New reals created at limit stages of iterated forcing

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Abstract

Assuming $\diamondsuit$, we construct a notion of forcing which iterates Souslin trees. This iteration codes any family of iteratively generic cofinal paths by a single real. The original construction due to R. Jensen starts in the constructible universe.

Introduction

We are interested in iterated forcing $\langle P_n, \dot{Q}_n \mid n < \omega \rangle$ such that for each $n < \omega$, $\Vdash_{P_n}$"$\dot{Q}_n$ is σ-Baire, i.e. adds no new countable sequences of ordinals" and yet if P_ω is any limit of the P_n, then P_ω is never σ-Baire.

In [DJ], a sequence of ω-many Souslin trees are constructed in the constructible universe L. They are connected in L so that any family of iteratively generic cofinal paths are coded by a single real.

We reformulate this construction by $\diamondsuit$. In doing so, we tentatively formulate a type of sequences of projections so that this remake gets included. However, we are unable to include the semiproper iteration of [M] which forces a stronger form of ψ_{AC}.

It appears that [DJ] constructs thin trees, while [M] does thick ones with a kind of homogenuity. A possible common thread is that any family of iteratively generic objects $\langle G_n \mid n < \omega \rangle$ is coded into a single real by types of sequences of projections. Hence no matter how we force with limit, we must add this new single real as long as $\langle G_n \mid n < \omega \rangle$ is new.

In §1, we quickly fix our notations. In §2, we formulate a general framework which explicates how codings take place. In §3, we carry a routine work to translate sequences of projections into the usual context of iterated forcing. For that we make use of the idea of forcing equivalence from [S]. In §4, we review $\diamondsuit$ and point out that for any notion of forcing P which has the c.c.c, is σ-Baire and of size at most ω_1, we may prepare a type of $\diamondsuit$-sequence in the ground model so that it remains so in any generic extension of P. In §5, we remake [JD] by $\diamondsuit$. In §6, we touch on our stronger form of ψ_{AC}. The principle ψ_{AC} is found in [W].

§1. Preliminaries

1.1 Definition. For a tree S and $x \in S$, we denote the *height* of x in S by $|x|$. So $|x| =$ the order-tye of $(\{\bar{x} \in S \mid \bar{x} <_S x\}, <_S)$. The height of S is denoted by $\mathrm{ht}(S)$. So there is no element $x \in S$ with $|x| = \mathrm{ht}(S)$. A *path* b of S means that b is a $<_S$-downward closed pairwise $<_S$-comparable subset of S. If the order-type of $(b, <_S)$ is α, then for $\beta < \alpha$, $b(\beta)$ denotes the β-th element of b. So $b(\beta) \in S_\beta$, where S_β denotes the β-th level of S. A tree S is *normal*, if

(1) For any $x \in S$ and $|x| < \alpha < \mathrm{ht}(S)$, there exists $x^* \in S_\alpha$ with $x <_S x^*$. (Dense)

(2) For any $x \in S$ with $|x| + 1 < \mathrm{ht}(S)$, we demand $|\mathrm{suc}_S(x)| = \omega$. ($\omega$-many successors)

(3) For any path b of S with no last element in b, there exists at most one element x in S which sits right above b. (At most one)

A *subtree* of ${}^{<\omega_1}\omega$ is a downward-closed subset of ${}^{<\omega_1}\omega$.

To construct limit levels of a tree, the following is basic.

1.2 Proposition. *Let T be a normal subtree of ${}^{<\omega_1}\omega$ such that $ht(T) = \alpha < \omega_1$ and α be a limit. Let $\{x_1, \cdots, x_l\}$ be a finite subset of ${}^{\alpha}\omega$. Let $y \in T$ with $|y| < \alpha$. Let $\langle A_n \mid n < \omega \rangle$ be a family of maximal antichains of T indexed by ω. Then there is $y^* \in {}^{\alpha}\omega$ such that $y \subset y^*$, $y^* \notin \{x_1, \cdots x_l\}$, for all $\beta < \alpha$, we have $y^* \lceil \beta \in T$ and for all $n < \omega$, we have $A_n \cap \{y^* \lceil \beta \mid \beta < \alpha\} \neq \emptyset$.*

Proof. Let $\langle \alpha_n \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals such that $\alpha_0 = |y|$ and $\sup\{\alpha_n \mid n < \omega\} = \alpha$. Construct $\langle y_n \mid n < \omega \rangle$ by recursion on n as follows;

