3.4(2),

$$p_i \equiv \bigwedge \text{for}((\bigcup_{X \in S \cap (G(n - 1) - G^*(n - 1))} \text{next}(X)) \cup (S \cap (\bigcup_{k=0}^{n-1} G^*(k)))).$$

We note that

$$(\bigcup_{X \in S \cap (G(n - 1) - G^*(n - 1))} \text{next}(X)) \cup (S \cap (\bigcup_{k=0}^{n-1} G^*(k))) \subseteq G^n.$$

If $A = B \land C$, then by the induction hypothesis, there exist subsets $S_B$ and $S_C$ of $G^n$ such that

$$B \equiv \bigwedge \text{for}(S_B), \quad \text{and} \quad C \equiv \bigwedge \text{for}(S_C).$$

Using Lemma 3.2,

$$B \land C \equiv \bigwedge \text{for}(S_B) \land \bigwedge \text{for}(S_C) \equiv \bigwedge \text{for}(S_B \cup S_C).$$

Similarly, if $A = B \lor C$, then

$$B \lor C \equiv \bigwedge \text{for}(S_B \cap S_C).$$

Also, if $A = B \supset C$, then using Lemma 3.13,

$$B \supset C \equiv (B \supset \bot) \lor C \equiv \bigwedge \text{for}((G^n - S_B) \cap S_C).$$

If $A = \square B$, then $B \in S^{n-1}(V)$, using the induction hypothesis, there exists a subset $S$ of $G^{n-1}$ such that

$$B \equiv \bigwedge \text{for}(S).$$

Hence

$$A = \square B \equiv (\bigwedge \neg \text{for}(S \cap G(n - 1))) \land (\bigwedge \neg \text{for}(S \cap (\bigcup_{k=0}^{n-2} G^*(k)))).$$
By Lemma 3.12 and Lemma 3.9,
\[
\bigwedge \square \text{for}(S \cap G(n-1)) \equiv \bigwedge \bigcup_{X \in S \cap G(n-1)} (\text{cf}(X) \land \bigwedge \{\text{for}(X_1) \mid X_1 \in G(n), \square \text{for}(X) \in \text{suc}(X_1)\}).
\]
\[
\equiv \bigwedge \text{for}(S_1 \cup S_2),
\]
where
\[
S_1 = \bigcup_{X \in S \cap G^*(n-1)} \{Y \in G^*(n-1) \mid (\text{ant}(X))^\square = (\text{suc}(Y))^\square\},
\]
\[
S_2 = \bigcup_{X \in S \cap G(n-1)} \{X_1 \in G(n) \mid \square \text{for}(X) \in \text{suc}(X_1)\}.
\]
On the other hand, by the induction hypothesis, there exists a subset \(T\) of \(G^{n-1}\) such that
\[
\bigwedge \square \text{for}(S \cap (\bigcup_{k=0}^{n-2} G^*(k))) \equiv \bigwedge \text{for}(T).
\]
Using Corollary 3.4(2),
\[
\bigwedge \square \text{for}(S \cap (\bigcup_{k=0}^{n-2} G^*(k))) \equiv \bigwedge \text{for}(T) \equiv \bigwedge \text{for}(S_3),
\]
where
\[
S_3 = (\bigcup_{X \in T \cap (G(n-1) - G^*(n-1))} \text{next}(X)) \lor (T \cap \bigcup_{k=0}^{n-1} G^*(k)).
\]
Hence
\[
A = \square B \equiv \bigwedge \text{for}(S_1 \cup S_2 \cup S_3)
\]
and we note that \(S_1 \cup S_2 \cup S_3 \subseteq G^n\).

4 On \text{prov}(X)

In Definition 4.1, we use the provability of \(S4\) to define \text{prov}(X) for \(X \in G(n)\). In this section, we give the set without using the provability of \(S4\).

Definition 4.1 For \(X \in G(n)\), we define \text{prov}_1(X), \text{prov}_2(X)\) and \text{prov}_3(X)\) as follows:
\[
\text{prov}_1(X) = \{(\Gamma \rightarrow \Delta, \square \text{for}(Y)) \in \text{next}^+(X) \mid Y \in G(n), (\text{ant}(X))^\square \not\subseteq (\text{ant}(Y))^\square\},
\]
\[
\text{prov}_2(X) = \{(\Gamma \rightarrow \Delta, \square \text{for}(\Gamma_0 \rightarrow \Delta_0, \square \text{for}(Y_0))) \in \text{next}^+(X) \mid Y_0 \in G(n-1) - G^*(n-1), (\Gamma_0 \rightarrow \Delta_0, \square \text{for}(Y_0)) \in G(n), \square \text{for}((Z \in \text{next}(Y_0) \mid \Gamma_0^\square \not\subseteq (\text{ant}(Z))^\square)) \subseteq \Gamma \cap \square \text{for}(G(n))\},
\]
\[
\text{prov}_3(X) = \{((\square \text{for}(Y), \Gamma \rightarrow \Delta, \square \text{for}(Z)) \in \text{next}^+(X) \mid Y, Z \in G^*(n), (\text{ant}(Y))^\square = (\text{ant}(Z))^\square\}.
\]

The purpose in this section is to prove

Theorem 4.2 For \(X \in G(n) - G^*(n)\),
\[
\text{prov}(X) = \text{prov}_1(X) \cup \text{prov}_2(X) \cup \text{prov}_3(X).
\]

To prove the theorem above, we need some lemmas.
Lemma 4.3  For $X \in G(n) - G^*(n)$,

$$\text{prov}_1(X) \subseteq \text{prov}(X).$$

Proof. Let $X_1$ be in $\text{prov}_1(X)$. Then $X_1 \in \text{next}^+(X)$ and there exist finite sets $\Gamma$ and $\Delta$ and a sequent $Y \in G(n)$ such that

1. $X_1 = (\exists \Gamma, \text{ant}(X) \rightarrow \text{suc}(X), \Delta, \square \text{for}(Y))$,
2. $(\text{ant}(X))^\square \nsubseteq (\text{ant}(Y))^\square$.

