A reflection combined with tail club guessing negates weak squares

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Abstract

We show a weak form of reflection principle combined with tail club guessing negates a class of weak squares.

Introduction

In [F] and [V], it is shown that the class of weak squares $\square^*_\kappa$ are all negated by the Strong Reflection Principle (SRP) of [B]. If weak square $\square^*_\kappa$ holds, then there exists a stationary subset $S \subseteq [\kappa]^\omega$ such that for any $\alpha < \kappa$, $S \cap [\alpha]^\omega$ is never stationary ([V]). But SRP implies every stationary subset $S \subseteq [\kappa]^\omega$ gets reflected in a stronger manner ([B]).

Meanwhile, [I] investigates tail club guessing in detail and constructs a model of set theory where a corresponding ideal is saturated. Remember SRP implies the non-stationary ideal on ω_1 is saturated ([B]). In [M], we launch a weak form of SRP compatible with tail club guessing on $A \subseteq \omega_1$ to see connection between [B] and [I]. It is consistent that SRP fails yet tail club guessing on $A \subseteq \omega_1$ and its associated SRP-like principle holds ([M]).

We record some of the consequences of this weak SRP-like principle of [M] combined with tail club guessing on $A \subseteq \omega_1$. More specifically, we first show (2.2 theorem) that for any regular cardinal $\kappa > \omega_1$ and any stationary subset $S \subseteq \{\alpha < \kappa \mid cf(\alpha) = \omega\}$, S gets reflected under our weak assumption. In particular, the ordinary square $\square_\kappa$ must fail. We further show (4.2 theorem) that the weak squares $\square^*_\kappa$ are all negated under this same assumption. To do so, we consider a closed game similar to [V]. Hence as far as $\square^*_\kappa$ are concerned, SRP and our weak SRP-like principle combined with tail club guessing have the same effects.

However, it is plausible under our weak assumption to have a stationary subset $S \subseteq [\omega_2]^\omega$ such that S does not get reflected to any $[\alpha]^\omega$ with $\alpha < \omega_2$. But I do not know how to construct this S. Recall that SRP eliminates every such S ([B]).

It is well-known that the Martin's Maximum (MM) implies SRP ([B]). It is easy to show that the Bounded Proper Forcing Axiom (BPFA) negates every possible tail club guessing on $A \subseteq \omega_1$. However, I do not know SRP alone negates every possible tail club guessing on $A \subseteq \omega_1$. We know $+$-type forcing axiom for a σ-closed p.o. set together with SRP eliminates every possible tail club guessing on $A \subseteq \omega_1$ ([M]).

§1. Tail club guessing and associated reflection principle

We list main notions and objects of our study. We first recap from [I] and [M].

1.1 Definition. $\langle C_\delta \mid \delta \in A\rangle$ is a *ladder system (on A)*, if

- $A \subseteq \{\delta < \omega_1 \mid \delta \text{ is limit}\}$,
- Each C_δ is a cofinal subset of δ with the order-type ω.

When we enumerate the elements of C_δ increasingly, we write $C_\delta = \{\delta_n \mid n < \omega\}$.

1.2 Definition. A ladder system $\langle C_\delta \mid \delta \in A\rangle$ is *tail club guessing (on A)*, if for all clubs $D \subseteq \omega_1$, there exist $\delta \in A$ such that $C_\delta \subseteq^* D$. This means there exists $m < \omega$ such that for all n with $m \leq n < \omega$, we have $\delta_n \in D$. We write

$$X^*(D) = \{\delta \in A \mid C_\delta \subseteq^* D\}.$$

Hence $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessig iff for all clubs D, we have $X^*(D) \neq \emptyset$.

We refer to a weak reflection of [M] as follows;

1.3 Definition. Let $\langle C_\delta \mid \delta \in A\rangle$ be tail club guessing. The *associated reflection principle* is the following statement.

Let (K, S, θ, a) be such that

- $K \supseteq \omega_1$,
- $S \subseteq [K]^\omega$,
- θ is a regular cardinal such that $K \in H_{|TC(K)|^+} \in H_{(2^{|TC(K)|})^+} \in H_\theta$,
- $a \in H_\theta$.

Then there exists $(D, \langle N_i \mid i < \omega_1\rangle)$ such that

- D is a club in ω_1,
- $\langle N_i \mid i < \omega_1\rangle$ is an $\in$-chain in H_θ with $a \in N_0$. Namely,
  - $(N_i, \in)$ is a countable elementary substructure of $(H_\theta, \in)$,
  - $\langle N_i \mid i \le j\rangle \in N_{j+1}$,
  - For limit i, we have $N_i = \bigcup\{N_j \mid j < i\}$,
- For $\delta \in X^*(D)$, either the following (1) or (2) holds.
  (1) $N_\delta \cap K \in S$,
  (2) For any $\in$-chain $\langle N'_n \mid n \le \omega\rangle$ in H_θ such that for all $n < \omega$, $N_{\delta_n} \subseteq_{\omega_1} N'_n$, we have $N'_\omega \cap K \notin S$, where $N_{\delta_n} \subseteq_{\omega_1} N'_n$ means $N_{\delta_n} \subseteq N'_n$ and $N_{\delta_n} \cap \omega_1 = N'_n \cap \omega_1$.

Notice that in (2), it suffices to prepare $\langle N'_n \mid m \le n < \omega\rangle$ for any $m < \omega$. This is because we may think of $N'_n = N_{\delta_n}$ for $n < m$. Then $N_{\delta_{m-1}} \in N_{\delta_m} \subseteq N'_m$ and so

$$N_{\delta_0} \in \cdots \in N_{\delta_{m-1}} \in N'_m \in N'_{m+1} \in \cdots$$

The following defines a class of weak squares $\square^*_\kappa$ found in [F] and [V]. If the usual square $\square_\kappa$ holds, then $\square^*_{\kappa^+}$ holds. Hence we may refer to $\square^*_\kappa$ a weak square.

