Provability logic and Grzegorczyk logic

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Abstract. Here we discuss the formulas having only one atomic formula $\perp$ in provability logic **GL** and the formulas having only one atomic formula p in Grzegorczyk logic **Grz**. It was defined a function f satisfying, for any formula A, $f(A) \in \mathbf{GL}$ if and only if $A \in \mathbf{Grz}$ (cf. Boolos [Boo93] and Goldblatt [Gol78]). While we define a function g satisfying, for any formula A having only one atomic formula $\perp$, $A \in \mathbf{GL}$ if and only if $g(A) \in \mathbf{Grz}$.

1 Introduction

We use lower case Latin letters $p, q, \cdots$ for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\perp$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). We use upper case Latin letters $A, B, \cdots$, possibly with suffixes, for formulas. We fix the enumeration **ENU** of formulas. For the finite non-empty set **S** of formulas, the expression

$$\bigwedge \mathbf{S}$$

denotes the formula

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n,$$

where $\mathbf{S} = \{A_1, \cdots, A_n\}$ and A_i occurs earlier than A_j in **ENU** if $i < j$. Also we put

$$\bigwedge \emptyset = \perp \supset \perp.$$

Definition 1.1. The depth $d(A)$ of a formula A is defined inductively as follows:
(1) $d(D) = 0$, for an atomic formula D,
(2) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
(3) $d(\Box B) = d(B) + 1$.

Let D be an atomic formula in $\{p, \perp\}$. By $\mathbf{S}(D)$, we mean the set of formulas constructed from D by using $\wedge$, $\vee$, $\supset$ and $\Box$. We put $\mathbf{S}^n(D) = \{B \in \mathbf{S}(D) \mid d(B) \leq n\}$.

By **GL**, we mean the smallest set of formulas containing all the tautologies and the axioms
$K : \Box(A \supset B) \supset (\Box A \supset \Box B)$,
$L : \Box(\Box A \supset A) \supset \Box A$ (Löb's axiom),
and closed under modus ponens and necessitation. By **Grz**, we mean the smallest set of formulas containing all the tautologies, K and the axioms
$T : \Box A \supset A$,
$grz : \Box(\Box(A \supset \Box A) \supset A) \supset \Box A$ (Grzegorczyk axiom),
and closed under modus ponens and necessitation.

For the terminology concerning Kripke models, we follow Chagrov and Zakharyaschev [CZ97].

Lemma 1.2.(cf. [CZ97])

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(1) $A \in \mathbf{GL}$ *if and only if* A *is valid in the class of finite Kripke frames with strict partial orders.*
(2) $A \in \mathbf{Grz}$ *if and only if* A *is valid in the class of finite Kripke frames with partial orders.*

Sequent systems for **GL** and **Grz** were described several papers. For example, Avron [Avr84] gave both systems, which we define below. For the terminology concerning sequent systems, we follow [Sas01]. By **GGL**, we mean the system obtained by adding the inference rule

$$\frac{\Box A, \Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box_{\mathbf{GL}})$$

to the system **LK** for the classical propositional logic. By **GGrz**, we mean the system obtained by adding the inference rules

$$\frac{\Box(A \supset \Box A), \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box_{\mathbf{Grz}}) \quad \text{and} \quad \frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to)$$

to the system **LK**.

Lemma 1.3.(cf. [Avr84], Valentini [Val83])
(1) $\bigwedge\Gamma \supset A \in \mathbf{GL}$ *if and only if* $\Gamma \to A \in \mathbf{GGL}$.
(2) $\bigwedge\Gamma \supset A \in \mathbf{Grz}$ *if and only if* $\Gamma \to A \in \mathbf{GGrz}$.
(3) **GGL** *and* **GGrz** *enjoy cut-elimination theorem.*

We can see similarity between **GL** and **Grz** in the Kripke semantics and sequent systems. We also note that

Lemma 1.4. *For* $k > 0, i > 0$, $\Box^k\bot \to \Box^{k+i}\bot$ *is provable in* **GGL** *and* **GGrz**.

Definition 1.5. A list $F_0, F_1, \cdots$ of formulas are defined inductively as follows:
(1) $F_0 = p$,
(2) $F_{k+1} = F_k \supset \Box F_k$.