Using Lemma 3.7, we have $X_1 \in S4$, and hence, we obtain the lemma. \hfill \Box

Lemma 4.4  For $X \in G(n) - G^*(n)$,

$$\text{prov}_2(X) \subseteq \text{prov}(X).$$

Proof. Let $X_1$ be in $\text{prov}_2(X)$. Then $X_1 \in \text{next}^+(X)$ and there exist finite sets $\Gamma$, $\Delta$, $\Gamma_0$ and $\Delta_0$ and a sequent $Y_0 \in G(n-1) - G^*(n-1)$ such that

1. $X_1 = (\Gamma \rightarrow \Delta, \square \text{for}(\Gamma_0 \rightarrow \Delta_0, \square \text{for}(Y_0)))$,
2. $(\Gamma_0 \rightarrow \Delta_0, \square \text{for}(Y_0)) \in G(n)$,
3. $\square \text{for}(\{Z \in \text{next}(Y_0) | \Gamma_0 \subseteq (\text{ant}(Z))^\square\}) \subseteq \Gamma \cap \square \text{for}(G(n))$.

By Corollary 3.4(1), we have

$$\text{for}(\text{next}(Y_0)) \rightarrow Y_0 \in S4.$$  

Using $(\square \rightarrow)$ and $(\rightarrow \square)$, possibly several times,

$$\square \text{for} \text{next}(Y_0) \rightarrow \square Y_0 \in S4.$$  

We define $Y$ as $Y = (\Gamma_0 \rightarrow \Delta_0, \square \text{for}(Y_0))$. Then $\text{ant}(Y) = \Gamma_0$ and

$$\square \text{for}(\text{next}(Y_0)) \rightarrow Y \in S4.$$  

So,

$$\square \text{for}(\{Z \in \text{next}(Y_0) | \Gamma_0 \subseteq (\text{ant}(Z))^\square\}), \square \text{for}(\{Z \in \text{next}(Y_0) | (\text{ant}(Y))^\square \nsubseteq (\text{ant}(Z))^\square\}) \rightarrow \text{for}(Y) \in S4.$$  

Using (3),

$$\Gamma, \square \text{for}(\{Z \in \text{next}(Y_0) | (\text{ant}(Y))^\square \nsubseteq (\text{ant}(Z))^\square\}) \rightarrow \text{for}(Y) \in S4.$$  

Using $(\lor \rightarrow)$, possibly several times,

$$\Gamma, \{\text{for}(Y) \lor \square \text{for}(Z) | Z \in \text{next}(Y_0), (\text{ant}(Y))^\square \nsubseteq (\text{ant}(Z))^\square\} \rightarrow \text{for}(Y) \in S4.$$  

Using Lemma 3.7 and (cut), possibly several times,

$$\Gamma \rightarrow \text{for}(Y) \in S4.$$  

So, we have $X_1 \in S4$, and hence, we obtain the lemma. \hfill \Box

Lemma 4.5  For $X \in G(n) - G^*(n)$,

$$\text{prov}_3(X) \subseteq \text{prov}(X).$$

Proof. By Lemma 3.8, we obtain the lemma. \hfill \Box

Lemma 4.6  Let $X$ be a sequent in $G(n + 1)$ and let $X_0$ be a sequent in $G(n)$. Then

$$X \in \text{next}(X_0) \text{ if and only if } X_0 = (\text{ant}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n).$$
**Lemma 4.7** Let $X$ be a sequent in $G(n+k)$. Then

1. for any $k > 0$, $(\text{ant}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n) \in G(n)$,
2. for any $k > 1$, $(\text{ant}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n) \in \text{suc}(X)$.
3. for any $k \geq 1$ and for any $X_0 \in G(n)$, $\text{ant}(X_0) \subseteq \text{ant}(X)$ and $\text{suc}(X_0) \subseteq \text{suc}(X)$ imply $\text{ant}(X) \cap BG_n = \text{ant}(X_0)$, $\text{suc}(X) \cap BG_n = \text{suc}(X_0)$ and $\square \text{for}(X_0) \in \text{suc}(X)$.

**Proof.** For (1). We use an induction on $k$.

Basis ($k = 0$). By $X \in G(n)$ and Lemma 3.6, $(\text{ant}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n) = X \in G(n)$.

Induction step ($k > 0$). By $X \in G(n+k)$, there exists a sequent $X_0 \in G(n+k-1)$ such that $X \in \text{next}(X_0)$. By the induction hypothesis, we have

$$(\text{ant}(X_0) \cap BG_n \rightarrow \text{suc}(X_0) \cap BG_n) \in G(n).$$

On the other hand, by Lemma 4.6,

$$\text{ant}(X_0) = \text{ant}(X) \cap BG_{n+k-1} \text{ and } \text{suc}(X_0) = \text{suc}(X) \cap BG_{n+k-1}.$$ 

So,

$$(\text{ant}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n) \in G(n).$$

Since $k > 1$, $BG_{n+k-1} \supseteq BG_n$. Hence we obtain (1).

For (2). We use an induction on $k$.

Basis ($k = 1$). By (1),

$$(\text{ant}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n) \in G(n).$$

Using Lemma 4.6,

$$X \in \text{next}(\text{ant}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n),$$

and using Definition 2.1, we obtain (2).

Induction step ($k > 1$). By $X \in G(n+k)$, there exists a sequent $X_0 \in G(n+k-1)$ such that $X \in \text{next}(X_0)$. By the induction hypothesis, we have

$$\square \text{for}(X_0) \cap BG_n \rightarrow \text{suc}(X_0) \cap BG_n) \in \text{suc}(X_0).$$

Similarly to (1), we have

$$\text{ant}(X_0) \cap BG_n = \text{ant}(X) \cap BG_{n-k-1} \cap BG_n = \text{ant}(X) \cap BG_n,$$

$$\text{suc}(X_0) \cap BG_n = \text{suc}(X) \cap BG_{n-k-1} \cap BG_n = \text{suc}(X) \cap BG_n,$$

and so,

$$\square \text{for}(X) \cap BG_n \rightarrow \text{suc}(X) \cap BG_n) \in \text{suc}(X_0).$$

By $X \in \text{next}(X_0)$, we have $\text{suc}(X_0) \subseteq \text{suc}(X)$, and so, we obtain (2).