1.4 Definition. Let κ be a regular cardinal with $\kappa > \omega_1$. The weak square $\square^*_\kappa$ holds, if there exists $\langle D_\gamma \mid \gamma < \kappa,\ \gamma \text{ is limit}\rangle$ such that

- Each D_γ is a club in γ,
- If $\overline{D_\gamma}$ denotes the set of limit points of D_γ below γ, then for any $\beta \in \overline{D_\gamma}$, we have $D_\beta = D_\gamma \cap \beta$ (coherence),
- There exists no club C of κ such that for all $\gamma \in \overline{C}$, we have $D_\gamma = C \cap \gamma$.

§2. 1 H lemma and reflecting stationary sets of ordinals with the cofinality ω

We prepare a lemma to enlarge elementary substructures. This is based on [B] and [I].

2.1 Lemma. (1 H lemma) *Let θ be a regular cardinal and $(N, \in)$ be an elementary substructure of $(H_\theta, \in)$. Let $s \in K \in N$ and set*

$$N(s) = \{f(s) \mid f \in N\}.$$

Then $(N(s), \in)$ is an elemetary substructure of $(H_\theta, \in)$ such that $\{s\} \cup N \subseteq N(s)$ holds.

Proof. Let $f_1, \cdots, f_n \in N$ so that $f_1(s), \cdots, f_n(s) \in N(s)$. Let $\varphi(v_1, \cdots, v_n, v)$ be a formula. Take $g \in H_\theta$ such that

$$H_\theta \models \text{“for any } a \in K \text{ and } b,\ \text{if } \varphi\bigl(f_1(a), \cdots, f_n(a), b\bigr) \text{ holds, then } \varphi\bigl(f_1(a), \cdots, f_n(a), g(a)\bigr)\text{”}.$$

This is possible as $K \in H_\theta$ and θ is regular. Hence a set of possible values of g is of size less than θ.

Since $(f_1, \cdots, f_n), K \in N$, we may assume $g \in N$. Now if $H_\theta \models$ "$\exists y\ \varphi(f_1(s), \cdots, f_n(s), y)$", then we have $H_\theta \models$ "$\varphi(f_1(s), \cdots, f_n(s), g(s))$". Hence by the Tarski's criterion, we conclude that $(N(s), \in)$ is an elementary substructure of $(H_\theta, \in)$.

For $b \in N$, let $f = \{(a, b) \mid a \in K\}$. Then $f \in N$ and $b = f(s) \in N(s)$. Hence $N \subseteq N(s)$.

Let $id = \{(a, a) \mid a \in K\}$. Then $id \in N$ and $s = id(s) \in N(s)$. Hence $s \in N(s)$ holds

□

Tail club guessing together with its associated reflection principle implies the ordinary reflection principle for stationary sets $S \subseteq \{\alpha < \kappa \mid cf(\alpha) = \omega\}$. Remember SRP implies this reflection of stationary sets and much more ([B]).

2.2 Theorem. *Let $\langle C_\delta \mid \delta \in A\rangle$ be tail club guessing. If the associated reflection principle holds, then for any regular cardinal $\kappa > \omega_1$ and any stationary $S \subseteq \{\alpha < \kappa \mid cf(\alpha) = \omega\}$, there exists $\gamma < \kappa$ such that $S \cap \gamma$ is stationary in γ.*

Proof. Fix κ and S and let

$$S^* = \{X \in [\kappa]^\omega \mid sup(X) \in S\}.$$

Let θ be a sufficiently large regular cardinal. Apply the associated reflection principle to $(\kappa, S^*, \theta, \kappa)$. Then we have a club D^0 and an $\in$-chain $\langle N_i \mid i < \omega_1\rangle$ in H_θ such that for each $\delta \in X^*(D^0)$, either the following (1) or (2) holds.

(1) $N_\delta \cap \kappa \in S^*$.

(2) For any $\in$-chain $\langle N'_n \mid n \leq \omega\rangle$ such that for all $n < \omega$, $N_{\delta_n} \subseteq_{\omega_1} N'_n$, we have $N'_\omega \cap \kappa \notin S^*$, where $C_\delta = \{\delta_n \mid n < \omega\}$ enumerated increasingly.

Let

$$B = \{\delta \in A \mid N_\delta \cap \kappa \in S^*\}$$

and let

$$\gamma = sup\{sup(N_i \cap \kappa) \mid i < \omega_1\}.$$

Then $\{sup(N_i \cap \kappa) \mid i < \omega_1\}$ is a club in γ and

$$\{sup(N_\delta \cap \kappa) \mid \delta \in B\} \subseteq S \cap \gamma.$$

Therefore the following suffices.

Claim. *B is positive. Namely, for any club $D^1 \subseteq \omega_1$, there exists $\delta \in B$ with $C_\delta \subseteq^* D^1$.*

Proof. Since S is stationary, we may take an elementary substructure M of H_θ such that $\langle N_i \mid i < \omega_1\rangle, D^0, D^1 \in M$ and $\omega_1 < M \cap \kappa \in S$. Since $cf(M \cap \kappa) = \omega$, we may fix $\langle s_n \mid n < \omega\rangle$ such that $\{s_n \mid n < \omega\}$ is cofinal in $M \cap \kappa$.

We then take a sequence of countable elementary substructures $\langle M^i \mid i < \omega_1\rangle$ of H_θ such that

- $\{D^0, \langle N_i \mid i < \omega_1\rangle, D^1\} \cup \{s_n \mid n < \omega\} \subset M^i \subset M$,
- $\langle M^i \cap \omega_1 \mid i < \omega_1\rangle$ is continously increasing. Hence it forms a club in ω_1.