- $y_0 = y$, $y_0 \subset y_1 \in T$, $\alpha_1 \leq |y_1| < \alpha$ and $y_1 \notin \{x_1 \lceil |y_1|, \ldots, x_l \lceil |y_1|\}$.
- There exists $\beta < |y_1|$ such that $y_1 \lceil \beta \in A_0$.
- $y_n \in T$, $\alpha_n \leq |y_n| < \alpha$.
- $y_n \subset y_{n+1} \in T$, $\alpha_{n+1} \leq |y_{n+1}| < \alpha$.
- There exists $\beta < |y_{n+1}|$ such that $y_{n+1} \lceil \beta \in A_n$.

It is straightforward to carry this construction due to (Dense) and (ω-many successor). Let

$$y^* = \bigcup \{y_n \mid n < \omega\}.$$

Then this y^* works.

□

The following is from [S].

1.3 Definition. Let P and Q be two notions of forcing. We say P and Q are *forcing equivalent*, if there exist a P-name $\tilde{G}_Q$ and a Q-name $\tilde{G}_P$ such that

(1) $V[G_P] \models$ "$\tilde{G}_Q[G_P]$ is a Q-generic filter over V".
(2) $V[G_Q] \models$ "$\tilde{G}_P[G_Q]$ is a P-generic filter over V".
(3) $V[G_P] \models$ "$\tilde{G}_P\big[\tilde{G}_Q[G_P]\big] = G_P$".
(4) $V[G_Q] \models$ "$\tilde{G}_Q\big[\tilde{G}_P[G_Q]\big] = G_Q$".

Where G_P and G_Q denote the respective generic filters over V. We denote $P \equiv Q$, if P and Q are forcing equivalent.

1.4 Proposition. *Let P and Q be notions of forcing such that $P \equiv Q$ with $\tilde{G}_Q$ and $\tilde{G}_P$. Then*

(1) $V[G_P] = V[\tilde{G}_Q[G_P]]$ for all P-generic filters G_P over V.
(2) $V[G_Q] = V[\tilde{G}_P[G_Q]]$ for all Q-generic filters G_Q over V.

Proof. Suffice to deal with (1). Let G_P be any P-generic filter over V and calculate $\tilde{G}_Q[G_P]$ which is Q-generic over V. Denote $G_Q = \tilde{G}_Q[G_P] \in V[G_P]$. Then we have

$$\tilde{G}_P[G_Q] = \tilde{G}_P\big[\tilde{G}_Q[G_P]\big] = G_P.$$

Hence $G_P \in V[G_Q]$ and so

$$V[G_P] = V[G_Q].$$

□

1.5 Proposition. *Let P, Q and R be notions of forcing. If $P \equiv Q \equiv R$, then $P \equiv R$. And so $\equiv$ is a class equivalence relation on the class of notions of forcing.*

Proof. Let $P \equiv Q$ with $\tilde{G}_Q$ and $\tilde{G}_P$. Let $Q \equiv R$ with $\bar{G}_R$ and $\bar{G}_Q$. Want to come up with a P-name $\underline{G}_R$ and an R-name $\underline{G}_P$.

Let

$$V[G_P] = V\big[\tilde{G}_Q[G_P]\big] \models \text{“}\underline{G}_R = \bar{G}_R\big[\tilde{G}_Q[G_P]\big]\text{”}.$$

and

$$V[G_R] = V\big[\bar{G}_Q[G_R]\big] \models \text{“}\underline{G}_P = \tilde{G}_P\big[\bar{G}_Q[G_R]\big]\text{”}.$$

Then

$$V[G_P] \models \text{"}\underline{G}_R \text{ is } R\text{−generic over } V\text{"}.$$

$$V[G_R] \models \text{"}\underline{G}_P \text{ is } P\text{−generic over } V\text{"}.$$

and

$$V[G_P] \models \text{"}\underline{G}_P\big[\underline{G}_R[G_P]\big] = \tilde{G}_P(\bar{G}_Q\bar{G}_R)\tilde{G}_Q[G_P] = \tilde{G}_P\tilde{G}_Q[G_P] = G_P\text{"}.$$

$$V[G_R] \models \text{"}\underline{G}_R\big[\underline{G}_P[G_R]\big] = \bar{G}_R(\tilde{G}_Q\tilde{G}_P)\bar{G}_Q[G_R] = \bar{G}_R\bar{G}_Q[G_R] = G_R\text{"}.$$

Hence $P \equiv R$.