For (3). By $\text{ant}(X_0) \subseteq \text{ant}(X)$ and $\text{suc}(X_0) \subseteq \text{suc}(X)$, we have

$$\text{ant}(X_0) \cap BG_n \subseteq \text{ant}(X) \cap BG_n \text{ and } \text{suc}(X_0) \cap BG_n \subseteq \text{suc}(X) \cap BG_n.$$ 

Using $X_1 \in G(n)$ and Lemma 3.6, we have

$$\text{ant}(X_0) \subseteq \text{ant}(X) \cap BG_n \text{ and } \text{suc}(X_0) \subseteq \text{suc}(X) \cap BG_n.$$ 

On the other hand, by (1) and Lemma 3.6, we have

$$\text{ant}(X_0) \cup \text{suc}(X_0) = (\text{ant}(X) \cap BG_n) \cup (\text{suc}(X) \cap BG_n) = BG_n,$$
\[ \text{ant}(X_0) \cap \text{suc}(X_0) = (\text{ant}(X) \cap \text{BG}_n) \cap (\text{suc}(X) \cap \text{BG}_n) = \emptyset. \]

Hence
\[ \text{ant}(X_0) = \text{ant}(X) \cap \text{BG}_n \text{ and } \text{suc}(X_0) = \text{suc}(X) \cap \text{BG}_n. \]

Using (2), we obtain \( \Box \text{for}(X_0) = \Box \text{for}(\text{ant}(X) \cap \text{BG}_n \rightarrow \text{suc}(X) \cap \text{BG}_n) \in \text{suc}(X). \)

**Definition 4.8** For \( X \in \text{G}(n) \), the saturation of \( X \), write \( \text{sat}(X) \), is defined as follows:

1. if \( n = 0 \), then \( \text{sat}(X) = X \),
2. if \( n > 0 \), then
\[ \text{sat}(X) = (\Gamma_d, \Gamma_c, \text{ant}(X), \{A \mid \Box A \in \text{ant}(X)\}) \rightarrow \text{suc}(X), \Delta_c, \Delta_d, \Delta_f, \]

where
\[
\begin{align*}
\Gamma_c &= \{ \bigwedge S \mid S \subseteq \text{ant}(X) - \Box \text{for}(\text{G}(n - 1)), \#(S) > 1 \}, \\
\Gamma_d &= \{ \bigvee S \mid S \cap (\text{ant}(X) - \Box \text{for}(\text{G}(n - 1))) \neq \emptyset, S \subseteq \text{BG}_{n-1}, \#(S) > 1 \}, \\
\Delta_c &= \{ \bigwedge S \mid S \cap (\text{suc}(X) - \Box \text{for}(\text{G}(n - 1))) \neq \emptyset, S \subseteq \text{BG}_{n-1}, \#(S) > 1 \}, \\
\Delta_d &= \{ \bigvee S \mid S \subseteq \text{suc}(X) - \Box \text{for}(\text{G}(n - 1)), \#(S) > 1 \}, \\
\Delta_f &= \{ \text{for}(\text{ant}(X) \cap \text{BG}_\ell \rightarrow \text{suc}(X) \cap \text{BG}_\ell) \mid \ell \leq n - 1, \text{ant}(X) \cap \text{BG}_\ell \neq \emptyset \}.
\end{align*}
\]

**Remark 4.9** Let \( X \) be a sequent in \( \text{G}(n) \). Then
\[ \text{ant}(X) \subseteq \text{ant}(\text{sat}(X)) \text{ and } \text{suc}(X) \subseteq \text{suc}(\text{sat}(X)). \]

**Lemma 4.10** Let \( X \) be a sequent in \( \text{G}(n) \). Then
\[ \text{ant}(\text{sat}(X)) \cap \text{suc}(\text{sat}(X)) = \emptyset. \]

**Proof.** We use \( \Gamma_c, \Gamma_d, \Delta_c, \Delta_d, \Delta_f \) as in Definition 4.8. We call a formula of the form \( C \wedge D \) a \( \wedge \)-formula. Similarly, we use \( \vee \)-formula, \( \supset \)-formula and \( \Box \)-formula. We note that
1. every member of \( \Gamma_c \cup \Delta_c \) is a \( \wedge \)-formula,
2. every member of \( \Gamma_d \cup \Delta_d \) is a \( \vee \)-formula,
3. every member of \( \Delta_f \) is a \( \supset \)-formula.

Also by Lemma 3.6,
4. every member of \( \text{ant}(X) \cup \text{suc}(X) \) is either a \( \Box \)-formula or a member of \( V \).

Suppose that \( A \in \text{ant}(\text{sat}(X)) \cap \text{suc}(\text{sat}(X)) \). Then
\[
\begin{align*}
(5) \ A \in \text{ant}(X) \cup \{ C \mid \Box C \in \text{ant}(X) \},
(6) \ A \in \text{suc}(X) \cup \Delta_c \cup \Delta_d \cup \Delta_f.
\end{align*}
\]

By (5), we divide the cases.

The case that \( A \in \Gamma_c \). By (1), (2), (3), (4) and (6), we have \( A \in \Gamma_c \cap \Delta_c \). So, there exist sets \( S \) and \( S' \) such that \( A = \bigwedge S = \bigwedge S', S \subseteq \text{ant}(X) \) and \( S' \cap \text{suc}(X) \neq \emptyset \). By \( A = \bigwedge S = \bigwedge S' \), we have \( S = S' \).

Using the other conditions, there exists a formula \( B \in S' \cap \text{suc}(X) = S \cap \text{suc}(X) \subseteq \text{ant}(X) \cap \text{suc}(X) \).
This is in contradiction with Lemma 3.6.

The case that \( A \in \Gamma_d \) can be shown similarly.

The case that \( A \in \text{ant}(X) \). By (1),(2),(3),(4) and (6), we have \( A \in \text{ant}(X) \cap \text{suc}(X) \), which is in contradiction with Lemma 3.6.

