On the structure corresponding to Lindenbaum algebra of Lewis logic **S4** [1]

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Abstract. The structure $\langle S/\equiv, \leq\rangle$ corresponds to Lindenbaum algebra of Lewis Logic **S4** if $S/\equiv$ is the quotient set of the set S of all formulas modulo the provability of **S4**, and $\leq$ is the derivation of **S4**. Here we treat the structure in the case that S is the set of formulas constructed from a finite set V of propositional variables and whose depth of $\Box$ is less than a given number n. It is known that this structure is Boolean (cf. Chagrov and Zakharyaschev [CZ97]). So, we have only to elucidate its generators. We give an inductive construction of concrete representatives for the generators of the Boolean. The case that V has only one variable has been treated in [Sas05]. We extend it to the case that V is a finite set. Also we construct representatives without the provability of **S4** while the result in [Sas05] almost depends on it.

1 Preliminaries

We use lower case Latin letters $p, q, p_1, p_2, \cdots$ for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\bot$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). We use upper case Latin letters $A, B, C, \cdots, A_0, A_1, A_2, \cdots$ for formulas. For a finite set S, $\#(S)$ denotes the number of elements S.

Let V be a finite set of propositional variables. $\mathbf{S}(V)$ denotes the set of formulas constructed from propositional variables in V and $\bot$ by using $\wedge$, $\vee$, $\supset$ and $\Box$. The depth $d(A)$ of a formula A is defined inductively as follows:

(1) $d(A) = 0$ if A is either a propositional variable or $\bot$,
(2) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
(3) $d(\Box B) = d(B) + 1$.

We define $\mathbf{S}^n(V)$ as $\mathbf{S}^n(V) = \{A \in \mathbf{S}(V) \mid d(A) \leq n\}$.

Let **ENU** be an enumeration of the formulas. For a non-empty finite set S of formulas, the expressions

$$\bigwedge S \qquad \text{and} \qquad \bigvee S$$

denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \qquad \text{and} \qquad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = S$ and A_i occurs earlier than A_{i+1} in **ENU**. Also the expressions

$$\bigwedge \emptyset \qquad \text{and} \qquad \bigvee \emptyset$$

denote the formulas $\bot \supset \bot$ and $\bot$, respectively.

By **S4**, we mean the smallest set of formulas containing all the tautologies and the axioms

$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$,
$T : \Box p \supset p$,
$4 : \Box p \supset \Box\Box p$

and closed under modus ponens, substitution and necessitation.

We introduce a sequent system for **S4** following Ohnishi and Matsumoto [OM57]. We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote

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the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. By a sequent, we mean the expression $(\Gamma \to \Delta)$. We often write $\Gamma \to \Delta$ instead of the expression with the parenthesis. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

We use upper case Latin letters $X, Y, Z, \cdots, X_0, X_1, X_2, \cdots$ for sequents. For a sequent $\Gamma \to \Delta$, we define $\mathbf{ant}(\Gamma \to \Delta)$ and $\mathbf{suc}(\Gamma \to \Delta)$ as follows:

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma, \qquad \mathbf{suc}(\Gamma \to \Delta) = \Delta.$$

Also for a sequent X and for a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as follows:

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \Gamma \neq \emptyset \\ \bigvee \Delta & \text{if } \Gamma = \emptyset, \end{cases}$$

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

By **GS4**, we mean the system defined by the following axioms and inference rules in the usual way.

Axioms of S4:

$$A \to A$$

$$\bot \to$$

Inference rules of S4:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to w)$$

$$\frac{\Gamma \to \Delta, A \qquad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}(cut)$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \qquad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \qquad B, \Pi \to \Lambda}{A \supset B, \Gamma, \Pi \to \Delta, \Lambda}(\supset \to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to \supset)$$

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box)$$

Lemma 1.1 ([OM57])
(1) $\Gamma \to \Delta \in$ **GS4** *if and only if* $\mathbf{for}(\Gamma \to \Delta) \in$ **S4**.
(2) *If* $\Gamma \to \Delta \in$ **GS4**, *then there exists a cut-free proof figure for* $\Gamma \to \Delta$ *in* **GS4**.

By the lemma above, we can identify **GS4** with **S4**. So, if there is no confusion, we use **S4** as the sequent system **GS4**.

2 Main results

Here we consider the structure $\langle \mathbf{S}^n(\{p_1 \cdots, p_m\})/\equiv, \leq \rangle$, where $A \equiv B$ if and only if $(A \supset B) \wedge (B \supset A) \in \mathbf{S4}$; and $[A] \leq [B]$ if and only if there exist $A' \in [A]$ and $B' \in [B]$ such that $A' \supset B' \in \mathbf{S4}$. Our main purpose is to give a concrete representative of each equivalence class of $\mathbf{S}^n(\{p_1 \cdots, p_m\})/\equiv$ in an inductive way and elucidate the structure. Since the structure is Boolean, we mainly construct representatives for generators. From now on, we fix the set $\{p_1, \cdots, p_m\}$ and write $\mathbf{V}$.

Definition 2.1 The sets $\mathbf{G}(n)$ and $\mathbf{G}^*(n)$ $(n = 0, 1, 2, \cdots)$ of sequents, and the mappings $\mathbf{next}^+$, $\mathbf{prov}$, $\mathbf{next}$ are defined inductively as follows:

$\mathbf{G}(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,

$\mathbf{G}^*(0) = \emptyset$,

$\mathbf{next}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}(n)), \Gamma \cap \Delta = \emptyset, \mathbf{for}(X) \in \Delta\}$, for $X \in \mathbf{G}(k)$,

$\mathbf{prov}(X) = \{Y \in \mathbf{next}^+(X) \mid Y \in \mathbf{S4}\}$, for $X \in \mathbf{G}(k)$,

$\mathbf{next}(X) = \mathbf{next}^+(X) - \mathbf{prov}(X)$, for $X \in \mathbf{G}(k)$,

$\mathbf{G}(k+1) = \bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}(X)$,

$\mathbf{G}^*(k+1) = \{X \in \mathbf{G}(k+1) \mid (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box}$ implies $(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(Y))^{\Box}$, for any $Y \in \mathbf{G}(k+1)\}$.

Here we use the provability of $\mathbf{S4}$, but in section 4, this provability will be replaced another conditions concerning the structure of sequents.

Definition 2.2 We define $\mathbf{G}^n$ as follows:

$$\mathbf{G}^n = \mathbf{G}(n) \cup \bigcup_{k=0}^{n-1} \mathbf{G}^*(k).$$

In the following theorem, it is shown that the above $\mathbf{G}^n$ is the set of representatives for the generators of $\langle \mathbf{S}^n(\{p_1, \cdots, p_m\}), \leq \rangle$.

Theorem 2.3

(1) $\mathbf{S}^n(\mathbf{V})/\equiv = \{[\bigwedge \mathbf{for}(\mathcal{S}))] \mid \mathcal{S} \subseteq \mathbf{G}^n\}$.

(2) *For subsets* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *of* $\mathbf{G}^n$,

(2.1) $\mathcal{S}_2 \subseteq \mathcal{S}_1$ *if and only if* $[\bigwedge \mathbf{for}(\mathcal{S}_2)] \leq [\bigwedge \mathbf{for}(\mathcal{S}_1)]$,

(2.2) $\mathcal{S}_1 = \mathcal{S}_2$ *if and only if* $[\bigwedge \mathbf{for}(\mathcal{S}_1)] = [\bigwedge \mathbf{for}(\mathcal{S}_2)]$.

In the next section, we prove (1) in the above theorem. In other words, we show that every equivalent class in $\mathbf{S}^n(\mathbf{V})/\equiv$ has a representative $\bigwedge \mathbf{for}(\mathcal{S})$ for some subset $\mathcal{S}$ of $\mathbf{G}^n$. Here we prove (2) in the above theorem. To prove (2), we need some lemmas.

Lemma 2.4

(1) $\mathbf{G}(n) \subseteq \mathbf{S}^n(\mathbf{V}) - \mathbf{S}^{n-1}(\mathbf{V})$.

(2) *every member of* $\mathbf{G}(n)$ *is not provable in* $\mathbf{S4}$.

Proof. By an induction on n. ⊣

Lemma 2.5 *For any* $X, Y \in \mathbf{G}^n$, $X \neq Y$ *implies* $\mathbf{for}(X) \vee \mathbf{for}(Y) \in \mathbf{S4}$.

Proof. We use an induction on n.

Basis($n = 0$). We have $X, Y \in \mathbf{G}^0 = \mathbf{G}(0)$. So, there exist subsets V_1, V_2 of $\mathbf{V}$ such that $X = (\mathbf{V} - V_1 \to V_1)$, $Y = (\mathbf{V} - V_2 \to V_2)$ and $V_1 \neq V_2$. By $V_1 \neq V_2$, we have either $V_1 \cap (\mathbf{V} - V_2) \neq \emptyset$ or $V_2 \cap (\mathbf{V} - V_1) \neq \emptyset$. Hence either $\mathbf{suc}(X) \cap \mathbf{ant}(Y) \neq \emptyset$ or $\mathbf{suc}(Y) \cap \mathbf{ant}(X) \neq \emptyset$, and so, we obtain the lemma.

Induction step($n \geq 1$). We divide the cases.

The case that $\{X, Y\} \subseteq \mathbf{G}(n)$. There exist sequents $X_0, Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$ and $Y \in \mathbf{next}(Y_0)$. So, there exist sets $\Gamma_X, \Gamma_Y, \Delta_X, \Delta_Y$ of formulas such that

(1) $X = (\Box\Gamma_X, \mathbf{ant}(X_0) \to \mathbf{suc}(X_0), \Box\Delta_X)$, $Y = (\Box\Gamma_Y, \mathbf{ant}(Y_0) \to \mathbf{suc}(Y_0), \Box\Delta_Y)$,

(2) $\Gamma_X \cup \Delta_X = \Gamma_Y \cup \Delta_Y = \mathbf{for}(\mathbf{G}(n-1))$,

(3) $\Gamma_X \cap \Delta_X = \Gamma_Y \cap \Delta_Y = \emptyset$,

(4) $\mathbf{for}(X_0) \in \Delta_X, \mathbf{for}(Y_0) \in \Delta_Y$.