□

1.6 Proposition. *Let P and Q be notions of forcing. If $P \equiv Q$ via $\tilde{G}_Q$ and $\tilde{G}_P$, then*

(1) For all $p \in P$, there exists $q \in Q$ such that $q \Vdash_Q$ "$p \in \tilde{G}_P$".

(2) For all $q \in Q$, there exists $p \in P$ such that $p \Vdash_P$ "$q \in \tilde{G}_Q$".

Proof. Suffice to show (1). Let $p \in G_P$. Then calculate $\tilde{G}_Q[G_P]$ and denote $G_Q = \tilde{G}_Q[G_P]$. Then

$$\tilde{G}_P[G_Q] = \tilde{G}_P\big[\tilde{G}_Q[G_P]\big] = G_P.$$

Hence

$$V[G_Q] \models \text{"}p \in \tilde{G}_P[G_Q].\text{"}$$

So $q \Vdash_Q$"$p \in \tilde{G}_P$" for some $q(\in G_Q)$.

□

1.7 Proposition. *Let P and Q be notions of forcing with $P \equiv Q$. Then*

(1) If P has the c.c.c, then so does Q.

(2) If P is σ-Baire, then so is Q.

Proof. For (1): Want to show Q has the c.c.c. Let $\langle q_i \mid i < \omega_1\rangle$ be given. For each $i < \omega_1$, take $p_i \in P$ such that $p_i \Vdash_P$"$q_i \in \tilde{G}_Q$". Since P has the c.c.c, there are i, j such that $i \neq j$ and p_i and p_j are compatible. Let $p \leq p_i, p_j$ in P. Then $p \Vdash_P$"$q_i, q_j \in \tilde{G}_Q$" and so q_i and q_j are compatible in Q.

For (2): Want to show Q is σ-Baire. Namely, no new countable sequences of ordinals are added. But $V[G_Q] = V\big[\tilde{G}_P[G_Q]\big]$ and P is σ-Baire. Hence Q is σ-Baire.

□

1.8 Question. Let P and Q be two notions of forcing. We denote $P < Q$, if there exists a Q-name $\tilde{G}_P$ such that $\Vdash_Q$"$\tilde{G}_P$ is P-generic over V" and for any $p \in P$, there exists $q \in Q$ such that $q \Vdash_Q$"$p \in \tilde{G}_P$". We have seen that if $P \equiv Q$, then $P < Q$ holds. Is it the case that if $P < Q$ and $Q < P$ hold, then $P \equiv Q$?

§2. General Framework

We formulate a type of sequences of projections. This is sufficient to cover the construction in this note.

2.1 Definition. Let S and T be trees of height ω_1 such that

(0) The 0-th levels consist of their roots. Denote $\{\mathrm{root}_S\} = S_0$ and $\{\mathrm{root}_T\} = T_0$. (Root)

(1) For all $x \in S$ and $\alpha < \omega_1$ with $|x| < \alpha$, there exists $x^* \in S$ s.t. $x <_S x^*$ and $|x^*| = \alpha$. Similarly for T, where $|x|$ and $|x^*|$ denote the height of x and x^* respectively. (Dense)

We say a triple (S, h, T) is a *step*, if h is a projection from T into S, i.e,

(1) $h(y_2) \geq_S h(y_1)$, if $y_2 \geq_T y_1$ for all $y_1, y_2 \in T$. (Order-preserving)

(2) If $x \geq_S h(y)$, then there exists $y' \in T$ s.t. $y' \geq_T y$ and $h(y') \geq_S x$ for all $y \in T$ and $x \in S$. (Reduction)

and h satisfies

(3) $h(\mathrm{root}_T) = \mathrm{root}_S$ and for all $y \in T$ with $|y| \geq 1$, it holds that $|h(y)| = |y| + 1$. (Ahead)

In the sequel, we write $y_2 \leq y_1$ to mean $y_2 \geq_T y_1$. We also denote $x_2 \leq x_1$ to mean $x_2 \geq_S x_1$ for S. This conforms to the usual convention in forcing arguments. And there should be no confusions. To explain the meaning of (Ahead), let b_T be a T-generic filter over the ground model V indexed by ω_1. So $b_T : \omega_1 \longrightarrow T$ and $b_T(\alpha) \in T_\alpha$. This makes sense by (Dense) on T. Let b_S be the induced S-generic filter from b_T by h. Then (Ahead) is a device to recover b_S from b_T in such a way that $b_S(\alpha+1) = h\big(b_T(\alpha)\big)$. We intend to provide details to this and other standard facts related to the projections for the sake of completeness.