The case that \( A \in \{ C \mid \Box C \in \text{ant}(X) \} \). We have \( \Box A \in \text{ant}(X) \), and using Lemma 3.6, \( n > 0 \). If \( A \in \Delta_f \), then by Lemma 4.7(2), \( \Box A \in \text{suc}(X) \), which is in contradiction with Lemma 3.6. So, using (6), \( A \in \text{suc}(X) \cup \Delta_c \cup \Delta_d \). On the other hand, by \( \Box A \in \text{ant}(X) \) and Lemma 3.6, there exist \( \ell \in \{ 0, \cdots, n-1 \} \)
Lemma 4.11 Let $X$ be a sequent in $G(n)$ and let be that $\Phi \subseteq \text{ant}(\text{sat}(X))$ and $\Psi \subseteq \text{suc}(\text{sat}(X))$. If $I$ is an inference rule in $S4$ except $(\to \Box)$ and $(\text{cut})$ whose lower sequent is $\Phi \to \Psi$, then $\Phi_1 \subseteq \text{ant}(\text{sat}(X))$ and $\Psi_1 \subseteq \text{suc}(\text{sat}(X))$, for some upper sequent $\Phi_1 \to \Psi_1$ of $I$.

**Proof.** We use $\Gamma_c, \Gamma_d, \Delta_c, \Delta_d, \Delta_f$ as in Definition 4.8. If $I$ is a weakening rule, then the lemma is clear, and so, we assume that $I$ is not a weakening rule. Let $A$ be the principal formula of $I$. We divide the cases.

The case that $A \in \Gamma_d$. There exist a set $S$ and a formula $B$ such that
\begin{align*}
(1.1) & \ A = (\bigvee S) \vee B, \\
(1.2) & \ (S \cup \{B\}) \cap (\text{ant}(X) - \Box \text{for}(G(n - 1))) \neq \emptyset, \\
(1.3) & \ S \cup \{B\} \subseteq B G_{n-1}, \\
(1.4) & \ #(S) > 0.
\end{align*}
Also $I$ is
\[
\begin{array}{c}
\frac{\bigvee S, \Phi^* \to \Psi \ B, \Phi^* \to \Psi}{\Phi \to \Psi}
\end{array}
\]
where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By (1.2), we have either $S \cap (\text{ant}(X) - \Box \text{for}(G(n - 1))) \neq \emptyset$ or $\{B\} \cap (\text{ant}(X) - \Box \text{for}(G(n - 1))) \neq \emptyset$. If $\{B\} \cap (\text{ant}(X) - \Box \text{for}(G(n - 1))) \neq \emptyset$, then $B \in \text{ant}(X) \subseteq \text{ant}(\text{sat}(X))$, and so, the left upper sequent satisfies the conditions. If $S \cap (\text{ant}(X) - \Box \text{for}(G(n - 1))) \neq \emptyset$ and $\#(S) = 1$, then $\bigvee S \in \text{ant}(X) \subseteq \text{ant}(\text{sat}(X))$, and so, the left upper sequent satisfies the conditions.

The case that $A \in \Delta_\ell$ can be shown similarly.

The case that $A \in \Gamma_c$. There exist a set $S$ and a formula $B$ such that
\begin{align*}
(2.1) & \ A = (\bigwedge S) \land B, \\
(2.2) & \ S \subseteq \text{ant}(X) - \Box \text{for}(G(n - 1)), \\
(2.3) & \ \{B\} \subseteq \text{ant}(X) - \Box \text{for}(G(n - 1)), \\
(2.4) & \ #(S) > 0.
\end{align*}
Also $I$ is either
\[
\begin{array}{c}
\frac{\bigwedge S, \Phi^* \to \Psi}{\Phi \to \Psi} \quad \text{or} \quad \frac{B, \Phi^* \to \Psi}{\Phi \to \Psi},
\end{array}
\]
where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By (2.3), $B \in \text{ant}(X) \subseteq \text{ant}(\text{sat}(X))$, So, the upper sequent $B, \Phi^* \to \Psi$ satisfies the conditions. By (2.2), if $\#(S) = 1$, then $\bigwedge S \in \text{ant}(X) \subseteq \text{ant}(\text{sat}(X))$; if not, $\bigwedge S \in \Gamma_c \subseteq \text{ant}(\text{sat}(X))$. So, the upper sequent $\bigwedge S, \Phi^* \to \Psi$ satisfies the conditions.

The case that $A \in \Delta_d$ can be shown similarly.

The case that $A \in \text{ant}(X) \cup \text{suc}(X)$. None of the member of $V$ is principal formula. So, by Lemma 3.6, $A = \Box B \in \text{ant}(X)$. Since $I$ is not $(\to \Box)$, $I$ is
\[
\begin{array}{c}
\frac{B, \Phi^* \to \Psi}{\Phi \to \Psi}.
\end{array}
\]
where Φ* ∈ {Φ, Φ − {A}}. By A = □B ∈ ant(X), we have B ∈ {C | □C ∈ ant(X)} ⊆ ant(sat(X)). So, the upper sequent satisfies the conditions.

The case that A ∈ {C | □C ∈ ant(X)}. We note that n > 0. By Lemma 3.6, there exist i ∈ {0, · · · , n − 1} and Y ∈ G(i) such that A = for(Y). We note that □A = □for(Y) ∈ ant(X). We define Z as Z = (ant(X) ∩ BG_i → suc(X) ∩ BG_i). Then by Lemma 4.7, Z ∈ G(i) and □for(Z) ∈ suc(X).

Using □for(Y) ∈ ant(X) and Lemma 3.6, we have Y ̸= Z. Using Lemma 3.6, we have ant(Y) ̸= ant(Z). In other words, ant(Y) ⊆ ant(Z) or ant(Z) ⊆ ant(Y). We divide the subcases.

The subcase that ant(Y) ⊆ ant(Z). We note that ant(Y) ̸= ∅. So, I is

\[
\Phi_1 \rightarrow \Psi_1, \bigwedge \text{ant}(Y) \bigvee \text{suc}(Y), \Phi_2 \rightarrow \Psi_2
\]

where Φ_1 ∪ Φ_2 ∈ {Φ, Φ − {A}} and Ψ_1 ∪ Ψ_2 = Ψ. On the other hand, by ant(Y) ⊆ ant(Z), there exists a formula B ∈ ant(Y) − ant(Z). Using Lemma 3.6,

\[B \in \text{ant}(Y) \cap \text{suc}(Z) \subseteq \text{ant}(Y) \cap (\text{suc}(X) \cap BG_i) \subseteq \text{ant}(Y) \cap (\text{suc}(X) − □\text{for}(G(n − 1))).\]

So, if #(ant(Y)) = 1, then ∆ ant(Y) = {B} ∈ suc(X); if not, ∆ ant(Y) ∈ ∆_c. Hence the left upper sequent of I satisfies the conditions.