Definition 1.6. A list $g_0, g_1, \cdots$ of functions from $\mathbf{S}(\bot)$ to $\mathbf{S}(p)$ are defined inductively as follows:
(1) $g_i(\bot) = \Box F_i$,
(2) $g_i(B \wedge C) = g_i(B) \wedge g_i(C)$,
(3) $g_i(B \vee C) = g_i(B) \vee g_i(C)$,
(4) $g_i(B \supset C) = g_i(B) \supset g_i(C)$,
(5) $g_i(\Box B) = \Box g_{i+1}(B)$.

The function g_0 transforms the formula $\Box(\Box\bot \supset \bot) \supset \Box\bot$, a instance of the axiom L, into a formula

$$g_0(\Box(\Box\bot \supset \bot) \supset \Box\bot) = \Box(\Box\Box(F_1 \supset \Box F_1) \supset \Box F_1) \supset \Box\Box F_1).$$

Here we note that the image is similar to $\Box(\Box(F_1 \supset \Box F_1) \supset F_1) \supset \Box F_1$, an instance of the axiom grz, and that the image and the instance are equivalent in **Grz**.

The main result is

Theorem 1.7. For any formula $A \in \mathbf{S}(\bot)$, and for any i,

$$A \in \mathbf{GL} \text{ if and only if } g_i(A) \in \mathbf{Grz}.$$

To prove the theorem, we use properties of the structures $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$ and $\langle \mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$, where for $\mathbf{L} \in \{\mathbf{GL}, \mathbf{Grz}\}$,
$A \equiv_{\mathbf{L}} B$ if and only if $(A \supset B) \wedge (B \supset A) \in \mathbf{L}$,
$[A] \leq_{\mathbf{L}} [B]$ if and only if $B \supset A \in \mathbf{L}$.

In the next section, we construct a representative of each equivalent class of the above two structures following [Boo93] and [Sas04]. In section 3, we prove Theorem 1.7 using the lemmas in section 2.

2 Construction of representatives

Here we construct a representative of each equivalent class in the quotient sets $\mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}$ and $\mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}$. It is known, however, two structures $\langle \mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}}, \leq_{\mathbf{GL}}\rangle$ and $\langle \mathbf{S}^n(p)/\equiv_{\mathbf{Grz}}, \leq_{\mathbf{Grz}}\rangle$ are boolean(cf. [CZ97]). So, we have only to construct representatives of generators of these two booleans. For representatives of generators of the structure for **GL**, we can refer [Boo93], and representatives for **Grz** was given in [Sas04].

Definition 2.1. For a formula A, $\Box^n A$ $(n = 0, 1, \cdots)$ are defined inductively as follows:
(1)$\Box^0 A = A$,
(2)$\Box^{k+1} A = \Box\Box^k A$.

Definition 2.2. The sets $\mathbf{G}_n$ $(n = 0, 1, 2, \cdots)$ of formulas are defined as follows:
$\mathbf{G}_0 = \{\bot\}$,
$\mathbf{G}_{k+1} = \{\Box^{k+1}\bot, \Box^{k+1}\bot \supset \Box^k\bot, \cdots, \Box\bot \supset \bot\}$.

Lemma 2.3.
(1) *None of the formulas in* $\mathbf{G}_n$ *is provable in* **GL**.
(2) *For* $k \leq n$, $\bigwedge\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\} \equiv_{\mathbf{GL}} \Box^k\bot$.
(3) *For any* $A, B \in \mathbf{G}_n$, $A \neq B$ *implies* $A \vee B \in \mathbf{GL}$.

Proof.
For (1): Let be that $M = \langle\{1, 2, \cdots, k+2\}, <, \models\rangle$, where $<$ is the ordinary strict order. Then we have

$$(M, k+2) \not\models \Box\bot \supset \bot, (M, k+1) \not\models \Box^2\bot \supset \Box\bot, \cdots, (M, 2) \not\models \Box^{k+1}\bot \supset \Box^k\bot, (M, 1) \not\models \Box^{k+1}\bot.$$

Using Lemma 1.2, we obtain (1).

For (2): By Lemma 1.4, we have $\Box^k\bot \to \bigwedge\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\} \in \mathbf{GGL}$. Using Lemma 1.3, we have $\Box^k\bot \supset \bigwedge\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\} \in \mathbf{GL}$.

We show $\bigwedge\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\} \supset \Box^k\bot \in \mathbf{GL}$ by an induction on n.

If $n = 0$, then the formula is a tautology.