2.2 Proposition. *Let (S, h, T) be a step.*

(1) The image of T under h, denoted by h“T, is dense in S.

(2) If b_S is an S-generic filter over V, then b_S is a cofinal path through S. We refere to this path as a generic cofinal path and denote the α-th element of b_S by $b_S(\alpha)$. So $b_S(\alpha) \in S_\alpha$, where S_α denotes the α-level of S.

Proof. For (1): Let $x \in S$. Since $x \leq \mathrm{root}_S = h(\mathrm{root}_T)$, we have $y \in T$ such that $h(y) \leq x$. Hence h“T is dense in S.

For (2): Since b_S is directed, it holds that every element of b_S is comparable. Since b_S is upward-closed in the notion of forcing S, we have that b_S is a downward-closed subset of the tree S. Hence b_S is a path. For any $\alpha < \omega_1$, $\{x \in S \mid \alpha < |x|\}$ is dense in S by (Dense). Hence b_S is cofinal through S.

□

We deal with the quotients.

2.3 Proposition. *Let (S, h, T) be a step. Let b_S be an S-generic filter over V. In $V[b_S]$, let*

$$T/b_S = \{y \in T \mid h(y) \in b_S\}.$$

Then T/b_S is a tree of height ω_1^V and satisfies (Dense). We simply denote T/b_S by $\hat{T}$.

Proof. Since $\mathrm{root}_S \in b_S$, we have $\mathrm{root}_T \in T/b_S$.

(T/b_S is downward-closed in T) Let $y_1 \in T/b_S$ and $y_2 \leq_T y_1$. Then $h(y_2) \leq_S h(y_1)$. But $h(y_1) \in b_S$ and so $h(y_2) \in b_S$. Hence $y_2 \in T/b_S$.

(T/b_S is dense) Let $y \in T/b_S$. So $h(y) \in b_S$. Fix any $\alpha < \omega_1$ and let $D = \{h(z) \in S \mid y <_T z, \alpha < |z| \text{ in } T\}$. Then this D is dense below $h(y)$ in S. This makes use of (projection) and (Dense). Hence there exists $z \in T/b_S$ such that $y <_T z$ and $|z| > \alpha$ in T/b_S. In particular, the height of T/b_S is ω_1^V.

□

2.4 Proposition. *$S * \hat{T} \equiv T$ holds by an $S * \hat{T}$-name $\tilde{b}_T$, T-names $\tilde{b}_S$ and $\tilde{b}_{\hat{T}}$ defined as follows;*

$$V[b_S][b_{\hat{T}}] \models \text{“}\tilde{b}_T = b_{\hat{T}}\text{”}.$$

$$V[b_T] \models \text{“}\tilde{b}_S = \text{ the downward closure of } \{h\big(b_T(\alpha)\big) \mid \alpha < \omega_1^V\},\ \tilde{b}_{\hat{T}} = b_T\text{”}.$$

Namely,

(1) $V[b_S][b_{\hat{T}}] \models$ "$\tilde{b}_T$ *is* T*-generic over* V".

(2) $V[b_T] \models$ "$\tilde{b}_S$ *is* S*-generic over* V *and* $\tilde{b}_{\hat{T}}$ *is* $\hat{T}[\tilde{b}_S] = T/\tilde{b}_S$*-generic over* $V[\tilde{b}_S]$".

(3) $V[b_S][b_{\hat{T}}] \models$ "$\tilde{b}_S[\tilde{b}_T] = b_S$ *and* $\tilde{b}_{\hat{T}}[\tilde{b}_T] = b_{\hat{T}}$".

(4) $V[b_T] \models$ "$\tilde{b}_T[\tilde{b}_S * \tilde{b}_{\hat{T}}] = b_T$".