If $X_0 \neq Y_0$, then by the induction hypothesis, $\mathbf{for}(X_0) \vee \mathbf{for}(Y_0) \in \mathbf{S4}$, and so, we obtain the lemma. Suppose that $X_0 = Y_0$. Then by $X \neq Y$, we have either $\Gamma_X \neq \Gamma_Y$ or $\Delta_X \neq \Delta_Y$, and using (2) and (3), we have both. Without loss of generality, we can suppose that $\Gamma_X \not\subseteq \Gamma_Y$. So, there exists a formula $A \in \Gamma_X - \Gamma_Y$, and using (2) and (3), $A \in \Gamma_X \cap \Delta_Y$. So, we have $\Box\Gamma_X \to \Box\Delta_Y \in \mathbf{S4}$. We note $\Box\Gamma_X, \mathbf{for}(X) \in \mathbf{S4}$ and $\Box\Delta_Y \to \mathbf{for}(Y) \in \mathbf{S4}$. Using (*cut*), possibly several times, we obtain $\to \mathbf{for}(X), \mathbf{for}(Y) \in \mathbf{S4}$, and hence we obtain the lemma. In the following we show how to use (*cut*) if each one of Γ_X and Δ_Y has only one element:

$$\dfrac{\dfrac{\to \Box\Gamma_X, \mathbf{for}(X) \quad \Box\Gamma_X \to \Box\Delta_Y}{\to \mathbf{for}(X), \Box\Delta_Y}\,(cut) \quad \Box\Delta_Y \to \mathbf{for}(Y)}{\to \mathbf{for}(X), \mathbf{for}(Y)}\,(cut).$$

The case that $\{X, Y\} \not\subseteq \mathbf{G}(n)$. There exists $Z \in \{X, Y\} - \mathbf{G}(n)$. Without loss of generality, we can suppose that $Z = Y \notin \mathbf{G}(n)$, and then $Y \in \bigcup_{k=0}^{n-1} \mathbf{G}^*(k) \subseteq \mathbf{G}(n-1)$. If $X \notin \mathbf{G}(n)$, then $X \in \bigcup_{k=0}^{n-1} \mathbf{G}^*(k) \subseteq \mathbf{G}(n-1)$. Using the induction hypothesis, we obtain the lemma. So, we assume that $X \in \mathbf{G}(n)$. Then there exist $X_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$. By $Y \in \bigcup_{k=0}^{n-1} \mathbf{G}^*(k)$ and Lemma 2.4(1), we have $Y \neq X_0$. By the induction hypothesis, we have $\mathbf{for}(X_0) \vee \mathbf{for}(Y) \in \mathbf{S4}$. We note that $\mathbf{for}(X_0) \vee \mathbf{for}(Y) \to \mathbf{for}(X) \vee \mathbf{for}(Y) \in \mathbf{S4}$. Using (*cut*), we obtain the lemma. ⊣

Proof of Theorem 2.3(2). (2.1) is clear. The "if part" of (2.2) is also clear. We show the "only if" part of (2.2). Suppose that $\mathcal{S}_1 \not\subseteq \mathcal{S}_2$. Then there exists a sequent $X \in \mathcal{S}_1 - \mathcal{S}_2$. By Lemma 2.5, we have $\mathbf{for}(X) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \in \mathbf{S4}$. By Lemma 2.4(2), we have $\mathbf{for}(X) \notin \mathbf{S4}$. Also we have $\bigwedge \mathbf{for}(\mathcal{S}_1) \to \mathbf{for}(X) \in \mathbf{S4}$. Hence considering the figure

$$\dfrac{\to \mathbf{for}(X) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \quad \dfrac{\mathbf{for}(X) \to \mathbf{for}(X) \quad \dfrac{\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \quad \bigwedge \mathbf{for}(\mathcal{S}_1) \to \mathbf{for}(X)}{\bigwedge \mathbf{for}(\mathcal{S}_2) \to \mathbf{for}(X)}\,(cut)}{\mathbf{for}(X) \vee \bigwedge \mathbf{for}(\mathcal{S}_2) \to \mathbf{for}(X)}\,(\vee \to)}{\to \mathbf{for}(X)}\,(cut),$$

we obtain $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \notin \mathbf{S4}$. Similarly, we can show that $\mathcal{S}_2 \not\subseteq \mathcal{S}_1$ implies $\bigwedge \mathbf{for}(\mathcal{S}_1) \to \bigwedge \mathbf{for}(\mathcal{S}_2) \notin \mathbf{S4}$. ⊣

3 Representatives of the equivalent classes in $\mathbf{S}^n(\mathbf{V})/\equiv$

Here we prove the following theorem.

Theorem 3.1 *For any $A \in \mathbf{S}^n(\mathbf{V})$, there exists a subset $\mathcal{S}$ of $\mathbf{G}^n$ such that $A \equiv \bigwedge \mathbf{for}(\mathcal{S})$.*

From the above theorem, we obtain Theorem 3.1(1), and that every equivalent class in $\mathbf{S}^n(\mathbf{V})/\equiv$ has a representative $\bigwedge \mathbf{for}(\mathcal{S})$ for some subset $\mathcal{S}$ of $\mathbf{G}^n$. To prove Theorem 3.1, we need some lemmas.

Lemma 3.2 *For any subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathbf{G}^n$,*
(1) $\bigwedge \mathcal{S}_1 \wedge \bigwedge \mathcal{S}_2 \equiv \bigwedge(\mathcal{S}_1 \cup \mathcal{S}_2)$,
(2) $\bigwedge \mathcal{S}_1 \vee \bigwedge \mathcal{S}_2 \equiv \bigwedge(\mathcal{S}_1 \cap \mathcal{S}_2)$.

Proof. (1) is clear. We show (2). Let A be in $\mathcal{S}_1$. Then by Lemma 2.5, we have $A \vee B \in \mathbf{S4}$ for any $B \in \mathcal{S}_2 - \{A\}$. So, if $A \in \mathcal{S}_2$, then

$$\begin{aligned} A \vee \bigwedge \mathcal{S}_2 &\equiv (A \vee A) \wedge (A \vee \bigwedge(\mathcal{S}_2 - \{A\}) \\ &\equiv A \wedge (A \vee \bigwedge(\mathcal{S}_2 - \{A\}) \\ &\equiv A; \end{aligned}$$

if not,

$$\begin{aligned} A \vee \bigwedge \mathcal{S}_2 &\equiv \bigwedge\{A \vee B \mid B \in \mathcal{S}_2\} \\ &\equiv p \supset p. \end{aligned}$$

Hence

$$\begin{aligned} \bigwedge \mathcal{S}_1 \vee \bigwedge \mathcal{S}_2 &\equiv \bigwedge\{A \vee \bigwedge \mathcal{S}_2 \mid A \in \mathcal{S}_1\} \\ &\equiv \bigwedge\{A \vee \bigwedge \mathcal{S}_2 \mid A \in \mathcal{S}_1 - \mathcal{S}_2\} \wedge \bigwedge\{A \vee \bigwedge \mathcal{S}_2 \mid A \in \mathcal{S}_1 \cap \mathcal{S}_2\} \\ &\equiv (p \supset p) \wedge \bigwedge\{A \mid A \in \mathcal{S}_1 \cap \mathcal{S}_2\} \\ &\equiv \bigwedge(\mathcal{S}_1 \cap \mathcal{S}_2). \end{aligned}$$

$\dashv$

Lemma 3.3 *Let $\Sigma, \Gamma, \Gamma_1, \Delta, \Delta_1$ be finite sets of formulas. Then for any subset $\Sigma' \subseteq \Sigma$,*

$$\Box\Sigma', \{\mathbf{for}(\Box\Gamma, \Box\Phi, \Gamma_1 \to \Delta_1, \Box\Psi, \Box\Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Proof. We define $\mathcal{S}$ as follows:

$$\mathcal{S} = \{\mathbf{for}(\Box\Gamma, \Box\Phi, \Gamma_1 \to \Delta_1, \Box\Psi, \Box\Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\},$$

and prove

$$\Box\Sigma', \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

We use an induction on $\#(\Sigma - \Sigma')$.

Basis($\Sigma' = \Sigma$). We note that

$$\mathbf{for}(\Box\Gamma, \Box\Sigma, \Gamma_1 \to \Delta_1, \Box\Delta) \in \mathcal{S}$$

and

$$\Box\Sigma, \mathbf{for}(\Box\Gamma, \Box\Sigma, \Gamma_1 \to \Delta_1, \Box\Delta), \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using weakening rule, we obtain the lemma.

Induction step($\Sigma' \neq \Sigma$)). By the induction hypothesis, for any $A \in \Sigma - \Sigma'$,

$$\Box(\Sigma' \cup \{A\}), \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using ($\vee \to$), possibly several times,

$$\Box\Sigma', \bigvee(\Box(\Sigma - \Sigma')), \mathcal{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using ($\vee \rightarrow$), possibly several times,

$$\Box\Sigma', \bigvee(\Delta_1 \cup \Box\Delta \cup \Box(\Sigma - \Sigma')), \mathcal{S}, \Box\Gamma, \Gamma_1 \rightarrow \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Using ($\supset\rightarrow$), possibly several times,

$$\Box\Sigma', \mathbf{for}(\Box\Gamma, \Gamma_1, \Box\Sigma' \rightarrow \Delta_1, \Box\Delta, \Box(\Sigma - \Sigma')), \mathcal{S}, \Box\Gamma, \Gamma_1 \rightarrow \Delta_1, \Box\Delta \in \mathbf{S4}.$$

We note that

$$\mathbf{for}(\Box\Gamma, \Gamma_1, \Box\Sigma' \rightarrow \Delta_1, \Box\Delta, \Box(\Sigma - \Sigma')) \in \mathcal{S},$$

and so,

$$\Box\Sigma', \mathcal{S}, \Box\Gamma, \Gamma_1 \rightarrow \Delta_1, \Box\Delta \in \mathbf{S4}.$$

⊣

Corollary 3.4 *Let X be a sequent in $\mathbf{G}(n)$ and let Y be a sequent in $\mathbf{G}_\ell$. Then*
(1) $\mathbf{for}(\mathbf{next}(X)) \rightarrow \mathbf{for}(X) \in \mathbf{S4}$,
(2) $\bigwedge \mathbf{for}(\mathbf{next}(X)) \equiv \mathbf{for}(X)$,
(3) $\{\mathbf{for}(Z) \mid Z \in \mathbf{next}(X), \Box\mathbf{for}(Y) \in \mathbf{suc}(Z)\} \rightarrow \mathbf{for}(X), \Box\mathbf{for}(Y) \in \mathbf{S4}$.

Proof. Considering the case that $\Gamma = \emptyset, \Gamma_1 = \mathbf{ant}(X), \Delta = \{\mathbf{for}(X)\}, \Delta_1 = \mathbf{suc}(X), \Sigma = \mathbf{for}(\mathbf{G}(n)) - \{\mathbf{for}(X)\}$ and $\Sigma' = \emptyset$ in the above lemma, we have $(\mathbf{for}(\mathbf{next}^+(X)) \rightarrow \mathbf{for}(X), \Box\mathbf{for}(X)) \in \mathbf{S4}$. Using (*cut*), possibly several times, we obtain (1). (2) follows from (1). Also considering the case that $\Gamma = \emptyset, \Gamma_1 = \mathbf{ant}(X), \Delta = \{\mathbf{for}(X), \mathbf{for}(Y)\}, \Delta_1 = \mathbf{suc}(X), \Sigma = \mathbf{for}(\mathbf{G}(n)) - \{\mathbf{for}(X), \mathbf{for}(Y)\}$ and $\Sigma' = \emptyset$ in the above lemma, we obtain (3) similarly to (1). ⊣

Definition 3.5 We define $\mathbf{BG}_\ell$ as follows:

$$\mathbf{BG}_\ell = \mathbf{V} \cup \bigcup_{i=0}^{\ell-1} \Box\mathbf{for}(\mathbf{G}(i)).$$

Lemma 3.6 *Let X be a sequent in $\mathbf{G}(n)$. Then*
(1) $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \mathbf{BG}_n$,
(2) $\mathbf{ant}(X) \cap \mathbf{suc}(X) = \emptyset$.