Suppose that $n > 0$. If $k = n$, then the formula is also a tautology. So, we assume that $k \leq n-1$. By the induction hypothesis, $\bigwedge\{\Box^{n-1}\bot, \Box^{n-1}\bot \supset \Box^{n-2}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\} \supset \Box^k\bot \in \mathbf{GL}$. Using Lemma 1.3,

$$\Box^{n-1}\bot, \Box^{n-1}\bot \supset \Box^{n-2}\bot, \cdots, \Box\bot \supset \bot \to \bot \in \mathbf{GGL}.$$

Using $\Box^n\bot \to \Box^n\bot \in \mathbf{GGL}$ and $(\supset\to)$, we have

$$\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box\bot \supset \bot \to \bot \in \mathbf{GGL}.$$

Using Lemma 1.3, the formula is provable in **GL**.

For (3): By Lemma 1.3 and Lemma 1.4. ⊣

By Lemma 2.3(3), we have

Corollary 2.4. *For any* $A, B \in \mathbf{G}_n$, $A \neq B$ *implies* $B \equiv_{\mathbf{GL}} A \supset B$.

Lemma 2.5. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be subsets of* $\mathbf{G}_n$. *Then*
(1) $(\bigwedge \mathbf{S}_1) \wedge (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_1 \cup \mathbf{S}_2)$,
(2) $(\bigwedge \mathbf{S}_1) \vee (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_1 \cap \mathbf{S}_2)$,
(3) $(\bigwedge \mathbf{S}_1) \supset (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_2 - \mathbf{S}_1)$,
(4) *if* $\mathbf{S}_1 \neq \emptyset$, *then* $\Box(\bigwedge \mathbf{S}_1) \equiv_{\mathbf{GL}} \Box^k\bot$, *where* $k = \min(\{n+1 \mid \Box^n\bot \in \mathbf{S}_1\} \cup \{i+1 \mid \Box^{i+1}\bot \supset \Box^i\bot \in \mathbf{S}_1\})$.

Proof. (1) is from associative law and commutative law of $\wedge$. For (2) and (3), we use Lemma 2.3(3) and Corollary 2.4, respectively. (4) was shown in [Boo93]. $\dashv$

Lemma 2.6. *Let A be a formula in $\mathbf{S}^n(\bot)$. Then there exists a subset $\mathbf{S}$ of $\mathbf{G}_n$ such that $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$.*

Proof. We use an induction on A. If $A = \bot$, then by Lemma 2.3(2),

$$\bigwedge \mathbf{G}_n = \bigwedge\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box\bot \supset \bot\} \equiv_{\mathbf{GL}} \bot = A.$$

If $A \neq \bot$, then by the induction hypothesis, Lemma 2.4 and Lemma 2.3(2), we obtain the lemma. $\dashv$

Lemma 2.7.
(1) $\mathbf{S}^n(\bot)/\equiv_{\mathbf{GL}} = \{[\bigwedge \mathbf{S}] \mid \mathbf{S} \subseteq \mathbf{G}_n^*\}$.
(2) *For subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_n^*$,*

(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge \mathbf{S}_1] \leq_{\mathbf{GL}} [\bigwedge \mathbf{S}_2]$,

(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge \mathbf{S}_1] = [\bigwedge \mathbf{S}_2]$.

Proof. (1) is from Lemma 2.6. We obtain (2.2) as a corollary of (2.1). The "only if" part of (1.1) is clear. We show the "if part" of (1.1). Suppose that $[\bigwedge \mathbf{S}_1] \leq_{\mathbf{GL}} [\bigwedge \mathbf{S}_2]$ and $\mathbf{S}_1 \not\subseteq \mathbf{S}_2$. By $\mathbf{S}_1 \not\subseteq \mathbf{S}_2$, there exists a formula A in $\mathbf{S}_1 - \mathbf{S}_2$. Using $[\bigwedge \mathbf{S}_1] \leq_{\mathbf{GL}} [\bigwedge \mathbf{S}_2]$, we have $\bigwedge \mathbf{S}_2 \supset A \in \mathbf{GL}$. Since $A \notin \mathbf{S}_2$, using Corollary 2.4, we have $\bigwedge \mathbf{S}_2 \supset A \equiv_{\mathbf{GL}} A$, and so, we have $A \in \mathbf{GL}$. This is in contradiction with Lemma 2.3(1). $\dashv$

Definition 2.8. The sets $\mathbf{G}_n^*$ $(n = 0, 1, 2, \cdots)$ of formulas are defined as follows:
$\mathbf{G}_0^* = \{F_0\}$,
$\mathbf{G}_1^* = \{F_0, F_1\}$,
$\mathbf{G}_{k+2}^* = \{F_{k+1}, F_{k+2}, \Box F_{k+1} \supset \Box F_k, \cdots, \Box F_1 \supset \Box F_0\}$

The following three lemmas were shown in [Sas04].