The subcase that ant(Z) ⊆ ant(Y) = ∅. By ant(Y) ̸= ∅, I is

\[
\Phi_1 \rightarrow \Psi_1, \bigwedge \text{ant}(Y) \bigvee \text{suc}(Y), \Phi_2 \rightarrow \Psi_2
\]

where Φ_1 ∪ Φ_2 ∈ {Φ, Φ − {A}} and Ψ_1 ∪ Ψ_2 = Ψ. On the other hand, by ant(Z) ⊆ ant(Y), there exists a formula B ∈ ant(Z) − ant(Y). Using Lemma 3.6,

\[B \in \text{ant}(Z) \cap \text{suc}(Y) \subseteq (\text{ant}(X) \cap BG_i) \cap \text{suc}(Y) \subseteq (\text{ant}(X) − □\text{for}(G(n − 1))) \cap \text{suc}(Y).\]

So, if #(suc(Y)) = 1, then V suc(Y) = {B} ∈ ant(X); if not, V suc(Y) ∈ Γ_d. Hence the right upper sequent satisfies the conditions.

The subcase that ant(Z) ⊆ ant(Y) = ∅. By ant(Y) = ∅ and Lemma 3.6, we have suc(Y) = BG_i.

If #(suc(Y)) = #(BG_i) = 1, then suc(Y) = {A} ⊆ V, and so, A is not a principal formula. So, we assume that #(suc(Y)) > 1. Then I is

\[
\bigvee (\text{suc}(Y) − \{C\}), \Phi^* \rightarrow \Psi \quad C, \Phi^* \rightarrow \Psi
\]

where Φ* ∈ {Φ, Φ − {A}} and V suc(Y) = (∧(suc(Y) − {C})) ∨ C. On the other hand, we note by ant(Z) ⊆ ant(Y), there exists a formula B ∈ ant(Z) − ant(Y). Using Lemma 3.6,

\[B \in \text{ant}(Z) \cap \text{suc}(Y) \subseteq (\text{ant}(X) \cap BG_i) \cap \text{suc}(Y) \subseteq (\text{ant}(X) − □\text{for}(G(n − 1))) \cap \text{suc}(Y).\]

So, if C = B, then C ∈ ant(X), and so, the right upper sequent satisfies the conditions. If C ̸= B, then \[B \in \text{suc}(Y) − \{C\} \text{ and } V (\text{suc}(Y) − \{C\}) \in \Gamma_d. \]

The case that A ∈ ∆_f. There exists ℓ ≤ n − 1 such that A = for(ant(X) ∩ S_ℓ → suc(X) ∩ S_ℓ) and ant(X) ∩ S_ℓ ̸= ∅. So, I is

\[
\bigwedge (\text{ant}(X) ∩ S_ℓ), \Phi \rightarrow \Psi^*, \bigvee (\text{suc}(X) ∩ S_ℓ)
\]

where Ψ* ∈ {Ψ, Ψ − {A}}. We note that ∆(ant(X) ∩ S_ℓ) ∈ ant(X) ∪ Γ_c and V (suc(X) ∩ S_ℓ) ∈ suc(X) ∪ Γ_d. So, the upper sequent of I satisfies the conditions.

Lemma 4.12 Let X be a sequent in G(n + k) and let Y be a sequent in G^*(n). If (ant(Y))^2 ≠ (ant(X))^2 ∩ BG_n. Then

\[\rightarrow \text{for}(Y), \Box \text{for}(X) \in S4.\]
**Proof.** We use an induction on $k$.

Basis ($k = 0$). By $X \in G(n)$ and Lemma 3.6, $(\text{ant}(Y))^\Box = (\text{ant}(X))^\Box \cap BG_n \neq (\text{ant}(X))^\Box$. Also by $Y \in G^*(n)$, we have

$$(\text{ant}(Y))^\Box \subseteq (\text{ant}(Z))^\Box \implies (\text{ant}(Y))^\Box = (\text{ant}(Z))^\Box,$$

for any $Z \in G(n)$.

Hence we have $(\text{ant}(Y))^\Box \not\subseteq (\text{ant}(X))^\Box$. Using Lemma 3.7, we obtain the lemma.

Induction step ($k > 0$). By $X \in G(n+k)$, there exists $X_0 \in G(n+k-1) - G^*(n+k-1)$ such that $X \in \text{next}(X_0)$. By Lemma 4.6,

$$(\text{ant}(X_0))^\Box \cap BG_n = (\text{ant}(X))^\Box \cap BG_n = (\text{ant}(X))^\Box \cap BG_n \neq (\text{ant}(Y))^\Box.$$  

So, by the induction hypothesis, we have

$$\rightarrow \text{for}(Y), \Box \text{for}(X_0) \in S4.$$

On the other hand, we note that $\Box \text{for}(X_0) \rightarrow \Box \text{for}(X) \in S4$, and using (cut), we obtain the lemma.  