Notice that $\langle M^i \mid i < \omega_1\rangle$ is not an $\in$-chain in H_θ and $M^i \cap \omega_1 \in D^0 \cap D^1$ holds.

Since $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing, there exists $\delta \in A$ such that $C_\delta \subseteq^* \{M^i \cap \omega_1 \mid i < \omega_1\}$. By reindexing, we may assume that $\{M_n \cap \omega_1 \mid n < \omega\}$ is an end-segment of C_δ.

Since $\langle N_i \mid i < \omega_1\rangle \in M_n$, we have $N_{M_n \cap \omega_1} \subseteq_{\omega_1} M_n$. Apply 2.1 lemma (1 H lemma) to enlarge each $N_{M_n \cap \omega_1}$ to

$$N'_n = N_{M_n \cap \omega_1}(\{s_0, \cdots, s_n\}).$$

This is possible, as $\{s_0, \cdots, s_n\} \in [\kappa]^{<\omega} \in N_{M_n \cap \omega_1}$. Since $N_{M_n \cap \omega_1} \cup \{s_0, s_1, s_2, \cdots\} \subseteq M_n$, we have

$$N'_n \subseteq M_n$$

and so

$$N_{M_n\cap\omega_1} \subseteq_{\omega_1} N'_n.$$

Since $N_{M_n\cap\omega_1} \in N_{M_{n+1}\cap\omega_1} \subseteq N'_{n+1}$ and $\{s_0,\cdots,s_n\} \subset \{s_0,\cdots,s_{n+1}\} \in N'_{n+1}$, we have

$$N_{M_n\cap\omega_1}, \{s_0,\cdots,s_n\} \in N'_{n+1}$$

and so

$$N'_n \in N'_{n+1}.$$

Let $N'_\omega = \bigcup\{N'_n \mid n<\omega\}$. Then $N'_\omega \subset M$ and $sup(N'_\omega \cap \kappa) = M\cap\kappa \in S$. Hence $N'_\omega \cap \kappa \in S^*$. Since $\delta \in X^*(D^0)$, we conclude $N_\delta \cap \kappa \in S^*$. Hence $\delta \in B$ and $C_\delta \subseteq^* D^1$.

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§3. **A closed game**

We consider a game similar to a closed game of [F] and [V]. We intend to formalize this subject in terms of sequences and trees of sequences. Hence a play is a sequence of specific types of objects listed and a strategy is an alternating tree in this note. If you are comfortable with the notion of closed games, then you may just observe that the game proposed here is closed for the player I. Hence this game is determined right away.

3.1 Definition. Let κ be a regular cardinal with $\kappa > \omega_1$, $f : [\kappa]^{<\omega} \longrightarrow \kappa$ and $\langle \delta_n \mid n<\omega\rangle$ be strictly increasing to $\delta < \omega_1$. We first define three unary predicates with the parameters f and $\langle \delta_n \mid n<\omega\rangle$. Notice that κ is definable from f.

A play b in the game $G(f, \langle\delta_n \mid n<\omega\rangle)$ means that b is a sequence of length ω such that

(1) $b = \langle (I_0,\alpha_0), \beta_0, (I_1,\alpha_1), \beta_1, \cdots, (I_k,\alpha_k), \beta_k, \cdots\rangle$,
(2) $\delta \subseteq I_0 = [x_0, y_0]$, $y_0 < \kappa$ and $\alpha_0 \in I_0$,
(3) $I_k = [x_k, y_k]$, $0 \le x_k < y_k < \kappa$ and $\alpha_k \in I_k$,
(4) $y_k < \beta_k < \kappa$ and β_k is f-closed.
(5) $\beta_k < x_{k+1}$ and $I_{k+1} = [x_{k+1}, y_{k+1}]$.

It is customary to view that a play in this game is played by two players I and II. The player I initiates a play. Then the player II follows. They take turn alternatingly so that

$$(I_0,\alpha_0), (I_1,\alpha_1), \cdots, (I_k,\alpha_k), \cdots$$

are played by the player I and

$$\beta_0, \beta_1, \cdots, \beta_k, \cdots$$

are played by the player II.

The player I *wins the play* b (*in the game*), if we define X_n by

$$X_n = \overline{\delta_n \cup \{\alpha_0,\alpha_1,\alpha_2,\cdots\}}$$

where $\overline{Z}$ denotes the f-closure of $Z \subset \kappa$. Then the following two are satisfied.

- For all $n<\omega$, $X_n \cap \omega_1 = \delta_n$,
- For all $n<\omega$, $X_n \subseteq I_0 \cup I_1 \cup I_2 \cup \cdots$.

The player II *wins the play* b (*in the game*), if the player I does not win the play b.

Notice that we always have

$$X_n = \bigcup\{\overline{\delta_n \cup \{\alpha_0,\cdots,\alpha_k\}} \mid k<\omega\}.$$

If I wins the play b, then since each β_k is f-closed, we have

$$\overline{\delta_n \cup \{\alpha_0, \cdots, \alpha_k\}} \subseteq X_n \cap \beta_k \subseteq I_0 \cup \cdots \cup I_k.$$

Since we prefer to consider this subject as matters on sequences and trees of sequences. Some of the intuitive notions are lost. In particular, the players I and II have no real meanings attached.