Lemma 2.9.
(1) $F_k \wedge F_{k+1} \equiv_{\mathbf{Grz}} \Box F_k$.
(2) *For* $k < n \neq 0$, $\bigwedge\{F_n, F_{n-1}, \Box F_{n-1} \supset \Box F_{n-2}, \cdots, \Box F_{k+1}\bot \supset \Box F_k\} \equiv_{\mathbf{Grz}} \Box F_k$.

Lemma 2.10. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be subsets of $\mathbf{G}_n^*$. Then*
(1) $(\bigwedge \mathbf{S}_1) \wedge (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge(\mathbf{S}_1 \cup \mathbf{S}_2)$,
(2) $(\bigwedge \mathbf{S}_1) \vee (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge(\mathbf{S}_1 \cap \mathbf{S}_2)$,
(3) $(\bigwedge \mathbf{S}_1) \supset (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge(\mathbf{S}_2 - \mathbf{S}_1)$,
(4) *if* $\mathbf{S}_1 \neq \emptyset$, *then* $\Box(\bigwedge \mathbf{S}_1) \equiv_{\mathbf{Grz}} \Box F_k$, *where* $k = \min(\{i \mid F_i \in \mathbf{S}_1\} \cup \{i \mid \Box F_{i+1} \supset \Box F_i \in \mathbf{S}_1\})$.

Lemma 2.11.
(1) $\mathbf{S}^n(p)/\equiv_{\mathbf{Grz}} = \{[\bigwedge \mathbf{S}] \mid \mathbf{S} \subseteq \mathbf{G}_n^*\}$.
(2) *For subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_n^*$,*

(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge \mathbf{S}_1] \leq_{\mathbf{Grz}} [\bigwedge \mathbf{S}_2]$,

(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge \mathbf{S}_1] = [\bigwedge \mathbf{S}_2]$.

3 Proof of the theorem

Here we give a proof of Theorem 1.7. We define a function h and show three lemmas. We put $g_i(\mathbf{S}) = \{g_i(A) \mid A \in \mathbf{S}\}$.

Definition 3.1. For a subset $\mathbf{S}$ of $\mathbf{G}_n$, we put

$$h_i(\mathbf{S}) = \{F_{n+i} \mid \Box^n\bot \in \mathbf{S}\} \cup \{F_{n+i+1} \mid \Box^n\bot \in \mathbf{S}\} \cup \bigcup_{k=1}^{n} \{\Box F_{k+i} \supset \Box F_{k+i-1} \mid \Box^k\bot \supset \Box^{k-1}\bot \in \mathbf{S}\}.$$

Lemma 3.2. *Let* $\mathbf{S}$ *and* $\mathbf{S}_1$ *be subsets of* $\mathbf{G}_n$*. Then for any* i*,*
(1) $h_i(\mathbf{S}) \subseteq \mathbf{G}^*_{n+i+1}$,
(2) $\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S})$,
(3) $\mathbf{S} \neq \mathbf{S}_1$ *implies* $\bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1)$.

Proof. (1) is clear from the definition. (2) is from Lemma 2.9(1). (3) is from (1) and Lemma 2.11(2.2). ⊣

Lemma 3.3. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be subsets of* $\mathbf{G}_n$*. Then for any* i*,*
(1) $(\bigwedge h_i(\mathbf{S}_1)) \wedge (\bigwedge h_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cup \mathbf{S}_2)$,
(2) $(\bigwedge h_i(\mathbf{S}_1)) \vee (\bigwedge h_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cap \mathbf{S}_2)$,
(3) $(\bigwedge h_i(\mathbf{S}_1)) \supset (\bigwedge h_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_2 - \mathbf{S}_1)$,
(4) *if* $\mathbf{S}_1 \neq \emptyset$*, then* $\Box(\bigwedge h_i(\mathbf{S}_1)) \equiv_{\mathbf{Grz}} \Box F_k$*, where* $k = \min(\{n+i \mid \Box^n\bot \in \mathbf{S}_1\} \cup \{j+i \mid \Box^{j+1}\bot \supset \Box^j\bot \in \mathbf{S}_1\})$.