Proof. By Lemma 2.4(1) and an induction on n. ⊣

Lemma 3.7 *Let X and Y be sequents in $\mathbf{G}(n)$. Then*

$$(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box \text{ implies } (\rightarrow \mathbf{for}(X), \Box\mathbf{for}(Y)) \in \mathbf{S4}.$$

Proof. By $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$, there exists a formula $\Box A \in (\mathbf{ant}(X))^\Box - (\mathbf{ant}(Y))^\Box$. Using Lemma 3.6, we have $\Box A \in (\mathbf{ant}(X))^\Box \cap (\mathbf{suc}(Y))^\Box$. So,

$$\Box A \rightarrow \mathbf{suc}(Y) \in \mathbf{S4}.$$

Hence

$$\Box A \rightarrow \mathbf{for}(Y) \in \mathbf{S4}.$$

Using ($\rightarrow \Box$),

$$\Box A \rightarrow \Box\mathbf{for}(Y) \in \mathbf{S4}.$$

Using weakening rule,

$$\mathbf{ant}(X) \rightarrow \mathbf{suc}(X), \Box\mathbf{for}(Y) \in \mathbf{S4}.$$

Hence we obtain the lemma. ⊣

Lemma 3.8 *Let X be a sequent in $\mathbf{G}^*(n)$ and let Y be a sequent in $\mathbf{G}(n)$ satisfying $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Then*

$$\Box\mathbf{for}(Y) \to \mathbf{for}(X) \in \mathbf{S4}.$$

Proof. If $n = 0$, then the lemma is clear from $\mathbf{G}^*(0) = \emptyset$. Also, if $X = Y$, then the lemma is clear. So, we assume $n > 0$ and $X \neq Y$. By $X \in \mathbf{G}^*(n)$ and $Y \in \mathbf{G}(n)$, there exist sequents $X_0, Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$ and $Y \in \mathbf{next}(Y_0)$. So, there exist four sets $\Gamma_X, \Gamma_Y, \Delta_X$ and Δ_Y such that

(1) $X = (\Box\Gamma_X, \mathbf{ant}(X_0) \to \mathbf{suc}(X_0), \Box\Delta_X)$, $Y = (\Box\Gamma_Y, \mathbf{ant}(Y_0) \to \mathbf{suc}(Y_0), \Box\Delta_Y)$,
(2) $\Gamma_X \cup \Delta_X = \Gamma_Y \cup \Delta_Y = \mathbf{for}(\mathbf{G}(n))$,
(3) $\Gamma_X \cap \Delta_X = \Gamma_Y \cap \Delta_Y = \emptyset$,
(4) $\mathbf{for}(X_0) \in \Delta_X$, $\mathbf{for}(Y_0) \in \Delta_Y$.

Also we have

(5) $X \notin \mathbf{S4}$, $Y \notin \mathbf{S4}$.

By $Y \in \mathbf{G}(n)$ and Corollary 3.4(1),

$$\mathbf{for}(\mathbf{next}(Y_0)) \to \mathbf{for}(Y_0) \in \mathbf{S4}.$$

Using $(\Box \to)$ and $(\to \Box)$,

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to \Box\mathbf{for}(Y_0) \in \mathbf{S4}.$$

By $\mathbf{ant}(X)^\Box = \mathbf{ant}(Y)^\Box$, (1) and Lemma 2.4(1), we have $\Gamma_X = \Gamma_Y$. Using (2),(3) and (4), we have $\Box\mathbf{for}(Y_0) \in \Box\Delta_Y = \Box\Delta_X$, and so, $\Box\mathbf{for}(Y_0) \to \mathbf{for}(X) \in \mathbf{S4}$. Using (*cut*),

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to \mathbf{for}(X) \in \mathbf{S4},$$

that is,

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Z))^\Box \text{ or } (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \to \mathbf{for}(X) \in \mathbf{S4}.$$

By $X \in \mathbf{G}^*(n)$, we have that $(\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Z))^\Box$ if and only if $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Z))^\Box$, and so,

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Z))^\Box \text{ or } (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \to \mathbf{for}(X) \in \mathbf{S4}.$$

By $\mathbf{ant}(X)^\Box = \mathbf{ant}(Y)^\Box$, (1) and Lemma 3.6, we have $\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Z))^\Box\} = \{Y\}$, and so,

$$\Box\mathbf{for}(Y), \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \to \mathbf{for}(X) \in \mathbf{S4}.$$

Using $(\vee \to)$, possibly several times,

$$\Box\mathbf{for}(Y), \{\mathbf{for}(X) \vee \Box\mathbf{for}(Z) \mid Z \in \mathbf{next}(Y_0), (\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\} \to \mathbf{for}(X) \in \mathbf{S4}.$$

Using Lemma 3.7, and (*cut*), possibly several times, we obtain the lemma. ⊣

Lemma 3.9 *Let X and Y be sequents in $\mathbf{G}(n)$ satisfying $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$. Then $X \in \mathbf{G}^*(n)$. if and only if $Y \in \mathbf{G}^*(n)$.*

Proof. From the definition of $\mathbf{G}^*(n)$,

$X \in \mathbf{G}^*(n)$ if and only if $(\mathbf{ant}(X))^\Box \subseteq (\mathbf{ant}(Z))^\Box$ implies $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Z))^\Box$, for any $Z \in \mathbf{G}(n)$,
$Y \in \mathbf{G}^*(n)$ if and only if $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box$ implies $(\mathbf{ant}(Y))^\Box = (\mathbf{ant}(Z))^\Box$, for any $Z \in \mathbf{G}(n)$.

Using $(\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box$, we obtain the lemma. ⊣

Definition 3.10 We define a mapping **cf** as follows:

$$\mathbf{cf}(X) = \begin{cases} \bigwedge\mathbf{for}(\{Y \in \mathbf{G}(n) \mid (\mathbf{ant}(X))^\Box = (\mathbf{ant}(Y))^\Box\}) & \text{if } X \in \mathbf{G}^*(n) \\ \bot \supset \bot & \text{if } X \in \mathbf{G}(n) - \mathbf{G}^*(n) \end{cases}$$

Lemma 3.11 *Let X be a sequent in $\mathbf{G}(n)$ and let Σ be a subset of $(\mathbf{ant}(X))^{\Box}$. Then*

$$\Sigma, \mathbf{cf}(X), \Phi \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

where $\Phi = \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n) - \mathbf{G}^*(n), (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(X))^{\Box}\}$.

Proof. We use an induction on $\omega n + \#((\mathbf{ant}(X))^{\Box} - \Sigma)$.

Basis($n = 0$). We note that $\mathbf{ant}(X)^{\Box} = \emptyset$ and for any $Y \in \mathbf{G}(0) - \mathbf{G}^*(0) = \mathbf{G}^*(0)$, $\mathbf{ant}(Y)^{\Box} = \emptyset$. Hence $\Phi = \mathbf{G}(0)$. So, it is not hard to see that $\Phi \to \in \mathbf{S4}$. Hence we obtain the lemma.

Induction step($n > 0$). By $n > 0$, there exists a sequent $X_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that $X \in \mathbf{next}(X_0)$. By the induction hypothesis,

$$\bot \supset \bot, \{\mathbf{for}(Y_0) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1), (\mathbf{ant}(Y_0))^{\Box} \subseteq (\mathbf{ant}(X_0))^{\Box}\} \to \Box\mathbf{for}(X_0) \in \mathbf{S4}.$$

Since $(\Box\mathbf{for}(X_0) \to \Box\mathbf{for}(X)), (\bot \to \bot) \in \mathbf{S4}$, using (cut), twice,

$$\{\mathbf{for}(Y_0) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1), (\mathbf{ant}(Y_0))^{\Box} \subseteq (\mathbf{ant}(X_0))^{\Box}\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using weakening rule,

$$\Sigma, \Phi, \{\mathbf{for}(Y_0) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)\} \to \Box\mathbf{for}(X) \in \mathbf{S4}. \tag{1}$$

On the other hand, by the induction hypothesis,

$$\Sigma, \mathbf{cf}(X), \Phi, A \to \Box\mathbf{for}(X) \in \mathbf{S4}, \tag{2}$$

for any formula $A \in (\mathbf{ant}(X))^{\Box} - \Sigma$. (2) also holds for any $A \in (\mathbf{suc}(X))^{\Box}$, and so, for any $A \in \mathbf{G}(n-1) - \Sigma$. Let Y be a sequent in $\mathbf{G}(n)$ such that $(\mathbf{ant}(Y))^{\Box} = \Sigma$. Then (2) holds for any $A \in \mathbf{G}(n-1) - (\mathbf{ant}(Y))^{\Box} = (\mathbf{suc}(Y))^{\Box}$. We note that $\mathbf{suc}(Y) = \{\mathbf{for}(Y_0)\} \cup (\mathbf{suc}(Y))^{\Box}$ if $Y \in \mathbf{next}(Y_0)$, so using (1) and $(\vee \to)$, possibly several times,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\bigvee \mathbf{suc}(Y) \mid Y \in \bigcup_{Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)} \mathbf{next}(Y_0), (\mathbf{ant}(Y))^{\Box} = \Sigma\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Also we have that $(\mathbf{ant}(Y))^{\Box} = \Sigma$ implies $\Sigma \to \bigwedge \mathbf{ant}(Y) \in \mathbf{S4}$, for any $Y \in \mathbf{G}(n)$; so using $(\supset\to)$,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(Y))^{\Box} = \Sigma\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using $(w \to)$, possibly several times,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(X))^{\Box}\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using the definition of Φ,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(X))^{\Box}\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using the definition of $\mathbf{G}^*(n)$,

$$\Sigma, \mathbf{cf}(X), \Phi, \{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(X))^{\Box}\} \to \Box\mathbf{for}(X) \in \mathbf{S4}. \tag{3}$$

If $X \notin \mathbf{G}^*(n)$, then by Lemma 3.9, $(\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(X))^{\Box}$ implies $Y \notin \mathbf{G}^*(n)$, and so,

$$\{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(X))^{\Box}\} = \emptyset \subseteq \mathbf{cf}(X).$$

If $X \in \mathbf{G}^*(n)$, then from Definition 3.10, we also have

$$\{\mathbf{for}(Y) \mid Y \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(X))^{\Box}\} \subseteq \mathbf{cf}(X).$$

So, the above condition also holds in any case. Using (3), we obtain the lemma. $\dashv$

Lemma 3.12 *Let X be a sequent in $\mathbf{G}(n)$. Then*

$$\Box\mathbf{for}(X) \equiv \mathbf{cf}(X) \wedge \bigwedge\{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n+1), \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\}.$$

Proof. By Lemma 3.8 and $(\to \wedge)$, possibly several times,

$$\Box\mathbf{for}(X) \to \mathbf{cf}(X) \in \mathbf{S4}.$$

Also we note that

$$\Box\mathbf{for}(X) \to \bigwedge\{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n+1), \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\} \in \mathbf{S4}.$$

Using $(\to \wedge)$,

$$\Box\mathbf{for}(X) \to \mathbf{cf}(X) \wedge \bigwedge\{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n+1), \Box\mathbf{for}(X) \in \mathbf{suc}(X_1)\} \in \mathbf{S4}.$$

We show the converse. By Corollary 3.4(3), for any $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$,

$$\{\mathbf{for}(Y_1) \mid Y_1 \in \mathbf{next}(Y), \Box\mathbf{for}(X) \in \mathbf{suc}(Y_1)\} \to \mathbf{for}(Y), \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using $(\to \wedge)$, possibly several times,

$$\{\mathbf{for}(Y_1) \mid Y_1 \in \bigcup_{Y \in \mathbf{G}(n)-\mathbf{G}^*(n)} \mathbf{next}(Y), \Box\mathbf{for}(X) \in \mathbf{suc}(Y_1)\} \to \bigwedge \mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)), \Box\mathbf{for}(X) \in \mathbf{S4}.$$

On the other hand, by Lemma 3.11,

$$\mathbf{cf}(X), \bigwedge \mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)) \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Using (cut),

$$\mathbf{cf}(X), \{\mathbf{for}(Y_1) \mid Y_1 \in \bigcup_{Y \in \mathbf{G}(n)-\mathbf{G}^*(n)} \mathbf{next}(Y), \Box\mathbf{for}(X) \in \mathbf{suc}(Y_1)\} \to \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Hence we obtain the lemma. $\dashv$

Lemma 3.13

$$\bot \equiv \bigwedge \mathbf{for}(\mathbf{G}^n).$$

Proof. We use an induction on n.