Proof. For (1): Since $\hat{T} = T/b_S$ is a tree of height ω_1^V and satisfies (Dense), we conclude $b_{\hat{T}}$ is a cofinal path through $\hat{T}$. So $b_{\hat{T}}$ is a cofinal path through T, too. Hence $b_{\hat{T}}$ is a directed and upward-closed subset of the notion of forcing T. It remains to observe that for any dense subset $D \in V$ of T, it holds that $D \cap b_{\hat{T}} \neq \emptyset$. To see this, it suffices to see $D \cap \hat{T}$ is dense in $\hat{T}$. To this end, let $y \in \hat{T}$. We have $h(y) \in b_S$. Let $D' = \{h(z) \mid z \in D, y \leq_T z\}$. Then this D' is dense below $h(y)$ in S. Hence we have $z \in D \cap \hat{T}$ with $z \leq y$ in $\hat{T}$.

For (2): In $V[b_T]$, let

$$b_S = \tilde{b}_S[b_T] = \{x \in S \mid x \leq_S h\big(b(\alpha)\big), \alpha < \omega_1^V\}.$$

(b_S is an S-generic filter over V): Want to show that this b_S is S-generic over V. Since b_S is a filter in S, it suffices to show that for any dense subset D of S, we have $D \cap b_S \neq \emptyset$.

Since h is a projection, we have $D' = \{y \in T \mid h(y) \leq d \text{ in } S,\ d \in D\}$ is dense in T. Hence we have $y \in b_T$ such that $h(y) \geq_S d$ for some $d \in D$. Hence $d \in D \cap b_S$.

($b_T \subseteq T/b_S$): For any $y \in b_T$, we have $h(y) \in b_S$ by the definition of b_S. So $y \in T/b_S$ by the definition of T/b_S. Hence b_T is a filter in T/b_S.

It suffices to show for any dense subset $\tilde{D} \in V[b_S]$ of T/b_S, we have $\tilde{D} \cap b_T \neq \emptyset$. Take $x \in b_S$ so that $x \Vdash_S$"$\tilde{D} \subseteq T/b_S$ is dense". Since $x \in b_S$, may take $y \in b_T$ such that $x \leq_S h(y)$. So $h(y) \Vdash_S$"$\tilde{D} \subseteq T/b_S$ is dense". We see $\{z \in T \mid d \geq z \text{ in } T,\ h(z) \Vdash_S \text{"}d \in \tilde{D}\text{"}\}$ is dense below y in T. Hence we have $z \in b_T$ and d such that $d \geq z$ in T and $h(z) \Vdash_S$"$d \in \tilde{D}$". Hence $h(z) \in b_S$ and $d \in \tilde{D} \cap b_T$.

For (3):

$$V[b_S][b_{\hat{T}}] \models \text{"}\tilde{b}_S\big[\tilde{b}_T[b_S * b_{\hat{T}}]\big] = \tilde{b}_S[b_{\hat{T}}] = \{x \in S \mid x \leq_S h(y), y \in b_{\hat{T}}\} = b_S\text{"}.$$

For the last equation, it suffices to see $\subseteq$. But $b_{\hat{T}} \subset T/b_S$ and so $h(y) \in b_S$ for all $y \in b_{\hat{T}}$.

$$V[b_S][b_{\hat{T}}] \models \text{"}\tilde{b}_{\hat{T}}\big[\tilde{b}_T[b_S * b_{\hat{T}}]\big] = \tilde{b}_T[b_S * b_{\hat{T}}] = b_{\hat{T}}\text{"}.$$

For (4):

$$V[b_T] \models \text{"}\tilde{b}_T\big[\tilde{b}_S[b_T] * \tilde{b}_{\hat{T}}[b_T]\big] = \tilde{b}_T\big[\tilde{b}_S[b_T] * b_T\big] = b_T\text{"}.$$

Hence $S * (T/b_S) \equiv T$ holds.

□

We record useful facts. Since h satisfies (Ahead), we further conclude (3) below.

2.5 Proposition. *(1)* $h(y) \Vdash_S$ "$y \in \hat{T}$".

(2) $y \Vdash_T$ "$h(y) \in \tilde{b}_S$".

(3) *If* $1 \leq \alpha = |y|$, *then* $y \Vdash_T$ "$h(y) = \tilde{b}_S(\alpha + 1)$, *the* $\alpha + 1$*-st element of the induced generic path* $\tilde{b}_S$".

(4) *In particular,* $\Vdash_T$ "$\tilde{b}_S(\alpha + 1) = h\big(b_T(\alpha)\big)$" *for* $\alpha \geq 1$.