Proof. We note that
$h_i(\mathbf{S}_1) \cup h_i(\mathbf{S}_2) = h_i(\mathbf{S}_1 \cup \mathbf{S}_2)$,
$h_i(\mathbf{S}_1) \cap h_i(\mathbf{S}_2) = h_i(\mathbf{S}_1 \cup \mathbf{S}_2)$,
$h_i(\mathbf{S}_2) - h_i(\mathbf{S}_1) = h_i(\mathbf{S}_2 - \mathbf{S}_1)$.
So, using Lemma 2.10 and Lemma 3.2(1) we obtain (1), (2) and (3).

For (4). By Lemma 2.10(4),

$$\Box(\bigwedge h_i(\mathbf{S}_1)) \equiv_{\mathbf{Grz}} \Box F_k,$$

where $k = \min(\{j \mid F_j \in h_i(\mathbf{S}_1)\} \cup \{j \mid \Box F_{j+1} \supset \Box F_j \in h_i(\mathbf{S}_1)\})$. So,

$$k = \min(\{n+i \mid \Box^n\bot \in \mathbf{S}_1\} \cup \{j+i \mid \Box^{j+1}\bot \supset \Box^j\bot \in \mathbf{S}_1\}).$$

Lemma 3.4. *Let* A *be a formula in* $\mathbf{S}^n(\bot)$ *and let* $\mathbf{S}$ *be a subset of* $\mathbf{G}_n$*. Then for any* i*,*

$$A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S} \text{ if and only if } g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}).$$

Proof. We use an induction on A.
Basis($A = \bot$): By Lemma 2.3, we have

$$A = \bot \equiv_{\mathbf{GL}} \bigwedge \mathbf{G}_n.$$

By Lemma 3.2(2) and Lemma 2.9(2), we have

$$\bigwedge g_i(\mathbf{G}_n) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{G}_n) = \bigwedge \mathbf{G}^*_{n+i+1} \equiv_{\mathbf{Grz}} \Box F_i = g_i(\bot) = g_i(A).$$

So, if $\mathbf{S} = \mathbf{G}^*_n$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Lemma 2.7(2.2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 3.2(2) and Lemma 3.2(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{G}_n) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$.

Induction step($A \neq \bot$): We divide the cases.

The case that $A = A_1 \wedge A_2$: We note $A_1, A_2 \in \mathbf{S}^n(\bot)$. So, by Lemma 2.5, there exist subsets $\mathbf{S}_1, \mathbf{S}_2$ of $\mathbf{G}_n$ such that

$$A_1 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_1, \quad A_2 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_2.$$

Using the induction hypothesis,

$$g_i(A_1) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_1), \quad g_i(A_2) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_2).$$

By Lemma 2.5(1),

$$A = A_1 \wedge A_2 \equiv_{\mathbf{GL}} (\bigwedge \mathbf{S}_1) \wedge (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_1 \cup \mathbf{S}_2).$$

Also by Lemma 3.2(2) and Lemma 3.3(1),

$$g_i(A) = g_i(A_1) \wedge g_i(A_2) \equiv_{\mathbf{Grz}} (\bigwedge g_i(\mathbf{S}_1)) \wedge (\bigwedge g_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} (\bigwedge h_i(\mathbf{S}_1)) \wedge (\bigwedge h_i(\mathbf{S}_2))$$

$$\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cup \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_1 \cup \mathbf{S}_2).$$

So, if $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Lemma 2.7(2.2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 3.2(2) and Lemma 3.2(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cup \mathbf{S}_2) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$.