**Corollary 4.13** Let $X$ be a sequent in $G(n+k)$ and let $Y$ be a sequent in $G^*(n)$. If $(\text{ant}(Y))^\Box \neq (\text{ant}(X))^\Box \cap BG_n$. Then

$$(\Box \text{for}(Y) \supset \text{for}(Y)) \equiv \text{for}(Y).$$

**Proof.** By Lemma 4.12 and (cut), we obtain the corollary.  

**Lemma 4.14** Let $X$ be a sequent in $G(n)$ and let $Y_i$ be a sequent in $G^*(k)$ ($k \in \{0,1,\ldots,n-1\}$). If $\Box \text{for}(Y_i) \in \text{suc}(X)$, then

$$\text{for}(\text{ant}(X))^\Box \rightarrow \text{for}(Y_i) \equiv \text{for}(Y_i).$$

**Proof.** We define $X_1$ as follows:

$$X_1 = (\text{ant}(X) \cap BG_k \rightarrow \text{suc}(X) \cap BG_k).$$

Then

$$(\text{ant}(X))^\Box = (\text{ant}(X) \cap BG_k)^\Box \cup (\text{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box \text{for}(G(i)))$$

$$= \text{ant}(X_1)^\Box \cup (\text{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box \text{for}(G(i))).$$

So, it is sufficient to show the following two:

1. for any $A \in (\text{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box \text{for}(G(i)))$, $A \supset \text{for}(Y_1) \equiv \text{for}(Y_1)$,
2. $\text{for}(\text{ant}(X_1))^\Box \rightarrow \text{for}(Y_i) \equiv \text{for}(Y_i)$.

For (1). There exist a number $i \in \{k,k+1,\ldots,n-1\}$ and a sequent $Z \in G(i)$ such that $A = \Box \text{for}(Z)$. If $(\text{ant}(Y_i))^\Box \neq (\text{ant}(Z))^\Box \cap BG_k$, then by Corollary 4.13, we obtain (1). So, we assume $(\text{ant}(Y_i))^\Box = (\text{ant}(Z))^\Box \cap BG_k$. We divide the cases.

The case that $i = k$. Then by Lemma 3.8, we have

$$\Box \text{for}(Z) \rightarrow \text{for}(Y_1) \in S4.$$

Using (→ □), we have

$$\Box \text{for}(Z) \rightarrow \Box \text{for}(Y_1) \in S4.$$

So, using $\Box \text{for}(Y_i) \in \text{suc}(X)$ and $\Box \text{for}(Z) = A \in \text{ant}(X)$, we have $X \in S4$. Using Lemma 2.4(2), $X \not\in G(n)$, which is in contradiction with $X \in G(n)$.
The case that $i > k$. We define $Z_1$ and $Z_2$ as follows:

$$Z_1 = (\text{ant}(Z) \cap BG_k \rightarrow \text{suc}(Z) \cap BG_k) \text{ and } Z_2 = (\text{ant}(Z) \cap BG_{k+1} \rightarrow \text{suc}(Z) \cap BG_{k+1}).$$

Then by Lemma 4.7, we have $Z_1 \in G(k)$ and $Z_2 \in G(k+1)$. Also by the assumption, we have $(\text{ant}(Y_1))^\triangledown \subseteq (\text{ant}(Z))^\triangledown \cap BG_k = (\text{ant}(Z))^\triangledown$, and using Lemma 3.9, $Z_1 \in G^*(k)$. On the other hand, by $Z_2 \in G(k+1)$, there exists a sequent $Z'_1 \in G(k) - G^*(k)$ such that $Z_2 \subseteq \text{next}(Z'_1)$. Using Lemma 4.6,

$$Z'_1 = (\text{ant}(Z_2) \cap BG_k \rightarrow \text{suc}(Z_2) \cap BG_k) = (\text{ant}(Z) \cap BG_{k+1} \cap BG_k \rightarrow \text{suc}(Z) \cap BG_{k+1} \cap BG_k) = (\text{ant}(Z) \cap BG_k \rightarrow \text{suc}(Z) \cap BG_k) = Z_1 \in G^*(k),$$

which is in contradiction with $Z'_1 \in G(k) - G^*(k)$.

For (2). Suppose that $(\text{ant}(X_1))^\triangledown \not\subseteq G^*(k)$. Then by Lemma 3.7, we have

$$\text{ant}(X_1) \rightarrow \text{suc}(X_1), \square \text{for}(Y_1) \in S^4.$$

So,

$$\text{ant}(X) \cap BG_k \rightarrow \text{suc}(X) \cap BG_k, \square \text{for}(Y_1) \in S^4.$$

Hence $X \in S^4$, which is in contradiction with Lemma 2.4(2) and $X \in G(n)$. So, we have $(\text{ant}(X_1))^\triangledown \subseteq (\text{ant}(Y_1))^\triangledown$, and so,

$$(\bigwedge((\text{ant}(X_1))^\triangledown) \supset \text{for}(Y_1)) \equiv ((\bigwedge((\text{ant}(X_1))^\triangledown) \cup \text{ant}(Y_1)) \supset \text{suc}(Y_1)) \equiv \bigwedge\text{ant}(Y_1) \supset \bigvee\text{suc}(Y_1).$$

Hence we obtain (2).

---

**Lemma 4.15** Let $X$ be a sequent in $G(n+k)$ and let $Y_0$ be a sequent in $G(n) - G^*(n)$. Let $X_1$ be a sequent in $\text{next}(X)$. If $\square \text{for}(Y_0) \in \text{suc}(X_1)$, then there exists a sequent $Y \in G^{n+k}$ such that $\square \text{for}(Y) \in \text{suc}(X_1)$, $\text{ant}(Y_0) \subseteq \text{ant}(Y)$ and $\text{suc}(Y_0) \subseteq \text{suc}(Y)$.

**Proof.** We use an induction on $k$.

Basis ($k = 0$). The lemma is clear from $Y_0 \in G(n)$ and $\square \text{for}(Y_0) \in \text{suc}(X_1)$.