An initial play p in the game $G(f, \langle \delta_n \mid n < \omega \rangle)$ means there exists a play b in the game $G(f, \langle \delta_n \mid n < \omega \rangle)$ such that $p = b \lceil k$ for some $k < \omega$. Let p_0 be an initial play of the game $G(f, \langle \delta_n \mid n < \omega \rangle)$ and G be a set of initial plays in the game closed downwards under taking initial segment. Namely, if $p \in G$ and $k \leq l(p)$, then $p \lceil k \in G$. Hence G is a tree with the strict inclusion. We define a binary predicate on p_0 and G. When we say G *is an alternating tree with the stem* p_0, it means that

(1) For all $p \in G$, we have either $p \subseteq p_0$ or $p_0 \subseteq p$.

(2) For any $p \in G$ with $l(p) = l(p_0) + 2l$ for some $l < \omega$, the set of successors of p in G, denoted by $suc_G(p)$, consists of all possible initial plays which extend p a step. Namely,

$$suc_G(p) = \{p^\frown \langle o \rangle \mid p^\frown \langle o \rangle \text{ is an initial play in the game}\}.$$

The exact types of o depend on $l(p_0)$.

(3) For any $p \in G$ with $l(p) = l(p_0) + (2l + 1)$ for some $l < \omega$, p has the only one successor in G. Namely,

$$|\, suc_G(p) \,| = 1.$$

Hence G forks as much as it can immediately after p_0. Then choose the only immediate successors. Then forks as much as it can. Then the only immediate successors. And so forth.

The set of alternating trees with the stem p_0 is denoted by

$$AT(p_0).$$

To have shorter notation, we introduce *the possible successive objects* $S(p)$ for initial plays p in the game.

$$S(p) = \begin{cases} \{(I, \alpha) \mid p^\frown \langle (I, \alpha) \rangle \text{ is an initial play in the game}\}, & \text{if } l(p) \text{ is even.} \\ \\ \{\beta \mid p^\frown \langle \beta \rangle \text{ is an initial play in the game}\}, & \text{if } l(p) \text{ is odd.} \end{cases}$$

For any alternating tree G, the set of cofinal branches b through G (*plays through* G) is denoted by $[G]$. Hence

$$[G] = \{b \mid \text{for all } k < \omega \ b \lceil k \in G\}.$$

For $k < \omega$, G_k denotes the set of the elements in G whose length are k.

$$G_k = \{p \in G \mid l(p) = k\}.$$

We lastly define two unary predicates. Let G be a set of initial plays which is closed under taking initial segment. G is *a winning tree for the player II*, if

- $G \in AT(\emptyset)$,
- For all $b \in [G]$, the player II wins the play b.

T is *a winning tree for the player I*, if

- There is $(I_0, \alpha_0) \in S(\emptyset)$ such that $T \in AT(\langle (I_0, \alpha_0) \rangle)$,
- For all $b \in [T]$, the player I wins the play b.

It is clear that these two kinds of trees are equivalent to winning strategies for II and I, respectively. However, we prefer this sort of static treatment of the subject in terms of trees.

Now we pay attention to a trivial but crutial fact. This is about three kinds of quantifiers on nodes, trees and branches.

3.2 Lemma. *Let p be any initial play (in the game). The following are equivalent.*

(1) $\exists\, G \in AT(p)\ \forall\, b \in [G]\ II$ *wins* b.

(2) $\forall\, o \in S(p)\ \exists\, o' \in S(p^\frown\langle o\rangle)\ \exists\, G' \in AT(p^\frown\langle o, o'\rangle)\ \forall\, b' \in [G']\ II$ *wins* b'.

Proof. It is immediate, if we recall the definition of alternating trees with stems.

□

3.3 Corollary. *Let p be any initial play. The following are equivalent.*

(1) $\forall\, G \in AT(p)\ \exists\, b \in [G]\ I$ *wins* b.

(2) $\exists\, o \in S(p)\ \forall\, o' \in S(p^\frown\langle o\rangle)\ \forall\, G' \in AT(p^\frown\langle o, o'\rangle)\ \exists\, b' \in [G']\ I$ *wins* b'.

□

3.4 Lemma. *The game $G(f, \langle\delta_n \mid n < \omega\rangle)$ is determined. Namely, either the following (1) or (2) holds.*

(1) The player II has a winning tree G.

(2) The player I has a winning tree T.

Proof. We argue in two cases.

Case 1. II has a winning tree: Then done.

Case 2. II does not have any winning tree: We construct a set of initial plays T which is closed under taking initial segment such that for all $k < \omega$

$IH(k)$: $\forall\, p \in T_{2k}\ \exists\, (I, \alpha) \in S(p)\ \forall \beta \in S(p^\frown\langle(I, \alpha)\rangle)\ \forall\, G \in AT(p^\frown\langle(I, \alpha), \beta\rangle)\ \exists\, b \in [G]\ I$ wins b.

We construct T_{2k-1}, T_{2k} by recursion on k.

$\underline{T_0}$: Since II does not have any winning tree, we have

$$\neg\big(\exists\, G \in AT(\emptyset)\ \forall\, b \in [G]\ II \text{ wins } b\big).$$

Hence by 3.3 corollary, we have

$$\exists\, (I_0, \alpha_0) \in S(\emptyset)\ \forall\, \beta \in S(\langle(I_0, \alpha_0)\rangle)\ \forall G' \in AT(\langle(I_0, \alpha_0), \beta_0\rangle)\ \exists\, b \in [G']\ I \text{ wins } b.$$

Let $T_0 = \{\emptyset\}$ (and $T_1 = \{\langle(I_0, \alpha_0)\rangle\}$ and $T_2 = \{\langle(I_0, \alpha_0), \beta_0\rangle \mid \beta_0 \in S(\langle(I_0, \alpha_0)\rangle)\}$).