Basis($n = 0$). It is not hard to see

$$\bot \equiv \bigwedge \mathbf{for}(\mathbf{G}^0).$$

Induction step($n > 0$). By the induction hypothesis,

$$\bot \equiv \bigwedge \mathbf{for}(\mathbf{G}^{n-1}).$$

So,

$$\bot \equiv \bigwedge \bigcup_{k=0}^{n-1} \mathbf{for}(\mathbf{G}^*(n-1)) \wedge \bigwedge \mathbf{for}(\mathbf{G}(n-1) - \mathbf{G}^*(n-1)).$$

Using Corollary 3.4(2),

$$\bot \equiv \bigwedge \bigcup_{k=0}^{n-1} \mathbf{for}(\mathbf{G}^*(n-1)) \wedge \bigwedge \mathbf{for}(\bigcup_{X \in \mathbf{G}(n-1)-\mathbf{G}^*(n-1)} \mathbf{next}(X)).$$

Hence

$$\bot \equiv \bigwedge \bigcup_{k=0}^{n-1} \mathbf{for}(\mathbf{G}^*(n-1)) \wedge \bigwedge \mathbf{for}(\mathbf{G}(n)) \equiv \bigwedge \mathbf{for}(\mathbf{G}^n).$$

$\dashv$

Lemma 3.14 *For a subset $\mathcal{S}$ of $\mathbf{G}^n$*

$$\bigwedge \mathbf{for}(\mathcal{S}) \supset \bot \equiv \bigwedge \mathbf{for}(\mathbf{G}^n - \mathcal{S}).$$

Proof. By Lemma 3.13,

$$\bigwedge \mathbf{for}(\mathbf{G}_n - \mathcal{S}) \to \bigwedge \mathbf{for}(\mathcal{S}) \supset \bot \in \mathbf{S4}.$$

By Lemma 2.5,

$$\to \bigwedge \mathbf{for}(\mathcal{S}), \bigwedge \mathbf{for}(\mathbf{G}^n - \mathcal{S}) \in \mathbf{S4}.$$

Using ($\supset\to$),

$$\bigwedge \mathbf{for}(\mathcal{S}) \supset \bot \to \bigwedge \mathbf{for}(\mathbf{G}^n - \mathcal{S}) \in \mathbf{S4}.$$

$\dashv$

Proof of Theorem 3.1. We use an induction on n.

Basis($n = 0$). The theorem follows from the results in Classical propositional logic.

Induction step($n > 0$). We use an induction on A.

If $A = \bot$, then from Lemma 3.13, we obtain the lemma.

If A is a propositional variable p_i, then by the induction hypothesis, there exists a subset $\mathcal{S} \subseteq \mathbf{G}^{n-1}$ such that $p_i \equiv \bigwedge \mathbf{for}(\mathcal{S})$. So,

$$p_i \equiv \bigwedge \mathbf{for}((\mathcal{S} \cap (\mathbf{G}(n-1) - \mathbf{G}^*(n-1))) \cup (\mathcal{S} \cap (\bigcup_{k=0}^{n-1} \mathbf{G}^*(k))).$$

Using Corollary 3.4(2),

$$p_i \equiv \bigwedge \mathbf{for}((\bigcup_{X \in \mathcal{S} \cap (\mathbf{G}(n-1) - \mathbf{G}^*(n-1))} \mathbf{next}(X)) \cup (\mathcal{S} \cap (\bigcup_{k=0}^{n-1} \mathbf{G}^*(k)))).$$

We note that

$$(\bigcup_{X \in \mathcal{S} \cap (\mathbf{G}(n-1) - \mathbf{G}^*(n-1))} \mathbf{next}(X)) \cup (\mathcal{S} \cap (\bigcup_{k=0}^{n-1} \mathbf{G}^*(k))) \subseteq \mathbf{G}^n.$$

If $A = B \wedge C$, then by the induction hypothesis, there exist subsets $\mathcal{S}_B$ and $\mathcal{S}_C$ of $\mathbf{G}^n$ such that

$$B \equiv \bigwedge \mathbf{for}(\mathcal{S}_B), \quad \text{and} \quad C \equiv \bigwedge \mathbf{for}(\mathcal{S}_C).$$

Using Lemma 3.2,

$$B \wedge C \equiv \bigwedge \mathbf{for}(\mathcal{S}_B) \wedge \bigwedge \mathbf{for}(\mathcal{S}_C) \equiv \bigwedge \mathbf{for}(\mathcal{S}_B \cup \mathcal{S}_C).$$

Similarly, if $A = B \vee C$, then

$$B \vee C \equiv \bigwedge \mathbf{for}(\mathcal{S}_B \cap \mathcal{S}_C).$$

Also, if $A = B \supset C$, then using Lemma 3.13,

$$B \supset C \equiv (B \supset \bot) \vee C \equiv \bigwedge \mathbf{for}((\mathbf{G}^n - \mathcal{S}_B) \cap \mathcal{S}_C).$$

If $A = \Box B$, then $B \in \mathbf{S}^{n-1}(\mathbf{V})$, using the induction hypothesis, there exists a subset $\mathcal{S}$ of $\mathbf{G}^{n-1}$ such that

$$B \equiv \bigwedge \mathbf{for}(\mathcal{S}).$$

Hence

$$A = \Box B \equiv (\bigwedge \Box \mathbf{for}(\mathcal{S} \cap \mathbf{G}(n-1))) \wedge (\bigwedge \Box \mathbf{for}(\mathcal{S} \cap (\bigcup_{k=0}^{n-2} \mathbf{G}^*(k)))).$$

By Lemma 3.12 and Lemma 3.9,

$$\bigwedge \Box \mathbf{for}(\mathcal{S} \cap \mathbf{G}(n-1)) \equiv \bigwedge \bigcup_{X \in \mathcal{S} \cap \mathbf{G}(n-1)} (\mathbf{cf}(X) \wedge \bigwedge \{\mathbf{for}(X_1) \mid X_1 \in \mathbf{G}(n), \Box \mathbf{for}(X) \in \mathbf{suc}(X_1)\}).$$

$$\equiv \bigwedge \mathbf{for}(\mathbf{S}_1 \cup \mathbf{S}_2),$$

where

$$\mathbf{S}_1 = \bigcup_{X \in \mathcal{S} \cap \mathbf{G}^*(n-1)} \{Y \in \mathbf{G}^*(n-1) \mid (\mathbf{ant}(X))^{\Box} = (\mathbf{suc}(Y))^{\Box}\},$$

$$\mathbf{S}_2 = \bigcup_{X \in \mathcal{S} \cap \mathbf{G}(n-1)} \{X_1 \in \mathbf{G}(n) \mid \Box \mathbf{for}(X) \in \mathbf{suc}(X_1)\}.$$

On the other hand, by the induction hypothesis, there exists a subset $\mathcal{T}$ of $\mathbf{G}^{n-1}$ such that

$$\bigwedge \Box \mathbf{for}(\mathcal{S} \cap (\bigcup_{k=0}^{n-2} \mathbf{G}^*(k))) \equiv \bigwedge \mathbf{for}(\mathcal{T}).$$

Using Corollary 3.4(2),

$$\bigwedge \Box \mathbf{for}(\mathcal{S} \cap (\bigcup_{k=0}^{n-2} \mathbf{G}^*(k))) \equiv \bigwedge \mathbf{for}(\mathcal{T}) \equiv \bigwedge \mathbf{for}(\mathbf{S}_3),$$

where

$$\mathbf{S}_3 = (\bigcup_{X \in \mathcal{T} \cap (\mathbf{G}(n-1) - \mathbf{G}^*(n-1))} \mathbf{next}(X)) \cup (\mathcal{T} \cap \bigcup_{k=0}^{n-1} \mathbf{G}^*(k)).$$

Hence

$$A = \Box B \equiv \bigwedge \mathbf{for}(\mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3)$$

and we note that $\mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3 \subseteq \mathbf{G}^n$. ⊣

4 On $\mathbf{prov}(X)$

In Definition 4.1, we use the provability of **S4** to define $\mathbf{prov}(X)$ for $X \in \mathbf{G}(n)$. In this section, we give the set without using the provability of **S4**.

Definition 4.1 For $X \in \mathbf{G}(n)$, we define $\mathbf{prov}_1(X)$, $\mathbf{prov}_2(X)$ and $\mathbf{prov}_3(X)$ as follows:

$$\mathbf{prov}_1(X) = \{(\Gamma \to \Delta, \Box \mathbf{for}(Y)) \in \mathbf{next}^+(X) \mid Y \in \mathbf{G}(n), (\mathbf{ant}(X))^{\Box} \not\subseteq (\mathbf{ant}(Y))^{\Box}\},$$

$$\mathbf{prov}_2(X) = \{(\Gamma \to \Delta, \Box \mathbf{for}(\Gamma_0 \to \Delta_0, \Box \mathbf{for}(Y_0))) \in \mathbf{next}^+(X) \mid Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1),$$
$$(\Gamma_0 \to \Delta_0, \Box \mathbf{for}(Y_0)) \in \mathbf{G}(n), \Box \mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid \Gamma_0^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}\}) \subseteq \Gamma \cap \Box \mathbf{for}(\mathbf{G}(n))\},$$

$$\mathbf{prov}_3(X) = \{(\Box \mathbf{for}(Y), \Gamma \to \Delta, \Box \mathbf{for}(Z)) \in \mathbf{next}^+(X) \mid Y, Z \in \mathbf{G}^*(n), (\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(Z))^{\Box}\}.$$

The purpose in this section is to prove

Theorem 4.2 *For $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,*

$$\mathbf{prov}(X) = \mathbf{prov}_1(X) \cup \mathbf{prov}_2(X) \cup \mathbf{prov}_3(X).$$

To prove the theorem above, we need some lemmas.