Proof. For (1): Let $y \in T$.

$$h(y) \in b_S \Longrightarrow T/b_S = \hat{T} = \{z \in T \mid h(z) \in b_S\} \Longrightarrow y \in \hat{T}.$$

For (2): Let $y \in T$.

$$y \in b_T \Longrightarrow h(y) \in \tilde{b}_S[b_T] = \{x \in S \mid x \leq_S h(z), z \in b_T\}.$$

For (3): Let $y \in T$ with $|y| = \alpha \geq 1$.

$$y \in b_T \Longrightarrow |h(y)| = |y| + 1 \Longrightarrow h(y) = \tilde{b}_S[b_T](\alpha + 1).$$

For (4): Let $\omega_1 > \alpha \geq 1$.

$$V[b_T] \models \text{“}\tilde{b}_S[b_T](\alpha + 1) = h\big(b_T(\alpha)\big)\text{”}.$$

□

2.6 Lemma. *Let (T_n, h_n) $(n = 1, 2, \cdots)$ satisfy*

- *(T_n, h_n, T_{n+1}) are steps.*
- *The T_n's satisfy the following;*

If $\alpha < \omega_1$ is any limit ordinal and b is any path of length α, then there exists at most one element at $(T_n)_\alpha$ above every member of b. (At most one)

Let, in any generic extension over V, b_n $(n = 1, 2, \cdots)$ be a generic cofinal path through T_n over V such that for all $\alpha \geq 1$,

$$b_n(\alpha + 1) = h_n\big(b_{n+1}(\alpha)\big).$$

Then we have

$$V\big[\langle b_n \mid n = 1, 2, \cdots\rangle\big] = V\big[\langle b_n(1) \mid n = 1, 2, \cdots\rangle\big].$$

Proof. We construct $\langle d_n\lceil\alpha \mid n = 1, 2, \cdots\rangle$ by recursion on α in $V[\langle b_n(1) \mid n = 1, 2, \cdots\rangle]$. We then show

$$d_n\lceil\alpha = b_n\lceil\alpha \quad (\text{for all } n = 1, 2, \cdots)$$

by induction on α . Hence $\langle b_n \mid n = 1, 2, \cdots\rangle \in V[\langle b_n(1) \mid n = 1, 2, \cdots\rangle]$.

(Construction) Let us define

$$\langle d_n(0), d_n(1)\rangle = \langle \mathrm{root}_{T_n}, b_n(1)\rangle = \langle b_n(0), b_n(1)\rangle.$$

(Successor stage) Suppose $\alpha \geq 1$ and we have constructed $\langle d_n(0), d_n(1), \cdots, d_n(\alpha)\rangle$. Want $d_n(\alpha + 1)$. Let

$$d_n(\alpha + 1) = h_n\big(d_{n+1}(\alpha)\big), \text{ if } d_{n+1}(\alpha) \in (T_{n+1})_\alpha.$$

Otherwise, $d_n(\alpha + 1)$ is undefined.

(Limit stage) Suppose α is a limit and we have constructed $d_n\lceil\alpha$. Want $d_n(\alpha)$. Let

$$d_n(\alpha) = \text{the unique element of } (T_n)_\alpha \text{ which sits above every element of the path } d_n\lceil\alpha, \text{ if this is possible.}$$

Otherwise $d_n(\alpha)$ is undefined.

This completes the construction.

(Induction) It is clear that

$$d_n\lceil 2 = b_n\lceil 2 \quad (\text{for all } n = 1, 2, \cdots).$$

Suppose $\alpha \geq 1$ and we have seen $d_n\lceil(\alpha+1) = b_n\lceil(\alpha+1)$ (for all $n = 1, 2, \cdots$). Want $d_n(\alpha+1) = b_n(\alpha+1)$. Since $d_{n+1}(\alpha) = b_{n+1}(\alpha) \in (T_{n+1})_\alpha$, we have

$$d_n(\alpha + 1) = h_n\big(d_{n+1}(\alpha)\big) = h_n\big(b_{n+1}(\alpha)\big) = b_n(\alpha + 1).$$

Suppose α is a limit and we have seen $d_n\lceil\alpha = b_n\lceil\alpha$ (for all $n = 1, 2, \cdots$). Want $d_n(\alpha) = b_n(\alpha)$. But $b_n(\alpha)$ is the unique element of $(T_n)_\alpha$ which sits above every element of the path $b_n\lceil\alpha = d_n\lceil\alpha$. We conclude that $d_n(\alpha) = b_n(\alpha)$.