$\underline{T_{2k} \longrightarrow T_{2k+1}, T_{2k+2}}$: Suppose we have constructed T_{2k} such that $IH(k)$ gets satisfied. By this assumption, it is immediate to construct T_{2k+1} and T_{2k+2} such that for each $p^\frown\langle(I, \alpha), \beta\rangle \in T_{2k+2}$, we have

$$\forall\, G \in AT(p^\frown\langle(I, \alpha), \beta\rangle)\ \exists\, b \in [G]\ I \text{ wins } b.$$

Hence by 3.3 corollary,

$$\exists\, (I', \alpha') \in S(p^\frown\langle(I, \alpha), \beta\rangle)\ \forall\, \beta' \in S(p^\frown\langle(I, \alpha), \beta, (I', \alpha')\rangle)\ \forall G' \in AT(p^\frown\langle(I, \alpha), \beta, (I', \alpha'), \beta'\rangle)$$

$$\text{we have } \exists\, b' \in [G']\ I \text{ wins } b'.$$

This completes the construction of $T \in AT(\langle(I_0, \alpha_0)\rangle)$. To finish the proof, we

Claim. *For all $b \in [T]$, I wins b. Hence T is a winning tree for I.*

Proof. By $IH(k)$, we see that for all $p \in T_{2k}$, there exists a play b' in the game such that p is an initial segment of b' and I wins the play b'. This play may not be in $[T]$. Therefore, given any play $b \in [T]$, for each $k < \omega$, $b\lceil 2k$ gets extended to a play b' which I wins. Hence for all n, we have

$$X_n \cap \omega_1 = \bigcup \{\overline{\delta_n \cup \{\alpha_0, \cdots, \alpha_k\}} \cap \omega_1 \mid k < \omega\} \subset \delta_n.$$

Also since all β_k's are f-closed, viewing as an initial segment of b',

$$\overline{\delta_n \cup \{\alpha_0, \cdots, \alpha_k\}} \subset I_0 \cup \cdots \cup I_k.$$

Hence I wins b.

$\square$

§4. Weak square $\square_\kappa^*$ may fail under tail club guessing

The following says how often the player I has a winning tree with respect to a given ladder system.

4.1 Lemma. *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system and $f : [\kappa]^{<\omega} \longrightarrow \kappa$. Let $\langle \delta_n \mid n < \omega\rangle$ increasingly enumerate each C_δ. Then there exists a club $D^f \subseteq \omega_1$ such that*

$$X^*(D^f) \subseteq \{\delta \in A \mid \exists\, m < \omega \ I \textit{ has a winning tree in the game } G(f, \langle \delta_n \mid m \le n < \omega\rangle)\}.$$

Proof. By contradiction. Suppose not. Then for all club $F \subseteq \omega_1$, there exists $\delta \in X^*(F)$ such that for any $m < \omega$, I does not have any winning tree in the game $G(f, \langle \delta_n \mid m \le n < \omega\rangle)$. Since these games are determined, this means II has winning trees in $G(f, \langle \delta_n \mid m \le n < \omega\rangle)$.

Let us set

$$B = \{\delta \in A \mid \forall\, m < \omega \ II \text{ has a winning tree in } G(f, \langle \delta_n \mid m \le n < \omega\rangle)\}.$$

Then this B is positive. Namely, $\langle C_\delta \mid \delta \in B\rangle$ is tail club guessing. Fix a correspondence

$$\langle \delta \mapsto \langle G^{(\delta,0)}, G^{(\delta,1)}, \cdots, G^{(\delta,m)}, \cdots\rangle \mid \delta \in B\rangle.$$

where $G^{(\delta,m)}$ denotes a winnig tree for II in the game $G(f, \langle \delta_n \mid m \le n < \omega\rangle)$.

Let λ be a sufficiently large regular cardinal and $\langle M_n \mid n < \omega\rangle$ be a sequence of elementary substructures of H_λ such that

(1) $\langle \delta \mapsto \langle G^{(\delta,0)}, G^{(\delta,1)}, \cdots, G^{(\delta,m)}, \cdots\rangle \mid \delta \in B\rangle \in M_0$,

(2) $M_n \in M_{n+1}$ and $\omega_1 \le M_n \cap \kappa < \kappa$ with $cf(M_n \cap \kappa) = \omega_1$.

Let

$$F = \{\delta < \omega_1 \mid \overline{\delta \cup \{M_0 \cap \kappa, M_1 \cap \kappa, \cdots\}} \cap \omega_1 = \delta\}.$$

Then F is a club. Since $\langle C_\delta \mid \delta \in B\rangle$ is tail club guessing, there exists $\delta \in B$ such that $C_\delta \subseteq^* F$. Take $m < \omega$ such that

$$\{\delta_n \mid m \le n < \omega\} \subset F.$$

Let

$$X_n = \overline{\delta_n \cup \{M_0 \cap \kappa, M_1 \cap \kappa, \cdots\}},$$

$$X_\omega = \bigcup \{X_n \mid m \le n < \omega\} = \overline{\delta \cup \{M_0 \cap \kappa, M_1 \cap \kappa, \cdots\}}.$$

Then for all n with $m \leq n < \omega$, we have

$$X_n \cap \omega_1 = \delta_n.$$

Since $\delta \in M_0$, we have $G^{(\delta,m)} \in M_0$. We construct a play $b \in [G^{(\delta,m)}]$. First we set (I_0, α_0) so that

- $X_\omega \cap [0, M_1 \cap \kappa) \subset I_0 = [0, y_0]$ and $\alpha_0 = M_0 \cap \kappa < y_0 < M_1 \cap \kappa$.

  Since$\langle (I_0, \alpha_0) \rangle \in G^{(\delta,m)} \cap M_1$, we have $\beta_0 < M_1 \cap \kappa$ such that

- $\langle (I_0, \alpha_0), \beta_0 \rangle \in G^{(\delta,m)} \cap M_1$.