Lemma 4.3 *For $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,*

$$\mathbf{prov}_1(X) \subseteq \mathbf{prov}(X).$$

Proof. Let X_1 be in $\mathbf{prov}_1(X)$. Then $X_1 \in \mathbf{next}^+(X)$ and there exist finite sets Γ and Δ and a sequent $Y \in \mathbf{G}(n)$ such that
(1) $X_1 = (\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta, \Box\mathbf{for}(Y))$,
(2) $(\mathbf{ant}(X))^\Box \not\subseteq (\mathbf{ant}(Y))^\Box$.
Using Lemma 3.7, we have $X_1 \in \mathbf{S4}$, and hence, we obtain the lemma. ⊣

Lemma 4.4 *For $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,*

$$\mathbf{prov}_2(X) \subseteq \mathbf{prov}(X).$$

Proof. Let X_1 be in $\mathbf{prov}_2(X)$. Then $X_1 \in \mathbf{next}^+(X)$ and there exist finite sets Γ, Δ, Γ_0 and Δ_0 and a sequent $Y_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that
(1) $X_1 = (\Gamma \to \Delta, \Box\mathbf{for}(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0)))$,
(2) $(\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0)) \in \mathbf{G}(n)$,
(3) $\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid \Gamma_0^\Box \subseteq (\mathbf{ant}(Z))^\Box\}) \subseteq \Gamma \cap \Box\mathbf{for}(\mathbf{G}(n))$.
By Corollary 3.4(1), we have

$$\mathbf{for}(\mathbf{next}(Y_0)) \to Y_0 \in \mathbf{S4}.$$

Using $(\Box \to)$ and $(\to \Box)$, possibly several times,

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to \Box Y_0 \in \mathbf{S4}.$$

We define Y as $Y = (\Gamma_0 \to \Delta_0, \Box\mathbf{for}(Y_0))$. Then $\mathbf{ant}(Y) = \Gamma_0$ and

$$\Box\mathbf{for}(\mathbf{next}(Y_0)) \to Y \in \mathbf{S4}.$$

So,

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid \Gamma_0^\Box \subseteq (\mathbf{ant}(Z))^\Box\}), \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \to \mathbf{for}(Y) \in \mathbf{S4}.$$

Using (3),

$$\Gamma, \Box\mathbf{for}(\{Z \in \mathbf{next}(Y_0) \mid (\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\}) \to \mathbf{for}(Y) \in \mathbf{S4}.$$

Using $(\vee \to)$, possibly several times,

$$\Gamma, \{\mathbf{for}(Y) \vee \Box\mathbf{for}(Z) \mid Z \in \mathbf{next}(Y_0), (\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box\} \to \mathbf{for}(Y) \in \mathbf{S4}.$$

Using Lemma 3.7 and (*cut*), possibly several times,

$$\Gamma \to \mathbf{for}(Y) \in \mathbf{S4}.$$

So, we have $X_1 \in \mathbf{S4}$, and hence, we obtain the lemma. ⊣

Lemma 4.5 *For $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$,*

$$\mathbf{prov}_3(X) \subseteq \mathbf{prov}(X).$$

Proof. By Lemma 3.8, we obtain the lemma. ⊣

Lemma 4.6 *Let X be a sequent in $\mathbf{G}(n+1)$ and let X_0 be a sequent in $\mathbf{G}(n)$. Then*

$$X \in \mathbf{next}(X_0) \text{ if and only if } X_0 = (\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n).$$

Proof. By Lemma 3.6 and Definition 2.1, we obtain the lemma. ⊣

Lemma 4.7 *Let X be a sequent in $\mathbf{G}(n+k)$. Then*

(1) *for any $k \geq 0$, $(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{G}(n)$,*

(2) *for any $k \geq 1$, $\Box\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{suc}(X)$.*

(3) *for any $k \geq 1$ and for any $X_0 \in \mathbf{G}(n)$, $\mathbf{ant}(X_0) \subseteq \mathbf{ant}(X)$ and $\mathbf{suc}(X_0) \subseteq \mathbf{suc}(X)$ imply $\mathbf{ant}(X) \cap \mathbf{BG}_n = \mathbf{ant}(X_0)$, $\mathbf{suc}(X) \cap \mathbf{BG}_n = \mathbf{suc}(X_0)$ and $\Box\mathbf{for}(X_0) \in \mathbf{suc}(X)$.*

Proof. For (1). We use an induction on k.

Basis($k = 0$). By $X \in \mathbf{G}(n)$ and Lemma 3.6, $(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) = X \in \mathbf{G}(n)$.

Induction step($k > 0$). By $X \in \mathbf{G}(n+k)$, there exists a sequent $X_0 \in \mathbf{G}(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. By the induction hypothesis, we have

$$(\mathbf{ant}(X_0) \cap \mathbf{BG}_n \to \mathbf{suc}(X_0) \cap \mathbf{BG}_n) \in \mathbf{G}(n).$$

On the other hand, by Lemma 4.6,

$$\mathbf{ant}(X_0) = \mathbf{ant}(X) \cap \mathbf{BG}_{n+k-1} \text{ and } \mathbf{suc}(X_0) = \mathbf{suc}(X) \cap \mathbf{BG}_{n+k-1}.$$

So,

$$(\mathbf{ant}(X) \cap \mathbf{BG}_{n+k-1} \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_{n+k-1} \cap \mathbf{BG}_n) \in \mathbf{G}(n).$$

Since $k \geq 1$, $\mathbf{BG}_{n+k-1} \supseteq \mathbf{BG}_n$. Hence we obtain (1).

For (2). We use an induction on k.

Basis($k = 1$). By (1),

$$(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{G}(n).$$

Using Lemma 4.6,

$$X \in \mathbf{next}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n),$$

and using Definition 2.1, we obtain (2).

Induction step($k > 1$). By $X \in \mathbf{G}(n+k)$, there exists a sequent $X_0 \in \mathbf{G}(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. By the induction hypothesis, we have

$$\Box\mathbf{for}(\mathbf{ant}(X_0) \cap \mathbf{BG}_n \to \mathbf{suc}(X_0) \cap \mathbf{BG}_n) \in \mathbf{suc}(X_0).$$

Similarly to (1), we have

$$\mathbf{ant}(X_0) \cap \mathbf{BG}_n = \mathbf{ant}(X) \cap \mathbf{BG}_{n-k-1} \cap \mathbf{BG}_n = \mathbf{ant}(X) \cap \mathbf{BG}_n,$$

$$\mathbf{suc}(X_0) \cap \mathbf{BG}_n = \mathbf{suc}(X) \cap \mathbf{BG}_{n-k-1} \cap \mathbf{BG}_n = \mathbf{suc}(X) \cap \mathbf{BG}_n,$$

and so,

$$\Box\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{suc}(X_0).$$

By $X \in \mathbf{next}(X_0)$, we have $\mathbf{suc}(X_0) \subseteq \mathbf{suc}(X)$, and so, we obtain (2).

For (3). By $\mathbf{ant}(X_0) \subseteq \mathbf{ant}(X)$ and $\mathbf{suc}(X_0) \subseteq \mathbf{suc}(X)$, we have

$$\mathbf{ant}(X_0) \cap \mathbf{BG}_n \subseteq \mathbf{ant}(X) \cap \mathbf{BG}_n \text{ and } \mathbf{suc}(X_0) \cap \mathbf{BG}_n \subseteq \mathbf{suc}(X) \cap \mathbf{BG}_n.$$

Using $X_1 \in \mathbf{G}(n)$ and Lemma 3.6, we have

$$\mathbf{ant}(X_0) \subseteq \mathbf{ant}(X) \cap \mathbf{BG}_n \text{ and } \mathbf{suc}(X_0) \subseteq \mathbf{suc}(X) \cap \mathbf{BG}_n.$$

On the other hand, by (1) and Lemma 3.6, we have

$$\mathbf{ant}(X_0) \cup \mathbf{suc}(X_0) = (\mathbf{ant}(X) \cap \mathbf{BG}_n) \cup (\mathbf{suc}(X) \cap \mathbf{BG}_n) = \mathbf{BG}_n,$$

$$\mathbf{ant}(X_0) \cap \mathbf{suc}(X_0) = (\mathbf{ant}(X) \cap \mathbf{BG}_n) \cap (\mathbf{suc}(X) \cap \mathbf{BG}_n) = \emptyset.$$

Hence

$$\mathbf{ant}(X_0) = \mathbf{ant}(X) \cap \mathbf{BG}_n \text{ and } \mathbf{suc}(X_0) = \mathbf{suc}(X) \cap \mathbf{BG}_n.$$

Using (2), we obtain $\Box\mathbf{for}(X_0) = \Box\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_n \to \mathbf{suc}(X) \cap \mathbf{BG}_n) \in \mathbf{suc}(X)$. ⊣

Definition 4.8 For $X \in \mathbf{G}(n)$, the saturation of X, write $\mathbf{sat}(X)$, is defined as follows:

(1) if $n = 0$, then

$$\mathbf{sat}(X) = X,$$

(2) if $n > 0$, then

$$\mathbf{sat}(X) = (\Gamma_d, \Gamma_c, \mathbf{ant}(X), \{A \mid \Box A \in \mathbf{ant}(X))\} \to \mathbf{suc}(X), \Delta_c, \Delta_d, \Delta_f),$$

where

$\Gamma_c = \{\bigwedge S \mid S \subseteq \mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1)), \#(S) > 1\}$,

$\Gamma_d = \{\bigvee S \mid S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset, S \subseteq \mathbf{BG}_{n-1}, \#(S) > 1\}$,

$\Delta_c = \{\bigwedge S \mid S \cap (\mathbf{suc}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset, S \subseteq \mathbf{BG}_{n-1}, \#(S) > 1\}$,

$\Delta_d = \{\bigvee S \mid S \subseteq \mathbf{suc}(X) - \Box\mathbf{for}(\mathbf{G}(n-1)), \#(S) > 1\}$,

$\Delta_f = \{\mathbf{for}(\mathbf{ant}(X) \cap \mathbf{BG}_\ell \to \mathbf{suc}(X) \cap \mathbf{BG}_\ell) \mid \ell \leq n-1, \mathbf{ant}(X) \cap \mathbf{BG}_\ell \neq \emptyset\}$.

Remark 4.9 *Let X be a sequent in $\mathbf{G}(n)$. Then*

$$\mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X)) \textit{ and } \mathbf{suc}(X) \subseteq \mathbf{suc}(\mathbf{sat}(X)).$$

Lemma 4.10 *Let X be a sequent in $\mathbf{G}(n)$. Then*

$$\mathbf{ant}(\mathbf{sat}(X)) \cap \mathbf{suc}(\mathbf{sat}(X)) = \emptyset.$$

Proof. We use $\Gamma_c, \Gamma_d, \Delta_c, \Delta_d.\Delta_f$ as in Definition 4.8. We call a formula of the form $C \wedge D$ a $\wedge$-formula. Similarly, we use $\vee$-formula, $\supset$-formula and $\Box$-formula, We note that

(1) every member of $\Gamma_c \cup \Delta_c$ is a $\wedge$-formula,

(2) every member of $\Gamma_d \cup \Delta_d$ is a $\vee$-formula,

(3) every member of Δ_f is a $\supset$-formula.

Also by Lemma 3.6,

(4) every member of $\mathbf{ant}(X) \cup \mathbf{suc}(X)$ is either a $\Box$-formula or a member of $\mathbf{V}$.

Suppose that $A \in \mathbf{ant}(\mathbf{sat}(X)) \cap \mathbf{suc}(\mathbf{sat}(X))$. Then

(5) $A \in \Gamma_d \cup \Gamma_c \cup \mathbf{ant}(X) \cup \{C \mid \Box C \in \mathbf{ant}(X))\}$,

(6) $A \in \mathbf{suc}(X) \cup \Delta_c \cup \Delta_d \cup \Delta_f$.

By (5), we divide the cases.

The case that $A \in \Gamma_c$. By (1), (2), (3), (4) and (6), we have $A \in \Gamma_c \cap \Delta_c$. So, there exist sets S and S' such that $A = \bigwedge S = \bigwedge S'$, $S \subseteq \mathbf{ant}(X)$ and $S' \cap \mathbf{suc}(X) \neq \emptyset$. By $A = \bigwedge S = \bigwedge S'$, we have $S = S'$. Using the other conditions, there exists a formula $B \in S' \cap \mathbf{suc}(X) = S \cap \mathbf{suc}(X) \subseteq \mathbf{ant}(X) \cap \mathbf{suc}(X)$. This is in contradiction with Lemma 3.6.