The case that $A = A_1 \vee A_2$: Similarly to the above case, there exist subsets $\mathbf{S}_1, \mathbf{S}_2$ of $\mathbf{G}_n$ such that

$$A_1 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_1, \quad A_2 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_2, \quad g_i(A_1) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_1), \quad g_i(A_2) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_2).$$

By Lemma 2.5(2),

$$A = A_1 \vee A_2 \equiv_{\mathbf{GL}} (\bigwedge \mathbf{S}_1) \vee (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_1 \cap \mathbf{S}_2).$$

Also by Lemma 3.2(2) and Lemma 3.3(2),

$$g_i(A) = g_i(A_1) \vee g_i(A_2) \equiv_{\mathbf{Grz}} (\bigwedge g_i(\mathbf{S}_1)) \vee (\bigwedge g_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} (\bigwedge h_i(\mathbf{S}_1)) \vee (\bigwedge h_i(\mathbf{S}_2))$$

$$\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cap \mathbf{S}_2) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_1 \cap \mathbf{S}_2).$$

So, if $\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_2$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Lemma 2.7(2.2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 3.2(2) and Lemma 3.2(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_1 \cap \mathbf{S}_2) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$.

The case that $A = A_1 \supset A_2$: Similarly to the above cases, there exist subsets $\mathbf{S}_1, \mathbf{S}_2$ of $\mathbf{G}_n$ such that

$$A_1 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_1, \quad A_2 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_2, \quad g_i(A_1) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_1), \quad g_i(A_2) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_2).$$

By Lemma 2.5(3),

$$A = A_1 \supset A_2 \equiv_{\mathbf{GL}} (\bigwedge \mathbf{S}_1) \supset (\bigwedge \mathbf{S}_2) \equiv_{\mathbf{GL}} \bigwedge(\mathbf{S}_2 - \mathbf{S}_1).$$

Also by Lemma 3.2(2) and Lemma 3.3(3),

$$g_i(A) = g_i(A_1) \supset g_i(A_2) \equiv_{\mathbf{Grz}} (\bigwedge g_i(\mathbf{S}_1)) \supset (\bigwedge g_i(\mathbf{S}_2)) \equiv_{\mathbf{Grz}} (\bigwedge h_i(\mathbf{S}_1)) \supset (\bigwedge h_i(\mathbf{S}_2))$$

$$\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_2 - \mathbf{S}_1) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S}_2 - \mathbf{S}_1).$$

So, if $\mathbf{S} = \mathbf{S}_2 - \mathbf{S}_1$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Lemma 2.7(2.2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 3.2(2) and Lemma 3.2(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}_2 - \mathbf{S}_1) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$.

The case that $A = \Box A_1$: Similarly to the above cases, there exists a subset $\mathbf{S}_1$ of $\mathbf{G}_{n-1}$ such that for any k,

$$A_1 \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}_1, \quad g_k(A_1) \equiv_{\mathbf{Grz}} \bigwedge g_k(\mathbf{S}_1).$$

If $\mathbf{S}_1 = \emptyset$, then we have

$$A = \Box A_1 \equiv_{\mathbf{GL}} \Box \bigwedge \emptyset \equiv_{\mathbf{GL}} \Box(\bot \supset \bot) \equiv_{\mathbf{GL}} \bot \supset \bot \equiv_{\mathbf{GL}} \bigwedge \emptyset$$

and

$$g_i(A) = g_i(\Box A_1) = \Box g_{i+1}(A_1) \equiv_{\mathbf{Grz}} \Box \bigwedge g_{i+1}(\emptyset) = \Box \bigwedge \emptyset \equiv_{\mathbf{Grz}} = \bigwedge \emptyset \equiv_{\mathbf{Grz}} \bigwedge h_i(\emptyset) \equiv_{\mathbf{Grz}} \bigwedge g_i(\emptyset)$$

So, if $\mathbf{S} = \emptyset$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Lemma 2.7(2.2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 3.2(2) and Lemma 3.2(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\emptyset) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$.

If $\mathbf{S}_1 = \{\Box^{n-1}\bot\}$, then we have

$$A = \Box A_1 \equiv_{\mathbf{GL}} \Box \bigwedge \{\Box^{n-1}\bot\} \equiv_{\mathbf{GL}} \bigwedge \{\Box^{n}\bot\}$$

and

$$g_i(A) = g_i(\Box A_1) = \Box g_{i+1}(A_1) \equiv_{\mathbf{Grz}} \Box \bigwedge g_{i+1}(\{\Box^{n-1}\bot\})$$

$$\equiv_{\mathbf{Grz}} \Box g_{i+1}(\Box^{n-1}\bot) \equiv_{\mathbf{Grz}} g_i(\Box^{n}\bot) \equiv_{\mathbf{Grz}} \bigwedge g_i(\{\Box^{n}\bot\}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\{\Box^{n}\bot\})$$

So, if $\mathbf{S} = \{\Box^{n}\bot\}$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Lemma 2.7(2.2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 3.2(2) and Lemma 3.2(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\{\Box^{n}\bot\}) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$.