  We then set (I_1, α_1) so that

- $X_\omega \cap [M_1 \cap \kappa, M_2 \cap \kappa) \subset I_1 = [M_1 \cap \kappa, y_1]$, $y_1 < M_2 \cap \kappa$ and $\alpha_1 = M_1 \cap \kappa$.

  Since $\langle (I_0, \alpha_0), \beta_0, (I_1, \alpha_1) \rangle \in G^{(\delta,m)} \cap M_2$, we have $\beta_1 < M_2 \cap \kappa$ such that

- $\langle (I_0, \alpha_0), \beta_0, (I_1, \alpha_1), \beta_1 \rangle \in G^{(\delta,m)} \cap M_2$.

  Suppose we have constructed

$$\langle (I_0, \alpha_0), \beta_0, (I_1, \alpha_1), \beta_1, \cdots, (I_n, \alpha_n), \beta_n \rangle \in G^{(\delta,m)} \cap M_{n+1}.$$

Then we set (I_{n+1}, α_{n+1}) so that

- $X_\omega \cap [M_{n+1} \cap \kappa, M_{n+2} \cap \kappa) \subset I_{n+1} = [M_{n+1} \cap \kappa, y_{n+1}]$, $y_{n+1} < M_{n+2} \cap \kappa$ and $\alpha_{n+1} = M_{n+1} \cap \kappa$.

  Since

$$\langle (I_0, \alpha_0), \beta_0, (I_1, \alpha_1), \beta_1, \cdots, (I_n, \alpha_n), \beta_n, (I_{n+1}, \alpha_{n+1}) \rangle \in G^{(\delta,m)} \cap M_{n+2},$$

we have $\beta_{n+1} < M_{n+2} \cap \kappa$ so that

$$\langle (I_0, \alpha_0), \beta_0, (I_1, \alpha_1), \beta_1, \cdots, (I_n, \alpha_n), \beta_n, (I_{n+1}, \alpha_{n+1}), \beta_{n+1} \rangle \in G^{(\delta,m)} \cap M_{n+2}.$$

This completes the construction of $b \in [G^{(\delta,m)}]$. Since $G^{(\delta,m)}$ is a winning tree for II, II wins this play b. But by construction,

- For all n with $m \leq n < \omega$, we have $X_n \cap \omega_1 = \delta_n$,
- For all n with $m \leq n < \omega$, we have $X_n \subset I_0 \cup I_1 \cup I_2 \cup \cdots$.

  Hence I wins this b. This is a contradiction.

□

Here is our main result of this note. Remember SRP negates every $\square^*_\kappa$. Tail club guessing together with its associated SRP-like principle does the same.

4.2 Theorem. *If a tail club guessing and its associated reflection principle hold, then for all regular cardinals $\kappa > \omega_1$, we do not have $\square^*_\kappa$.*

The following suffices and is a rendition of [F] and [V] in our context.

4.3 Lemma. *Let $\langle C_\delta \mid \delta \in A \rangle$ be tail club guessing and its associated reflection principle hold. Let $\langle D_\gamma \mid \gamma < \kappa, \gamma \text{ is limit} \rangle$ be a $\square^*_\kappa$-sequence. Let us set*

$$S = \{X \in [\kappa]^\omega \mid sup(X) = \gamma \text{ for some } \gamma < \kappa \text{ and } X \cap D_\gamma \text{ is bounded below } \gamma\}.$$

Let θ be a sufficiently large regular cardinal and apply the associated reflection principle to $(\kappa, S, \theta, \kappa)$. Then

(1) There exists a club D^0 and an $\in$-chain $\langle N_i \mid i < \omega_1 \rangle$ in H_θ such that for all $\delta \in X^*(D^0)$, we have either the following (1.1) or (1.2).

(1.1) $N_\delta \cap \kappa \in S$.

(1.2) For any $\in$*-chain* $\langle N'_n \mid n \leq \omega\rangle$ *in* H_θ *such that for all* $n < \omega$, $N_{\delta_n} \subseteq_{\omega_1} N'_n$, *we have* $N'_\omega \cap \kappa \notin S$, *where* δ_n *increasingly enumerates each* C_δ.

Furthermore, there exists a club $D^f \subseteq D^0$ *such that*

(2) $X^*(D^f) \subseteq \{\delta \in A \mid N_\delta \cap \kappa \in S\}$.

However

(3) $\{\delta \in A \mid N_\delta \cap \kappa \in S\}$ *is not stationary.*

Hence $\square^*_\kappa$ *does not hold.*

Proof. Let us set S and θ and take D^0 and $\langle N_i \mid i < \omega_1\rangle$ as specified. Let $f : [\kappa]^{<\omega} \longrightarrow \kappa$ be such that

$$C(f) \subseteq \{N \cap \kappa \mid \langle N_i \mid i < \omega_1\rangle \in N \prec H_\theta\}.$$

where $C(f) = \{X \in [\kappa]^\omega \mid X \text{ is } f\text{-closed}\}$ and $N \prec H_\theta$ abbreviates that $(N, \in)$ is a countable elementary substructure of $(H_\theta, \in)$.

Now apply 4.1 lemma to this f. We get a club D^f such that

$$X^*(D^f) \subseteq \{\delta \in A \mid \exists m < \omega\ I \text{ has a winnig tree in the game } G(f, \langle \delta_n \mid m \leq n < \omega\rangle)\}.$$

We may assume $D^f \subseteq D^0$. Let us fix a map

$$\langle \delta \mapsto T^{(\delta, m_\delta)} \mid \delta \in X^*(D^f)\rangle,$$

where $T^{(\delta,m_\delta)}$ is a winning tree for I in $G(f, \langle \delta_n \mid m_\delta \leq n < \omega\rangle)$.