The case that $A \in \Gamma_d$ can be shown similarly.

The case that $A \in \mathbf{ant}(X)$. By (1),(2),(3),(4) and (6), we have $A \in \mathbf{ant}(X) \cap \mathbf{suc}(X)$, which is in contradiction with Lemma 3.6.

The case that $A \in \{C \mid \Box C \in \mathbf{ant}(X)\}$. We have $\Box A \in \mathbf{ant}(X)$, and using Lemma 3.6, $n > 0$. If $A \in \Delta_f$, then by Lemma 4.7(2), $\Box A \in \mathbf{suc}(X)$, which is in contradiction with Lemma 3.6. So, using (6), $A \in \mathbf{suc}(X) \cup \Delta_c \cup \Delta_d$. On the other hand, by $\Box A \in \mathbf{ant}(X)$ and Lemma 3.6, there exist $\ell \in \{0, \cdots, n-1\}$

and $Y \in \mathbf{G}(\ell)$ such that $A = \mathbf{for}(Y)$. By $A \in \mathbf{suc}(X) \cup \Delta_c \cup \Delta_d$, (1), (2) and (4), A is not a $\supset$-formula, and we have, $\mathbf{ant}(Y) = \emptyset$ and $A = \mathbf{for}(Y) = \bigvee \mathbf{suc}(Y)$.

If $\#(\mathbf{suc}(Y)) = 1$, then $\mathbf{BG}_\ell \supseteq \mathbf{suc}(Y) = \{\mathbf{for}(Y)\} = \{A\}$, and using (6), $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$. If $\#(\mathbf{suc}(Y)) > 1$, then $\mathbf{for}(Y)$ is a $\vee$-formula, and using (1) and (4), we have $A = \mathbf{for}(Y) = \bigvee \mathbf{suc}(Y) \in \Delta_d$, and so, $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$. Hence in any case, $\mathbf{suc}(Y) \subseteq \mathbf{suc}(X)$. Also we note that $\mathbf{ant}(Y) = \emptyset \subseteq \mathbf{ant}(X)$. So, using Lemma 4.7(3), we have $\Box A = \Box\mathbf{for}(Y) \in \mathbf{suc}(X)$. This is in contradiction with $\Box A \in \mathbf{ant}(X)$ and Lemma 3.6. $\dashv$

Lemma 4.11 *Let X be a sequent in $\mathbf{G}(n)$ and let be that $\Phi \subseteq \mathbf{ant}(\mathbf{sat}(X))$ and $\Psi \subseteq \mathbf{suc}(\mathbf{sat}(X))$. If I is an inference rule in* **S4** *except $(\to \Box)$ and (cut) whose lower sequent is $\Phi \to \Psi$, then $\Phi_1 \subseteq \mathbf{ant}(\mathbf{sat}(X))$ and $\Psi_1 \subseteq \mathbf{suc}(\mathbf{sat}(X))$, for some upper sequent $\Phi_1 \to \Psi_1$ of I.*

Proof. We use $\Gamma_c, \Gamma_d, \Delta_c, \Delta_d.\Delta_f$ as in Definition 4.8. If I is a weakening rule, then the lemma is clear, and so, we assume that I is not a weakening rule. Let A be the principal formula of I. We divide the cases.

The case that $A \in \Gamma_d$. There exist a set S and a formula B such that
(1.1) $A = (\bigvee S) \vee B$,
(1.2) $(S \cup \{B\}) \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1)) \neq \emptyset$,
(1.3) $S \cup \{B\} \subseteq \mathbf{BG}_{n-1}$,
(1.4) $\#(S) > 0$.
Also I is

$$\frac{\bigvee S, \Phi^* \to \Psi \qquad B, \Phi^* \to \Psi}{\Phi \to \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By (1.2), we have either $S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$ or $\{B\} \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$. If $\{B\} \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$, then $B \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$, and so, the left upper sequent satisfies the conditions. If $S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$ and $\#(S) = 1$, then $\bigvee S \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$, and so, the left upper sequent satisfies the conditions. If $S \cap (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \neq \emptyset$ and $\#(S) > 1$, then using (1.3), $\bigvee S \in \Gamma_d \subseteq \mathbf{ant}(\mathbf{sat}(X))$, and so, the left upper sequent satisfies the conditions.

The case that $A \in \Delta_c$ can be shown similarly.

The case that $A \in \Gamma_c$. There exist a set S and a formula B such that
(2.1) $A = (\bigwedge S) \wedge B$,
(2.2) $S \subseteq \mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))$,
(2.3) $\{B\} \subseteq \mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))$,
(2.4) $\#(\mathbf{S}) > 0$.
Also I is either

$$\frac{\bigwedge S, \Phi^* \to \Psi}{\Phi \to \Psi} \quad \text{or} \quad \frac{B, \Phi^* \to \Psi}{\Phi \to \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By (2.3), $B \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$, So, the upper sequent $B, \Phi^* \to \Psi$ satisfies the conditions. By (2.2), if $\#(S) = 1$, then $\bigwedge S \in \mathbf{ant}(X) \subseteq \mathbf{ant}(\mathbf{sat}(X))$; if not, $\bigwedge S \in \Gamma_c \subseteq \mathbf{ant}(\mathbf{sat}(X))$. So, the upper sequent $\bigwedge S, \Phi^* \to \Psi$ satisfies the conditions.

The case that $A \in \Delta_d$ can be shown similarly.

The case that $A \in \mathbf{ant}(X) \cup \mathbf{suc}(X)$. None of the member of $\mathbf{V}$ is principal formula. So, by Lemma 3.6, $A = \Box B \in \mathbf{ant}(X)$. Since I is not $(\to \Box)$, I is

$$\frac{B, \Phi^* \to \Psi}{\Phi \to \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$. By $A = \Box B \in \mathbf{ant}(X)$, we have $B \in \{C \mid \Box C \in \mathbf{ant}(X)\} \subseteq \mathbf{ant}(\mathbf{sat}(X))$. So, the upper sequent satisfies the conditions.

The case that $A \in \{C \mid \Box C \in \mathbf{ant}(X)\}$. We note that $n > 0$. By Lemma 3.6, there exist $i \in \{0, \cdots, n-1\}$ and $Y \in \mathbf{G}(i)$ such that $A = \mathbf{for}(Y)$. We note that $\Box A = \Box\mathbf{for}(Y) \in \mathbf{ant}(X)$. We define Z as $Z = (\mathbf{ant}(X) \cap \mathbf{BG}_i \to \mathbf{suc}(X) \cap \mathbf{BG}_i)$. Then by Lemma 4.7, $Z \in \mathbf{G}(i)$ and $\Box\mathbf{for}(Z) \in \mathbf{suc}(X)$. Using $\Box\mathbf{for}(Y) \in \mathbf{ant}(X)$ and Lemma 3.6, we have $Y \neq Z$. Using Lemma 3.6, we have $\mathbf{ant}(Y) \neq \mathbf{ant}(Z)$. In other words, $\mathbf{ant}(Y) \not\subseteq \mathbf{ant}(Z)$ or $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y)$. We divide the subcases.

The subcase that $\mathbf{ant}(Y) \not\subseteq \mathbf{ant}(Z)$. We note that $\mathbf{ant}(Y) \neq \emptyset$. So, I is

$$\frac{\Phi_1 \to \Psi_1, \bigwedge \mathbf{ant}(Y) \qquad \bigvee \mathbf{suc}(Y), \Phi_2 \to \Psi_2}{\Phi \to \Psi},$$

where $\Phi_1 \cup \Phi_2 \in \{\Phi, \Phi - \{A\}\}$ and $\Psi_1 \cup \Psi_2 = \Psi$. On the other hand, by $\mathbf{ant}(Y) \not\subseteq \mathbf{ant}(Z)$, there exists a formula $B \in \mathbf{ant}(Y) - \mathbf{ant}(Z)$. Using Lemma 3.6,

$$B \in \mathbf{ant}(Y) \cap \mathbf{suc}(Z) \subseteq \mathbf{ant}(Y) \cap (\mathbf{suc}(X) \cap \mathbf{BG}_i) \subseteq \mathbf{ant}(Y) \cap (\mathbf{suc}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))).$$

So, if $\#(\mathbf{ant}(Y)) = 1$, then $\bigwedge \mathbf{ant}(Y) = \{B\} \in \mathbf{suc}(X)$; if not, $\bigwedge \mathbf{ant}(Y) \in \Delta_c$. Hence the left upper sequent of I satisfies the conditions.

The subcase that $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y) \neq \emptyset$. By $\mathbf{ant}(Y) \neq \emptyset$, I is

$$\frac{\Phi_1 \to \Psi_1, \bigwedge \mathbf{ant}(Y) \qquad \bigvee \mathbf{suc}(Y), \Phi_2 \to \Psi_2}{\Phi \to \Psi},$$

where $\Phi_1 \cup \Phi_2 \in \{\Phi, \Phi - \{A\}\}$ and $\Psi_1 \cup \Psi_2 = \Psi$. On the other hand, by $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y)$, there exists a formula $B \in \mathbf{ant}(Z) - \mathbf{ant}(Y)$. Using Lemma 3.6,

$$B \in \mathbf{ant}(Z) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) \cap \mathbf{BG}_i) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \cap \mathbf{suc}(Y).$$

So, if $\#(\mathbf{suc}(Y)) = 1$, then $\bigvee \mathbf{suc}(Y) = \{B\} \in \mathbf{ant}(X)$; if not, $\bigvee \mathbf{suc}(Y) \in \Gamma_d$. Hence the right upper sequent satisfies the conditions.

The subcase that $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y) = \emptyset$. By $\mathbf{ant}(Y) = \emptyset$ and Lemma 3.6, we have $\mathbf{suc}(Y) = \mathbf{BG}_i$. If $\#(\mathbf{suc}(Y)) = \#(\mathbf{BG}_i) = 1$, then $\mathbf{suc}(Y) = \{A\} \subseteq \mathbf{V}$, and so, A is not a principal formula. So, we assume that $\#(\mathbf{suc}(Y)) > 1$. Then I is

$$\frac{\bigvee(\mathbf{suc}(Y) - \{C\}), \Phi^* \to \Psi \qquad C, \Phi^* \to \Psi}{\Phi \to \Psi},$$

where $\Phi^* \in \{\Phi, \Phi - \{A\}\}$ and $\bigvee \mathbf{suc}(Y) = (\bigvee(\mathbf{suc}(Y) - \{C\})) \vee C$. On the other hand, we note by $\mathbf{ant}(Z) \not\subseteq \mathbf{ant}(Y)$, there exists a formula $B \in \mathbf{ant}(Z) - \mathbf{ant}(Y)$. Using Lemma 3.6,

$$B \in \mathbf{ant}(Z) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) \cap \mathbf{BG}_i) \cap \mathbf{suc}(Y) \subseteq (\mathbf{ant}(X) - \Box\mathbf{for}(\mathbf{G}(n-1))) \cap \mathbf{suc}(Y).$$

So, if $C = B$, then $C \in \mathbf{ant}(X)$, and so, the right upper sequent satisfies the conditions. If $C \neq B$, then $B \in \mathbf{suc}(Y) - \{C\}$ and $\bigvee(\mathbf{suc}(Y) - \{C\}) \in \Gamma_d$. So, the left upper sequent satisfies the conditions.