This completes the induction.

□

Given a nice tree S, we want to define a nice tree T and h so that (S, h, T) is a step. For this, we list necessities on the steps. We build T and h via some $\langle x \mapsto t_x \mid x \in S\rangle$ which satisfies the listed properties and more.

2.7 Proposition. *Let (S, h, T) be a step. Then for each $x \in h``T \setminus \{\mathrm{root}_S\}$, we may define*

$$t_x = \text{ the downward closure of } \{y \in T \mid h(y) = x\}.$$

Then we have

(1) t_x is a tree of height $|x|$, the height of x in S.

(2) If $x_1 <_S x_2$, then t_{x_1} gets end-extended to t_{x_2}. (Coherence)

(3) If $x_1 \neq x_2$ in $S_{\alpha+1} \cap h``T$ with $1 \leq \alpha$, then

$$(t_{x_1})_\alpha \cap (t_{x_2})_\alpha = \emptyset.$$

(Forking)

Proof. For (1): Let $h(y) = x$. Then $|y| + 1 = |x|$. Hence t_x is a tree with $\mathrm{ht}(t_x) = |x|$.

For (2): Let $x_1 <_S x_2$, $|x_1| = \alpha_1 + 1$, $|x_2| = \alpha_2 + 1$. We first observe $(t_{x_2})_{\alpha_1} \subseteq (t_{x_1})_{\alpha_1}$. To this end, let $h(y_2) = x_2$ and $y_1 <_T y_2$ with $|y_1| = \alpha_1$ so that $y_1 \in (t_{x_2})_{\alpha_1}$. Then $h(y_1) <_S h(y_2) = x_2$ with $|h(y_1)| = \alpha_1 + 1$. Hence $h(y_1) = x_1$ and so $y_1 \in (t_{x_1})_{\alpha_1}$.

Conversely, we show $(t_{x_2})_{\alpha_1} \supseteq (t_{x_1})_{\alpha_1}$. Let $y_1 \in (t_{x_1})_{\alpha_1}$. So $h(y_1) = x_1$. Take $z \in T$ such that $x_2 \leq_S h(z)$ and $y_1 <_T z$. Let $y_2 \leq_T z$ with $|y_2| = \alpha_2$. Then $|h(y_2)| = \alpha_2 + 1$ and $h(y_2), x_2 \leq_S h(z)$. Hence $h(y_2) = x_2$ and so $y_2 \in t_{x_2}$. Hence $y_1 \in (t_{x_2})_{\alpha_1}$.

For (3): Let $h(y_1) = x_1$ and $h(y_2) = x_2$. Since $x_1 \neq x_2$, we have $y_1 \neq y_2$. Hence $(t_{x_1})_\alpha \cap (t_{x_2})_\alpha = \emptyset$.

□

We record is a sufficient condition to get a step. However the T below does not satisfy (At most one) at all, we see no use of this observation.

2.8 Proposition. *Let S be a tree such that S satisfies $\mathrm{root}_S = \emptyset$, (Dense) and is of height ω_1. Let $\dot{T}$ be an S-name such that in V^S, $\dot{T} \subset {}^{<\omega_1^V} V$ is a tree such that $\dot{T}$ satisfies $\mathrm{root}_{\dot{T}} = \emptyset$, $<_{\dot{T}}$ is the strict inclusion $\subset$, (Dense) and is of height ω_1^V. We further assume that*

- *If $w \Vdash_S$ "$t \in \dot{T}_\alpha$", $|w| \geq \alpha + 1$ and $\alpha \geq 1$, then $w' \Vdash_S$ "$t \in \dot{T}_\alpha$", where $w' \in S_{\alpha+1}$ with $w' \leq_S w$.*
- *If $a \Vdash_S$ "$t \in \dot{T}_\alpha$", then a decides $\{y \in \dot{T} \mid y \subset t\}$.*

Let $T = \{(s, \check{t}) \mid 1 \leq \alpha < \omega_1, s \in S_{\alpha+1}, s \Vdash_S$ "$\check{t} \in \dot{T}_\alpha$" $\} \cup \{(\emptyset, \emptyset)\}$. Then T is dense in $S * \dot{T}$ and define $h : T \longrightarrow S$ by $h(s, \check{t}) = s$. Then (S, h, T) is a step such that $\Vdash_S$ "T/b_S is isomorphic to $\dot{T}$".