Suppose that $\mathbf{S}_1 \notin \{\emptyset, \{\Box^{n-1}\bot\}\} = \mathcal{P}(\{\Box^{n-1}\bot\})$. Then we have $\mathbf{S}_1 \not\subseteq \{\Box^{n-1}\bot\}$. Since $\mathbf{S}_1 \subseteq \mathbf{G}_{n-1}$, we have $\emptyset \neq \mathbf{S}_1 - \{\Box^{n-1}\bot\} \subseteq \{\Box^{n-1}\bot \supset \Box^{n-1}\bot, \cdots, \Box\bot \supset \bot\}$. So, there exists the minimum k of $\{\ell \mid \Box^{\ell}\bot \supset \Box^{\ell-1} \in \mathbf{S}_1\}$. By Lemma 2.5(4) and Lemma 2.3(2),

$$A = \Box A_1 \equiv_{\mathbf{GL}} \Box \bigwedge \mathbf{S}_1 \equiv_{\mathbf{GL}} \Box^{k}\bot \equiv_{\mathbf{GL}} \bigwedge \{\Box^{n}\bot, \Box^{n}\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^{k}\bot\}.$$

Also by Lemma 3.2(2) and Lemma 3.3(4),

$$g_i(A) = g_i(\Box A_1) = \Box g_{i+1}(A_1) \equiv_{\mathbf{Grz}} \Box \bigwedge g_{i+1}(\mathbf{S}_1) \equiv_{\mathbf{Grz}} \Box \bigwedge h_{i+1}(\mathbf{S}_1) \equiv_{\mathbf{Grz}} \Box F_{k+i}$$

$$\equiv_{\mathbf{Grz}} \bigwedge \{F_{n+i+1}, F_{n+i}, \Box F_{n+i} \supset \Box F_{n+i-1}, \cdots, \Box F_{k+i+1} \supset \Box F_{k+i}\}$$

$$\equiv_{\mathbf{Grz}} \bigwedge h_i(\{\Box^{n}\bot, \Box^{n}\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^{k}\bot\})$$

$$\equiv_{\mathbf{Grz}} \bigwedge g_i(\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\})$$

So, if $\mathbf{S} = \{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\}$, then we have both of $A \equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$ and $g_i(A) \equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. If not, then by Lemma 2.7(2.2), $A \not\equiv_{\mathbf{GL}} \bigwedge \mathbf{S}$, and by Lemma 3.2(2) and Lemma 3.2(3),

$$\bigwedge g_i(\mathbf{S}) \equiv_{\mathbf{Grz}} \bigwedge h_i(\mathbf{S}) \not\equiv_{\mathbf{Grz}} \bigwedge h_i(\{\Box^n\bot, \Box^n\bot \supset \Box^{n-1}\bot, \cdots, \Box^{k+1}\bot \supset \Box^k\bot\}) \equiv_{\mathbf{Grz}} g_i(A),$$

and so, $g_i(A) \not\equiv_{\mathbf{Grz}} \bigwedge g_i(\mathbf{S})$. ⊣

Considering the case that $\mathbf{S} = \emptyset$ in Lemma 3.4, we obtain Theorem 1.7.

References

[Avr84] A. Avron, *On modal systems having arithmetical interpretations*, The Journal of Symbolic Logic, 49, 1984, pp. 935–942.

[Boo93] G. Boolos, *The logic of provability*, Cambridge University Press, 1993.

[CZ97] A. Chagrov and M. Zakharyaschev, *Modal Logic*, Oxford University Press, 1997.

[Gol78] R. Goldblatt, *Arithmetical necessity, provability and intuitionistic logic*, Theoria 44, pp. 38–46, 1978.

[Sas01] K. Sasaki, *Logics and Provability* , ILLC Dissertation Series DS-2001-07, Institute for Logic, Language and Computation, University of Amsterdam, 2001.

[Sas04] K. Sasaki, *Formulas with only one variable in Grzegorczyk logic* , Journal of the Nanzan Academic Society Mathematical Sciences and Information Engineering, 4, Nanzan University, pp. 57–69, 2004.

[Val83] S. Valentini, *The modal logic of provability: cut-elimination*, Journal of Philosophical logic, 12, 1983, pp. 471–476.