Induction step ($k > 0$). By $X \in G(n+k)$, there exists a sequent $X_0 \in G(n+k-1) - G^*(n+k-1)$ such that $X \in \text{next}(X_0)$. Also by $k > 0$ and Lemma 3.6, $\square \text{for}(Y_0) \in \text{suc}(X_1) \cap \square \text{for}(G(n) - G^*(n)) = \text{suc}(X) \cap \square \text{for}(G(n) - G^*(n))$. Using the induction hypothesis, there exists a sequent $Y_2 \in G^{n+k-1}$

such that $\square \text{for}(Y_2) \in \text{suc}(X)$, $\text{ant}(Y_0) \subseteq \text{ant}(Y_2)$ and $\text{suc}(Y_0) \subseteq \text{suc}(Y_2)$. If $Y_2 \in \bigcup_{i=0}^{n+k-1} G^*(i)$, then $Y_2 \in G^{n+k}$, and we obtain the lemma. So, we assume that $Y_2 \in G(n+k-1) - G^*(n+k-1)$. On the other hand, by Lemma 2.4 and Lemma 4.4, we have $X_1 \not\in \text{prov}_2(X)$. Using the four conditions

$$\square \text{for}(X) \in \text{suc}(X_1),$$

$$\square \text{for}(Y_2) \in \text{suc}(X),$$

$$Y_2 \in G(n+k-1) - G^*(n+k-1)$$

and $X \in G(n+k)$, we have

$$\square \text{for}([Z \in \text{next}(Y_2) \mid (\text{ant}(X))^\triangledown \subseteq (\text{ant}(Z))^\triangledown]) \not\subseteq \text{ant}(X_1) \cap \square \text{for}(G(n+k)).$$

So, there exists a sequent $Y \in \text{next}(Y_2)$ such that $(\text{ant}(X))^\triangledown \subseteq (\text{ant}(Y))^\triangledown$ and $\square \text{for}(Y) \not\in \text{ant}(X_1) \cap \square \text{for}(G(n+k))$. By $Y \in \text{next}(Y_2)$, we have $Y \in G(n+k) \subseteq G^{n+k}$. Using $\square \text{for}(Y) \not\in \text{ant}(X_1) \cap \square \text{for}(G(n+k))$ and Lemma 3.6, we have $\square \text{for}(Y) \not\in \text{ant}(X_1)$ and $\square \text{for}(Y) \in \text{suc}(X_1)$. Also by $Y \in \text{next}(Y_2)$, we have $\text{ant}(Y_2) \subseteq \text{ant}(Y)$ and $\text{suc}(Y_2) \subseteq \text{suc}(Y)$. Hence we have $\text{ant}(Y_0) \subseteq \text{ant}(Y)$ and $\text{suc}(Y_0) \subseteq \text{suc}(Y)$. --
Lemma 4.16 Let $X$ and $Y$ be sequents in $G(n) - G^*(n)$ and let $X_1$ be a sequent in $next^+(X) - (prov_2(X) \cup prov_1(X) \cup prov_3(X))$. If $\square \text{for}(Y) \in suc(X_1)$, then

$$(\Gamma, \text{ant}(X_1) \cap \square \text{for}(G(n)), \text{ant}(Y) \rightarrow suc(Y), \Delta_Y) \in next^+(X) - (prov_2(X) \cup prov_1(X) \cup prov_3(X)),$$

where

$$\Delta_Y = \{ \square \text{for}(Z) \in suc(X_1) \cap \square \text{for}(G(n)) \mid \text{ant}(Y)^\circ \subseteq \text{ant}(Z)^\circ \}$$

and

$$\Gamma_Y = \{ \square \text{for}(Z) \in suc(X_1) \cap \square \text{for}(G(n)) \mid \text{ant}(Y)^\circ \not\subseteq \text{ant}(Z)^\circ \}.$$

Proof. We define the sequent $Y_1$ as follows:

$$Y_1 = (\Gamma, \text{ant}(X_1) \cap \square \text{for}(G(n)), \text{ant}(Y) \rightarrow suc(Y), \Delta_Y).$$

It is not hard to see that $Y_1 \in next^+(Y)$. So, it is sufficient to show the following three:

1. $Y_1 \not\in prov_1(Y),$
2. $Y_1 \not\in prov_2(Y),$
3. $Y_1 \not\in prov_3(Y).$

For (1). Suppose that $Y_1 \in prov_1(Y)$. Then there exists a sequent $Z \in G(n)$ such that $\square \text{for}(Z) \in suc(Y_1), (\text{ant}(Y))^\circ \not\subseteq (\text{ant}(Z))^\circ$. By Lemma 3.6, we have $\square \text{for}(Z) \not\in BG_n \supseteq suc(Y)$, and using $\square \text{for}(Z) \in suc(Y_1) = suc(Y) \cup \Delta_Y$, we have $\square \text{for}(Z) \in \Delta_Y$. So, $(\text{ant}(Y))^\circ \not\subseteq (\text{ant}(Z))^\circ$. This is in contradiction with $(\text{ant}(Y))^\circ \not\subseteq (\text{ant}(Z))^\circ$.

For (2). Suppose that $Y_1 \in prov_2(Y)$. Then there exist sequents $Z \in G(n)$ and $Z_0 \in G(n-1) - G^*(n-1)$ such that

1. $\square \text{for}(Z) \in suc(Y_1),$
2. $\square \text{for}(Z_0) \in suc(Z),$
3. $\square \text{for}(\{ Z' \in next(Z_0) \mid (\text{ant}(Z))^\circ \subseteq (\text{ant}(Z'))^\circ \}) \subseteq suc(Y_1) \cap \square \text{for}(G(n)).$

Similarly to (1), by (2.1), we have

$$\square \text{for}(Z) \in suc(X_1).$$

Also by Lemma 3.6, we have $suc(Y_1) \cap \square \text{for}(G(n)) = \Delta_Y$, and using (2.3), we have

$$\square \text{for}(\{ Z' \in next(Z_0) \mid (\text{ant}(Z))^\circ \subseteq (\text{ant}(Z'))^\circ \}) \subseteq \Delta_Y \subseteq suc(X_1) \cap G(n).$$

By (2.4), (2.2), (2.5) and $X_1 \in next^+(X)$, we obtain $X_1 \in prov_2(X)$, which is in contradiction with $X_1 \not\in prov_2(X)$.

For (3). Suppose that $Y_1 \in prov_3(Y)$. Then there exist sequents $Z, Z' \in G^*(n)$ such that

1. $\square \text{for}(Z) \in \text{ant}(Y_1),$
2. $\square \text{for}(Z') \in suc(Y_1)$
3. $(\text{ant}(Z))^\circ = (\text{ant}(Z'))^\circ$.