Let us also fix a sequence

$$\langle M^\gamma \mid \gamma < \kappa\rangle$$

such that

- $\langle \delta \mapsto T^{(\delta,m_\delta)} \mid \delta \in X^*(D^f)\rangle \in M^0$,
- M^γ is an elementary substructure of H_θ, $|M^\gamma| < \kappa$ and $\omega_1 \leq M^\gamma \cap \kappa < \kappa$,
- $M^\gamma \subset M^{\gamma+1}$ and $M^\gamma \in M^{\gamma+1}$ (strictly increasing),
- For limit γ, $M^\gamma = \bigcup\{M^{\gamma'} \mid \gamma' < \gamma\}$ (continuous).

We then take an elementary substructure M of H_θ such that

- $|M| < \kappa$, $M \cap \kappa < \kappa$ and $cf(M \cap \kappa) = \omega_1$,
- $\langle D_\gamma \mid \gamma < \kappa, \gamma \text{ is limit}\rangle$, $\langle M^\gamma \mid \gamma < \kappa\rangle \in M$.

Let

$$C = \{M^\gamma \cap \kappa \mid \gamma < \kappa\}.$$

Then C is a club in κ with $C \in M$. Let $\gamma^* = M \cap \kappa$ and $\overline{C} = \{\gamma < \kappa \mid C \cap \gamma \text{ is cofinal in } \gamma \in C\}$. Hence $\overline{C}$ denotes the set of limit points of C and $\overline{C} \in M$ holds.

Claim 1. $C \cap \gamma^* \not\subseteq^* D_{\gamma^*}$. *Namely, there exist cofinally many* $\alpha \in C \cap \gamma^*$ *below* γ^* *which are not in* D_{γ^*}.

Proof. By contradiction. Suppose $C \cap \gamma^* \subseteq^* D_{\gamma^*}$. Take $\xi < \gamma^*$ such that $C \cap [\xi, \gamma^*) \subset D_{\gamma^*}$. Notice that $\xi \in M$. Let

$$D^* = \bigcup\{D_\gamma \mid \xi < \gamma \in \overline{C}\}.$$

Then $D^* \in M$.

Subclaim. $M \models$ *"For* $\gamma_1 < \gamma_2$ *such that* $\gamma_1, \gamma_2 \in \overline{C}$ *and* $\xi < \gamma_1, \gamma_2$, *we have* $D_{\gamma_1} = D_{\gamma_2} \cap \gamma_1$*".*

Proof. Let $\gamma_1 < \gamma_2 < \gamma^*$ such that $\gamma_1, \gamma_2 \in \overline{C}$ and $\xi < \gamma_1, \gamma_2$. Since $C \cap [\xi, \gamma^*) \subset D_{\gamma^*}$, we have $\gamma_1, \gamma_2 \in \overline{D_{\gamma^*}}$. Hence $D_{\gamma_1} = D_{\gamma^*} \cap \gamma_1$ and $D_{\gamma_2} = D_{\gamma^*} \cap \gamma_2$. Therefore, $D_{\gamma_1} = D_{\gamma_2} \cap \gamma_1$.

Subclaim. *$D^* = \bigcup\{D_\gamma \mid \xi < \gamma \in \overline{C}\}$ is a club in κ.*

Proof. By elementarity, for $\gamma_1 < \gamma_2$ such that $\gamma_1, \gamma_2 \in \overline{C}$ and $\xi < \gamma_1, \gamma_2$, we have $D_{\gamma_1} = D_{\gamma_2} \cap \gamma_1$.

Subclaim. *If $\gamma \in \overline{D^*}$, then $D^* \cap \gamma = D_\gamma$. Hence $\langle D_\gamma \mid \gamma < \kappa, \text{ limit } \gamma\rangle$ is not a $\square^*_\kappa$-sequence.*

Proof. $D^* \cap \gamma = D_{\gamma_1} \cap \gamma$ for some γ_1 with $\gamma < \gamma_1$. Then $\gamma \in \overline{D_{\gamma_1}}$ and so $D_\gamma = D_{\gamma_1} \cap \gamma$. Hence $D^* \cap \gamma = D_\gamma$ holds.

□

Claim 2. *There exists $\langle \gamma_n, \eta_n \mid n < \omega\rangle$ such that*

- *$\gamma_n \in (C \cap \gamma^*) \setminus D_{\gamma^*}$ and $\eta_n \in D_{\gamma^*}$,*
- *$\gamma_n < \eta_n < \gamma_{n+1}$.*

Proof. By claim 1, $(C \cap \gamma^*) \setminus D_{\gamma^*}$ is cofinal below γ^*. Hence we may recursively construct γ_n and η_n.

□

Claim 3. *Let $\eta = sup\{\eta_n \mid n < \omega\}$. Then since $cf(\gamma^*) = \omega_1$, we have $\eta \in \overline{D_{\gamma^*}} \cap \gamma^*$. Hence $D_\eta = D_{\gamma^*} \cap \eta$. Therefore, we may assume*

- *$\gamma_n = M_n \cap \kappa \in (C \cap \eta) \setminus D_\eta$,*
- *$\eta_n \in D_\eta$,*
- *M_n is an elementary substructure of H_θ and $M_n \in M_{n+1}$.*

Proof. $\gamma_n = M^\alpha \cap \kappa$ for some ordinal $\alpha < \kappa$. Just reindex them.

□

Claim 4. *$X^*(D^f) \subseteq \{\delta \in A \mid N_\delta \cap \kappa \in S\}$.*

Proof. Let $\delta \in X^*(D^f)$ and take $m = m_\delta < \omega$ so that the player I has the winning tree $T^{(\delta,m)} = T^{(\delta,m_\delta)} \in M_0$. Play $G(f, \langle \delta_n \mid m \leq n < \omega\rangle)$ to construct a play $b \in [T^{(\delta,m)}]$ such that

- $\langle (I_0, \alpha_0)\rangle \in T^{(\delta,m)} \cap M_0$.