The case that $A \in \Delta_f$. There exists $\ell \leq n-1$ such that $A = \mathbf{for}(\mathbf{ant}(X) \cap \mathbf{S}_\ell \to \mathbf{suc}(X) \cap \mathbf{S}_\ell)$ and $\mathbf{ant}(X) \cap \mathbf{S}_\ell \neq \emptyset$. So, I is

$$\frac{\bigwedge(\mathbf{ant}(X) \cap \mathcal{S}_\ell), \Phi \to \Psi^*, \bigvee(\mathbf{suc}(X) \cap \mathbf{S}_\ell)}{\Phi \to \Psi},$$

where $\Psi^* \in \{\Psi, \Psi - \{A\}\}$. We note that $\bigwedge(\mathbf{ant}(X) \cap \mathcal{S}_\ell) \in \mathbf{ant}(X) \cup \Gamma_c$ and $\bigvee(\mathbf{suc}(X) \cap \mathbf{S}_\ell) \in \mathbf{suc}(X) \cup \Gamma_d$. So, the upper sequent of I satisfies the conditions. $\dashv$

Lemma 4.12 *Let X be a sequent in $\mathbf{G}(n+k)$ and let Y be a sequent in $\mathbf{G}^*(n)$. If $(\mathbf{ant}(Y))^\Box \neq (\mathbf{ant}(X))^\Box \cap \mathbf{BG}_n$. Then*

$$\to \mathbf{for}(Y), \Box\mathbf{for}(X) \in \mathbf{S4}.$$

Proof. We use an induction on k.

Basis($k = 0$). By $X \in \mathbf{G}(n)$ and Lemma 3.6, $(\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(X))^{\Box} \cap \mathbf{BG}_n \neq (\mathbf{ant}(X))^{\Box}$. Also by $Y \in \mathbf{G}^*(n)$, we have

$$(\mathbf{ant}(Y))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box} \text{ implies } (\mathbf{ant}(Y))^{\Box} = (\mathbf{ant}(Z))^{\Box}, \text{ for any } Z \in \mathbf{G}(n).$$

Hence we have $(\mathbf{ant}(Y))^{\Box} \not\subseteq (\mathbf{ant}(X))^{\Box}$. Using Lemma 3.7, we obtain the lemma.

Induction step($k > 0$). By $X \in \mathbf{G}(n+k)$, there exists $X_0 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. By Lemma 4.6,

$$(\mathbf{ant}(X_0))^{\Box} \cap \mathbf{BG}_n = (\mathbf{ant}(X))^{\Box} \cap \mathbf{BG}_{n+k-1} \cap \mathbf{BG}_n = (\mathbf{ant}(X))^{\Box} \cap \mathbf{BG}_n \neq (\mathbf{ant}(Y))^{\Box}.$$

So, by the induction hypothesis, we have

$$\to \mathbf{for}(Y), \Box\mathbf{for}(X_0) \in \mathbf{S4}.$$

On the other hand, we note that $\Box\mathbf{for}(X_0) \to \Box\mathbf{for}(X) \in \mathbf{S4}$, and using (*cut*), we obtain the lemma. ⊣

Corollary 4.13 *Let X be a sequent in $\mathbf{G}(n+k)$ and let Y be a sequent in $\mathbf{G}^*(n)$. If $(\mathbf{ant}(Y))^{\Box} \neq (\mathbf{ant}(X))^{\Box} \cap \mathbf{BG}_n$. Then*

$$(\Box\mathbf{for}(X) \supset \mathbf{for}(Y)) \equiv \mathbf{for}(Y).$$

Proof. By Lemma 4.12 and (*cut*), we obtain the corollary. ⊣

Lemma 4.14 *Let X be a sequent in $\mathbf{G}(n)$ and let Y_1 be a sequent in $\mathbf{G}^*(k)$ ($k \in \{0, 1, \cdots, n-1\}$). If $\Box\mathbf{for}(Y_1) \in \mathbf{suc}(X)$, then*

$$\mathbf{for}(\mathbf{ant}(X)^{\Box} \to \mathbf{for}(Y_1)) \equiv \mathbf{for}(Y_1).$$

Proof. We define X_1 as follows:

$$X_1 = (\mathbf{ant}(X) \cap \mathbf{BG}_k \to \mathbf{suc}(X) \cap \mathbf{BG}_k).$$

Then

$$(\mathbf{ant}(X))^{\Box} = (\mathbf{ant}(X) \cap \mathbf{BG}_k)^{\Box} \cup (\mathbf{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)))$$

$$= \mathbf{ant}(X_1)^{\Box} \cup (\mathbf{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box\mathbf{for}(\mathbf{G}(i))).$$

So, it is sufficient to show the following two:

(1) for any $A \in \mathbf{ant}(X) \cap \bigcup_{i=k}^{n-1} \Box\mathbf{for}(\mathbf{G}(i)))$, $A \supset \mathbf{for}(Y1) \equiv \mathbf{for}(Y_1)$,

(2) $\mathbf{for}(\mathbf{ant}(X_1)^{\Box} \to \mathbf{for}(Y_1)) \equiv \mathbf{for}(Y_1)$.

For (1). There exist a number $i \in \{k, k+1, \cdots, n-1\}$ and a sequent $Z \in \mathbf{G}(i)$ such that $A = \Box\mathbf{for}(Z)$. If $(\mathbf{ant}(Y_1))^{\Box} \neq (\mathbf{ant}(Z))^{\Box} \cap \mathbf{BG}_k$, then by Corollary 4.13, we obtain (1). So, we assume $(\mathbf{ant}(Y_1))^{\Box} = (\mathbf{ant}(Z))^{\Box} \cap \mathbf{BG}_k$. We divide the cases.

The case that $i = k$. Then by Lemma 3.8, we have

$$\Box\mathbf{for}(Z) \to \mathbf{for}(Y_1) \in \mathbf{S4}.$$

Using $(\to \Box)$, we have

$$\Box\mathbf{for}(Z) \to \Box\mathbf{for}(Y_1) \in \mathbf{S4}.$$

So, using $\Box\mathbf{for}(Y_1) \in \mathbf{suc}(X)$ and $\Box\mathbf{for}(Z) = A \in \mathbf{ant}(X)$, we have $X \in \mathbf{S4}$. Using Lemma 2.4(2), $X \notin \mathbf{G}(n)$, which is in contradiction with $X \in \mathbf{G}(n)$.

The case that $i > k$. We define Z_1 and Z_2 as follows:

$$Z_1 = (\mathbf{ant}(Z) \cap \mathbf{BG}_k \rightarrow \mathbf{suc}(Z) \cap \mathbf{BG}_k) \text{ and } Z_2 = (\mathbf{ant}(Z) \cap \mathbf{BG}_{k+1} \rightarrow \mathbf{suc}(Z) \cap \mathbf{BG}_{k+1}).$$

Then by Lemma 4.7, we have $Z_1 \in \mathbf{G}(k)$ and $Z_2 \in \mathbf{G}(k+1)$. Also by the assumption, we have $(\mathbf{ant}(Y_1))^{\Box} = (\mathbf{ant}(Z))^{\Box} \cap \mathbf{BG}_k = (\mathbf{ant}(Z_1))^{\Box}$, and using Lemma 3.9, $Z_1 \in \mathbf{G}^*(k)$. On the other hand, by $Z_2 \in \mathbf{G}(k+1)$, there exists a sequent $Z_1' \in \mathbf{G}(k) - \mathbf{G}^*(k)$ such that $Z_2 \in \mathbf{next}(Z_1')$. Using Lemma 4.6,

$$\begin{aligned} Z_1' &= (\mathbf{ant}(Z_2) \cap \mathbf{BG}_k \rightarrow \mathbf{suc}(Z_2) \cap \mathbf{BG}_k) \\ &= (\mathbf{ant}(Z) \cap \mathbf{BG}_{k+1} \cap \mathbf{BG}_k \rightarrow \mathbf{suc}(Z) \cap \mathbf{BG}_{k+1} \cap \mathbf{BG}_k) \\ &= (\mathbf{ant}(Z) \cap \mathbf{BG}_k \rightarrow \mathbf{suc}(Z) \cap \mathbf{BG}_k) = Z_1 \in \mathbf{G}^*(k), \end{aligned}$$

which is in contradiction with $Z_1' \in \mathbf{G}(k) - \mathbf{G}^*(k)$.

For (2). Suppose that $(\mathbf{ant}(X_1))^{\Box} \not\subseteq (\mathbf{ant}(Y_1))^{\Box}$. Then by Lemma 3.7, we have

$$\mathbf{ant}(X_1) \rightarrow \mathbf{suc}(X_1), \Box\mathbf{for}(Y_1) \in \mathbf{S4}.$$

So,

$$\mathbf{ant}(X) \cap \mathbf{BG}_k \rightarrow \mathbf{suc}(X) \cap \mathbf{BG}_k, \Box\mathbf{for}(Y_1) \in \mathbf{S4}.$$

Hence $X \in \mathbf{S4}$, which is in contradiction with Lemma 2.4(2) and $X \in \mathbf{G}(n)$. So, we have $(\mathbf{ant}(X_1))^{\Box} \subseteq (\mathbf{ant}(Y_1))^{\Box}$, and so,

$$(\bigwedge(\mathbf{ant}(X_1))^{\Box}) \supset \mathbf{for}(Y_1) \equiv (\bigwedge(\mathbf{ant}(X_1))^{\Box} \cup \mathbf{ant}(Y_1)) \supset \bigvee \mathbf{suc}(Y_1) \equiv \bigwedge \mathbf{ant}(Y_1) \supset \bigvee \mathbf{suc}(Y_1).$$

Hence we obtain (2). $\dashv$

Lemma 4.15 *Let X be a sequent in $\mathbf{G}(n+k)$ and let Y_0 be a sequent in $\mathbf{G}(n)-\mathbf{G}^*(n)$. Let X_1 be a sequent in $\mathbf{next}(X)$. If $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)$, then there exists a sequent $Y \in \mathbf{G}^{n+k}$ such that $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$, $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y)$.*

Proof. We use an induction on k.

Basis($k = 0$). The lemma is clear from $Y_0 \in \mathbf{G}(n)$ and $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)$.