Similarly to (1), by (3.2), we have

$$(\square \text{for}(Z') \in \Delta_Y \subseteq suc(X_1)).$$

By $\square \text{for}(Z') \in \Delta_Y$, we have $(\text{ant}(Y))^\circ \subseteq (\text{ant}(Z'))^\circ$. Using (3.3), $(\text{ant}(Y))^\circ \subseteq (\text{ant}(Z))^\circ$. So, we have $\square \text{for}(Z') \not\in \text{ant}(Y)$. Using (3.1), we have $\square \text{for}(Z') \in \text{ant}(X_1) \cup \text{ant}(Y)$. Similarly to (1), we have

$$(\square \text{for}(Z') \in \text{ant}(X_1)).$$

By (3.4), (3.5), (3.3) and $X_1 \in next^+(X)$, we obtain $X_1 \in prov_3(X)$, which is in contradiction with $X_1 \not\in prov_3(X)$. $\blacksquare$

Lemma 4.17 Let $\mathcal{P}$ be a cut-free proof figure in S4 whose end sequent is $\Phi \rightarrow \Psi$. Then for any $X \in G(n) - G^*(n)$ and for any $X_1 \in next^+(X) - (prov_2(X) \cup prov_1(X) \cup prov_3(X))$,

$$(\Phi \rightarrow \Psi) \not\in \{ (\Phi^* \rightarrow \Psi^*) \mid \Phi^* \subseteq \text{ant}(\text{sat}(X_1)), \Psi^* \subseteq \text{ant}(\text{sat}(X_1)) \}.$$

Proof. We use an induction on $\mathcal{P}$. 19
Using (2) and Lemma 4.15, there exists a sequent $Y$

We divide the cases.

By (6), we have

and let $I$ be the inference rule introducing the end sequent in $\mathcal{P}$.

If $I$ is not $\{\rightarrow \Box\}$, then by Lemma 4.11, an upper sequent $I$ belongs to

This is in contradiction with the induction hypothesis.

So, we assume that $I$ is $\{\rightarrow \Box\}$. Then there exist a set $\Gamma$ and a sequent $Y_0$ such that

(1) $\Gamma \subseteq \text{ant}(X_1)$,

(2) $\Box \text{for}(Y_0) \in \text{suc}(X_1)$,

(3) $(\Phi \rightarrow \Psi) = (\Gamma \rightarrow \Box \text{for}(Y_0))$,

(4) $I \quad \Gamma \rightarrow \Box \text{for}(Y_0)$

We divide the cases.

The case that $Y_0 \in G^*(k)$ for some $k \leq n$. By Lemma 4.14, (1) and (2),

for($\Gamma \rightarrow \text{for}(Y_0)$) $\equiv$ for($\text{ant}(X_1)^0 \rightarrow \text{for}(Y_0)$) $\equiv$ for($Y_0$).

Using (4), we have $Y_0 \in S_4$, which is in contradiction with $Y_0 \in G^*(k)$ and Lemma 2.4(2).

The case that $Y_0 \not\in G^*(k)$ for any $k \leq n$. Then by Lemma 3.6, $Y_0 \in G(k) - G^*(k)$ for some $k \leq n$.

Using (2) and Lemma 4.15, there exists a sequent $Y \in G^n$ such that

(5) $\Box \text{for}(Y) \in \text{suc}(X_1)$,

(6) $\text{ant}(Y_0) \subseteq \text{ant}(Y)$ and $\text{suc}(Y_0) \subseteq \text{suc}(Y)$.

By (6), we have for($Y_0$) $\rightarrow$ for($Y$) $\in S_4$, and using (4), we have $\Gamma \rightarrow \text{for}(Y) \in S_4$. If $Y \in G^*(i)$ for some $i \leq n$, then using (1), (5) and Lemma 4.14, we obtain a contradiction similarly to the above case.

So, by $Y \in G^n$, we can assume that $Y \in G(n) - G^*(n)$. Then by (5) and Lemma 4.16,

$Y_1 = (\Gamma_Y \cap \Box \text{for}(G(n)), \text{ant}(Y) \rightarrow \text{suc}(Y), \Delta_Y \in \text{next}^+(X) - (\text{prov}_2(X) \cup \text{prov}_4(X) \cup \text{prov}_3(X))$, where $\Delta_Y$ and $\Gamma_Y$ are as in Lemma 4.16. By (6), we have $\text{ant}(Y_0) \subseteq \text{ant}(Y_1)$ and $\text{suc}(Y_0) \subseteq \text{suc}(Y_1)$.

Using Lemma 4.7(3),

for($Y_0$) $= \text{for}(\text{ant}(Y_0) \rightarrow \text{suc}(Y_0)) = \text{for}(\text{ant}(Y_1) \cap \text{BG}_k \rightarrow \text{suc}(Y_1) \cap \text{BG}_k) \in \text{suc}(\text{sat}(Y_1))$.

On the other hand, by $\Box \text{for}(Y) \in \text{suc}(X_1)$ and $Y \in G(n)$, we have

$(\text{ant}(X))^0 \not\subseteq (\text{ant}(Y))^0$ implies $X_1 \in \text{prov}_4(X)$.

So, using $X_1 \not\in \text{prov}_4(X)$, we have

$\Gamma \subseteq (\text{ant}(X))^0 \subseteq (\text{ant}(Y))^0 \subseteq \text{ant}(\text{sat}(Y_1))$.

So, the upper sequent of $I$ belongs to

$\{(\Phi^* \rightarrow \Psi^*) | \Phi^* \subseteq \text{ant}(\text{sat}(Y_1)), \Psi^* \subseteq \text{suc}(\text{sat}(Y_1))\}$

for $Y_1 \in \text{next}^+(X) - (\text{prov}_2(X) \cup \text{prov}_4(X) \cup \text{prov}_3(X))$. This is in contradiction with the induction hypothesis.

By the above lemma and Lemma 1.1(2), we obtain
Corollary 4.18 Let $X$ be a sequent in $G(n) - G^*(n)$. Then
\[ \text{prov}_2(X) \cup \text{prov}_1(X) \cup \text{prov}_3(X) \supseteq \text{prov}(X). \]

From Lemma 4.3, Lemma 4.4, Lemma 4.5 and Corollary 4.18, we obtain Theorem 4.2.

References


[Sas05] K. Sasaki, Formulas with only one variable in Lewis logic $S_4$, Academia Mathematical Sciences and Information Engineering, 5, Nanzan University, pp. 39–48.