Then choose β_0 so that

$$\eta_0 \in D_\eta \cap (M_0 \cap \kappa,\ M_1 \cap \kappa) \subset \beta_0 < M_1 \cap \kappa.$$

This is possible, as $M_1 \cap \kappa \notin D_\eta$. Then since $\langle (I_0, \alpha_0), \beta_0\rangle \in T^{(\delta,m)} \cap M_1$, we have $(I_1, \alpha_1) \in M_1$ such that

$$\langle (I_0, \alpha_0), \beta_0, (I_1, \alpha_1)\rangle \in T^{(\delta,m)} \cap M_1.$$

Suppose we have constructed up to $(I_n, \alpha_n) \in M_n$. We then prepare β_n so that

$$\eta_n \in D_\gamma \cap (M_n \cap \kappa,\ M_{n+1} \cap \kappa) \subset \beta_n < M_{n+1} \cap \kappa.$$

This is possible, as $M_{n+1} \cap \kappa \notin D_\eta$. Then since $\langle (I_0, \alpha_0), \beta_0, \cdots, (I_n, \alpha_n), \beta_n\rangle \in T^{(\delta,m)} \cap M_{n+1}$, we have $(I_{n+1}, \alpha_{n+1}) \in M_{n+1}$ such that

$$\langle (I_0, \alpha_0), \beta_0, (I_1, \alpha_1), \cdots, (I_n, \alpha_n), \beta_n, (I_{n+1}, \alpha_{n+1})\rangle \in T^{(\delta,m)} \cap M_{n+1}.$$

This completes the construction of $b \in [T^{(\delta,m)}]$. For n with $m \leq n < \omega$, let

$$X_n = \overline{\delta_n \cup \{\alpha_0, \alpha_1, \alpha_2, \cdots\}}$$

and let

$$X_\omega = \bigcup \{X_n \mid m \le n < \omega\}.$$

Since $T^{(\delta,m)}$ is a winnig tree for I, the following two hold.

- For all n with $m \le n < \omega$, we have $X_n \cap \omega_1 = \delta_n$,
- For all n with $m \le n < \omega$, we have $X_n \subset I_0 \cup I_1 \cup I_2 \cup \cdots$.

By construction, $X_\omega \cap D_\eta$ is bounded below η. Hence

$$X_\omega \in S.$$

For each n with $m \le n < \omega$, since X_n is f-closed, there exists $\overline{N_n}$ such that $\langle N_i \mid i < \omega_1 \rangle \in \overline{N_n} \prec H_\theta$ and $\overline{N_n} \cap \kappa = X_n$. Hence we have

$$N_{\delta_n} \subseteq_{\omega_1} \overline{N_n}.$$

Notice that $\overline{N_n}$'s do not form an $\in$-chain. So we must reconstruct them. Recall that α_n are strictly increasing cofinally in X_ω. Let $\langle s_k \mid k < \omega \rangle$ be such that

- $s_k \in [\{\alpha_n \mid n < \omega\}]^{<\omega}$,
- $s_k \subset \overline{N_k} \cap \kappa = X_k$,
- $s_k \subseteq s_{k+1}$,
- $\bigcup \{s_k \mid k < \omega\} = \{\alpha_n \mid n < \omega\}$.

By applying 2.1 lemma (1 H lemma), for all n with $m \le n < \omega$, we construct

$$N'_n = \{g(s_n) \mid g \in N_{\delta_n}\}.$$

Let $N'_\omega = \bigcup \{N'_n \mid m \le n < \omega\}$. Notice that $s_n \in \overline{N_n}$. Hence we have

- $N'_n \prec H_\theta$ and $\{s_n\} \cup N_{\delta_n} \subseteq N'_n \subseteq \overline{N_n}$,

Since $N_{\delta_n} \in N_{\delta_{n+1}} \subseteq N'_{n+1}$ and $s_n \subseteq s_{n+1} \in N'_{n+1}$, we have $N_{\delta_n}, s_n \in N'_{n+1} \prec H_\theta$ and so

- $N'_n \in N'_{n+1}$,
- $sup(N'_\omega \cap \kappa) = sup(X_\omega) = \eta$ and $N'_\omega \cap \kappa \subseteq X_\omega$.

Since

- $N_{\delta_n} \subseteq_{\omega_1} N'_n$ for all n with $m \le n < \omega$,
- $N'_\omega \cap \kappa \in S$,

We conclude $N_\delta \cap \kappa \in S$.

□

Claim 5. *$\{\delta \in A \mid N_\delta \cap \kappa \in S\}$ is not stationary.*

Proof. Let $\gamma^i = sup(N_i \cap \kappa)$. Then $\langle \gamma^i \mid i < \omega_1 \rangle$ is a club in $\gamma = \sup\{\gamma^i \mid i < \omega_1\}$. Since D_γ is a club in γ and $cf(\gamma) = \omega_1$, the following J is a club in ω_1.

$$J = \{j < \omega_1 \mid (D_\gamma \cap \{\gamma^i \mid i < \omega_1\}) \text{ is cofinal below } \gamma^j\}.$$

For each $j \in J$, we have $D_{\gamma^j} = D_\gamma \cap \gamma^j$. Then $N_j \cap D_{\gamma^j}$ is a cofinal subset of γ^j. Hence $N_j \cap \kappa \notin S$. Therefore,

$$J \cap \{\delta \in A \mid N_\delta \cap \kappa \in S\} = \emptyset.$$

□

□

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