Induction step($k > 0$). By $X \in \mathbf{G}(n+k)$, there exists a sequent $X_0 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$ such that $X \in \mathbf{next}(X_0)$. Also by $k > 0$ and Lemma 3.6, $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n)) = \mathbf{suc}(X) \cap \Box\mathbf{for}(\mathbf{G}(n) - \mathbf{G}^*(n))$. Using the induction hypothesis, there exists a sequent $Y_2 \in \mathbf{G}^{n+k-1}$ such that $\Box\mathbf{for}(Y_2) \in \mathbf{suc}(X)$, $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y_2)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y_2)$. If $Y_2 \in \bigcup_{i=0}^{n+k-1} \mathbf{G}^*(i)$, then $Y_2 \in \mathbf{G}^{n+k}$, and we obtain the lemma. So, we assume that $Y_2 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$. On the other hand, by Lemma 2.4 and Lemma 4.4, we have $X_1 \notin \mathbf{prov}_2(X)$. Using the four conditions

$\Box\mathbf{for}(X) \in \mathbf{suc}(X_1)$,
$\Box\mathbf{for}(Y_2) \in \mathbf{suc}(X)$,
$Y_2 \in \mathbf{G}(n+k-1) - \mathbf{G}^*(n+k-1)$ and
$X \in \mathbf{G}(n+k)$,

we have

$$\Box\mathbf{for}(\{Z \in \mathbf{next}(Y_2) \mid (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Z))^{\Box}\}) \not\subseteq \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n+k)).$$

So, there exists a sequent $Y \in \mathbf{next}(Y_2)$ such that $(\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box}$ and $\Box\mathbf{for}(Y) \notin \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n+k))$. By $Y \in \mathbf{next}(Y_2)$, we have $Y \in \mathbf{G}(n+k) \subseteq \mathbf{G}^{n+k}$. Using $\Box\mathbf{for}(Y) \notin \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n+k))$ and Lemma 3.6, we have $\Box\mathbf{for}(Y) \notin \mathbf{ant}(X_1)$ and $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$. Also by $Y \in \mathbf{next}(Y_2)$, we have $\mathbf{ant}(Y_2) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_2) \subseteq \mathbf{suc}(Y)$. Hence we have $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y)$. $\dashv$

Lemma 4.16 *Let X and Y be sequents in $\mathbf{G}(n) - \mathbf{G}^*(n)$ and let X_1 be a sequent in $\mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X))$. If $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$, then*

$$(\Gamma_Y, \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Delta_Y) \in \mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X)),$$

where
$\Delta_Y = \{\Box\mathbf{for}(Z) \in \mathbf{suc}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid \mathbf{ant}(Y)^\Box \subseteq \mathbf{ant}(Z)^\Box\}$ *and*
$\Gamma_Y = \{\Box\mathbf{for}(Z) \in \mathbf{suc}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)) \mid \mathbf{ant}(Y)^\Box \not\subseteq \mathbf{ant}(Z)^\Box\}$.

Proof. we define the sequent Y_1 as follows:

$$Y_1 = (\Gamma_Y, \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Delta_Y).$$

It is not hard to see that $Y_1 \in \mathbf{next}^+(Y)$. So, it is sufficient to show the following three:
(1) $Y_1 \notin \mathbf{prov}_1(Y)$,
(2) $Y_1 \notin \mathbf{prov}_2(Y)$,
(3) $Y_1 \notin \mathbf{prov}_3(Y)$.

For (1). Suppose that $Y_1 \in \mathbf{prov}_1(Y)$. Then there exists a sequent $Z \in \mathbf{G}(n)$ such that $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_1)$, $(\mathbf{ant}(Y))^\Box \neq\subseteq (\mathbf{ant}(Z))^\Box$. By Lemma 3.6, we have $\Box\mathbf{for}(Z) \notin \mathbf{BG}_n \supseteq \mathbf{suc}(Y)$, and using $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_1) = \mathbf{suc}(Y) \cup \Delta_Y$, we have $\Box\mathbf{for}(Z) \in \Delta_Y$. So, $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box$. This is in contradiction with $(\mathbf{ant}(Y))^\Box \not\subseteq (\mathbf{ant}(Z))^\Box$.

For(2). Suppose that $Y_1 \in \mathbf{prov}_2(Y)$. Then there exist sequents $Z \in \mathbf{G}(n)$ and $Z_0 \in \mathbf{G}(n-1) - \mathbf{G}^*(n-1)$ such that
(2.1) $\Box\mathbf{for}(Z) \in \mathbf{suc}(Y_1)$,
(2.2) $\Box\mathbf{for}(Z_0) \in \mathbf{suc}(Z)$,
(2.3) $\Box\mathbf{for}(\{Z' \in \mathbf{next}(Z_0) \mid (\mathbf{ant}(Z))^\Box \subseteq (\mathbf{ant}(Z'))^\Box\}) \subseteq \mathbf{suc}(Y_1) \cap \Box\mathbf{for}(\mathbf{G}(n))$.
Similarly to (1), by (2.1), we have
(2.4) $\Box\mathbf{for}(Z) \in \mathbf{suc}(X_1)$,
Also by Lemma 3.6, we have $\mathbf{suc}(Y_1) \cap \Box\mathbf{for}(\mathbf{G}(n)) = \Delta_Y$, and using (2.3), we have
(2.5) $\Box\mathbf{for}(\{Z' \in \mathbf{next}(Z_0) \mid (\mathbf{ant}(Z))^\Box \subseteq (\mathbf{ant}(Z'))^\Box\}) \subseteq \Delta_Y \subseteq \mathbf{suc}(X_1) \cap \mathbf{G}(n)$.
By (2.4), (2.2), (2.5) and $X_1 \in \mathbf{next}^+(X)$, we obtain $X_1 \in \mathbf{prov}_2(X)$, which is in contradiction with $X_1 \notin \mathbf{prov}_2(X)$.

For (3). Suppose that $Y_1 \in \mathbf{prov}_3(Y)$. Then there exist sequents $Z, Z' \in \mathbf{G}^*(n)$ such that
(3.1) $\Box\mathbf{for}(Z) \in \mathbf{ant}(Y_1)$,
(3.2) $\Box\mathbf{for}(Z') \in \mathbf{suc}(Y_1)$
(3.3) $(\mathbf{ant}(Z))^\Box = (\mathbf{ant}(Z'))^\Box$.
Similarly to (1), by (3.2), we have
(3.4) $\Box\mathbf{for}(Z') \in \Delta_Y \subseteq \mathbf{suc}(X_1)$.
By $\Box\mathbf{for}(Z') \in \Delta_Y$, we have $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z'))^\Box$. Using (3.3), $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(Z))^\Box$. So, we have $\Box\mathbf{for}(Z') \notin \Gamma_Y$. Using (3.1), we have $\Box\mathbf{for}(Z') \in \mathbf{ant}(X_1) \cup \mathbf{ant}(Y)$. Similarly to (1), we have
(3.5) $\Box\mathbf{for}(Z') \in \mathbf{ant}(X_1)$.
By (3.4), (3.5), (3.3) and $X_1 \in \mathbf{next}^+(X)$, we obtain $X_1 \in \mathbf{prov}_3(X)$, which is in contradiction with $X_1 \notin \mathbf{prov}_3(X)$. ⊣

Lemma 4.17 *Let $\mathcal{P}$ be a cut-free proof figure in* **S4** *whose end sequent is $\Phi \to \Psi$. Then for any $X \in \mathbf{G}(n) - \mathbf{G}^*(n)$ and for any $X_1 \in \mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X))$,*

$$(\Phi \to \Psi) \notin \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

Proof. We use an induction on $\mathcal{P}$.

Basis($\mathcal{P}$ consists of an axiom). Suppose that

$$(\Phi \to \Psi) \in \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

Then by Lemma 4.10, $\Phi \cap \Psi = \emptyset$, which is not an axiom.

Induction step ($\mathcal{P}$ has the inference rule introducing the end sequent). Suppose that

$$(\Phi \to \Psi) \in \{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

and let I be the inference rule introducing the end sequent in $\mathcal{P}$.

If I is not $(\to \Box)$, then by Lemma 4.11, an upper sequent I belongs to

$$\{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(X_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(X_1))\}.$$

This is in contradiction with the induction hypothesis.

So, we assume that I is $(\to \Box)$. Then there exist a set Γ and a sequent Y_0 such that

(1) $\Gamma \subseteq \mathbf{ant}(X_1)^{\Box}$,

(2) $\Box\mathbf{for}(Y_0) \in \mathbf{suc}(X_1)^{\Box}$,

(3) $(\Phi \to \Psi) = (\Gamma \to \Box\mathbf{for}(Y_0))$,

(4) I is $\dfrac{\Gamma \to \mathbf{for}(Y_0)}{\Gamma \to \Box\mathbf{for}(Y_0)}$.

We divide the cases.

The case that $Y_0 \in \mathbf{G}^*(k)$ for some $k \leq n$. By Lemma 4.14, (1) and (2),

$$\mathbf{for}(\Gamma \to \mathbf{for}(Y_0)) \equiv \mathbf{for}(\mathbf{ant}(X_1)^{\Box} \to \mathbf{for}(Y_0)) \equiv \mathbf{for}(Y_0).$$

Using (4), we have $Y_0 \in \mathbf{S4}$, which is in contradiction with $Y_0 \in \mathbf{G}^*(k)$ and Lemma 2.4(2).

The case that $Y_0 \notin \mathbf{G}^*(k)$ for any $k \leq n$. Then by Lemma 3.6, $Y_0 \in \mathbf{G}(k) - \mathbf{G}^*(k)$ for some $k \leq n$. Using (2) and Lemma 4.15, there exists a sequent $Y \in \mathbf{G}^n$ such that

(5) $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$,

(6) $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y)$.

By (6), we have $\mathbf{for}(Y_0) \to \mathbf{for}(Y) \in \mathbf{S4}$, and using (4), we have $\Gamma \to \mathbf{for}(Y) \in \mathbf{S4}$. If $Y \in \mathbf{G}^*(i)$ for some $i \leq n$, then using (1), (5) and Lemma 4.14, we obtain a contradiction similarly to the above case. So, by $Y \in \mathbf{G}^n$ we can assume that $Y \in \mathbf{G}(n) - \mathbf{G}^*(n)$. Then by (5) and Lemma 4.16,

$$Y_1 = (\Gamma_Y, \mathbf{ant}(X_1) \cap \Box\mathbf{for}(\mathbf{G}(n)), \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Delta_Y) \in \mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X)),$$

where Δ_Y and Γ_Y are as in Lemma 4.16. By (6), we have $\mathbf{ant}(Y_0) \subseteq \mathbf{ant}(Y_1)$ and $\mathbf{suc}(Y_0) \subseteq \mathbf{suc}(Y_1)$. Using Lemma 4.7(3),

$$\mathbf{for}(Y_0) = \mathbf{for}(\mathbf{ant}(Y_0) \to \mathbf{suc}(Y_0)) = \mathbf{for}(\mathbf{ant}(Y_1) \cap \mathbf{BG}_k \to \mathbf{suc}(Y_1) \cap \mathbf{BG}_k) \in \mathbf{suc}(\mathbf{sat}(Y_1)).$$

On the other hand, by $\Box\mathbf{for}(Y) \in \mathbf{suc}(X_1)$ and $Y \in \mathbf{G}(n)$, we have

$$(\mathbf{ant}(X))^{\Box} \not\subseteq (\mathbf{ant}(Y))^{\Box} \text{ implies } X_1 \in \mathbf{prov}_1(X).$$

So, using $X_1 \notin \mathbf{prov}_1(X)$, we have

$$\Gamma \subseteq (\mathbf{ant}(X))^{\Box} \subseteq (\mathbf{ant}(Y))^{\Box} \subseteq \mathbf{ant}(\mathbf{sat}(Y_1)).$$

So, the upper sequent of I belongs to

$$\{(\Phi^* \to \Psi^*) \mid \Phi^* \subseteq \mathbf{ant}(\mathbf{sat}(Y_1)), \Psi^* \subseteq \mathbf{suc}(\mathbf{sat}(Y_1))\}$$

for $Y_1 \in \mathbf{next}^+(X) - (\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X))$. This is in contradiction with the induction hypothesis. ⊣

By the above lemma and Lemma 1.1(2), we obtain

Corollary 4.18 *Let X be a sequent in $\mathbf{G}(n) - \mathbf{G}^*(n)$. Then*

$$\mathbf{prov}_2(X) \cup \mathbf{prov}_1(X) \cup \mathbf{prov}_3(X) \supseteq \mathbf{prov}(X).$$

From Lemma 4.3, Lemma 4.4, Lemma 4.5 and Corollary 4.18, we obtain Theorem 4.2.

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