Proof. For $(s_1, \check{t}_1)$, $(s_2, \check{t}_2) \in T$, get the ordering $(s_2, \check{t}_2) <_T (s_1, \check{t}_1)$ iff $s_1 <_S s_2$ and $t_1 \subset t_2$. So $(s_2, \check{t}_2) \leq_T (s_1, \check{t}_1)$ iff $s_1 \leq_S s_2$ and $t_1 \subseteq t_2$. Notice that neither ($s_1 = s_2$ and $t_1 \subset t_2$) nor ($s_1 <_S s_2$ and $t_1 = t_2$) do not occure due to their heights.

It is easy to see that T is a tree which satisfies (Root), (Dense) and is of height ω_1. Note that T may not satisfy (At most one). We observe h is a projection.

(Order-preserving) Let $(s_2, \check{t}_2) \leq_T (s_1, \check{t}_1)$. Then $s_2 \leq_S s_1$.

(Reduction) Let $s \leq_S s'$ and $s = h(s, \check{t})$. Since T is a dense subset in $S * \dot{T}$, get $(s'', \check{t}'') \in T$ such that $(s'', \check{t}'') \leq (s', \check{t})$ in $S * \dot{T}$. So $s' \leq_S s''$, $h(s'', \check{t}'') = s''$ and $(s'', \check{t}'') \leq (s, \check{t})$ in T.

(Ahead) $h(\emptyset, \emptyset) = \emptyset$ and for all $(s, \check{t}) \in T$ with $|(s, \check{t})| \geq 1$ in T, we have $|s| = |(s, \check{t})| + 1$. This is because

$$\big\{(s', \check{t}') \in T \mid (s', \check{t}') <_T (s, \check{t})\big\} = \big\{(s_{\beta+1}, \check{t}_\beta) \mid 1 \leq \beta < \alpha\big\} \cup \big\{(\emptyset, \emptyset)\big\}$$

where $|s| = \alpha + 1$ in S with necessarily $\alpha \geq 1$, $s_{\beta+1}$ denotes the element in $S_{\beta+1}$ below s and $t_\beta = \beta$-th element in $\dot{T}$ below t (in V^S). Hence $|(s, \check{t})|$ in T is $1 + (\alpha - 1) = \alpha$. Since $|s|$ in S is $\alpha + 1$, we are done.

Lastly, let b_S be S-generic over V. In $V[b_S]$, we have

$$T/b_S = \big\{(s, \check{t}) \in T \mid h(s, \check{t}) = s \in b_S\big\} = \big\{(b_S(\alpha + 1), \check{t}) \mid 1 \leq \alpha < \omega_1, t \in \dot{T}_\alpha\big\} \cup \big\{(\emptyset, \emptyset)\big\}.$$

Hence T/b_S and $\dot{T}$ are isomorphic.

□

§3. **Routine Translations**

We turn everything so far developed into the context of ordinary iterated forcing construction. Therefore, we expect lots of straightforward routines.

3.1 Lemma. *Let P be a p.o. set and (S, h, T) be a step. Let $P \equiv S$ via an S-name $\underline{G}_P$ and a P-name $\underline{b}_S$ and $S * (T/b_S) \equiv T$ via an $S * (T/b_S)$-name $\tilde{b}_T$ and T-names $\tilde{b}_S$ and $\tilde{b}_{\hat{T}}$, where $\hat{T}$ denotes T/b_S. Let $\dot{T} = T/\underline{b}_S$ in $V[G_P]$, where G_P be any P-generic filter over V. Then*

$$P * \dot{T} \equiv T$$

via a $P * \dot{T}$-name $\bar{b}_T$ and T-names $\bar{G}_P$ and $\bar{b}_{\dot{T}}$, where we set

$$V[G_P][b_{\dot{T}}] \models \text{“}\bar{b}_T = b_{\dot{T}}\text{”}.$$

$$V[b_T] \models \text{“}\bar{G}_P = \underline{G}_P[\tilde{b}_S],\ \bar{b}_{\dot{T}} = b_T\text{”}.$$

In particular, we have the following for $\alpha \geq 1$.

$$V[G_P][\dot{b}_T] \models \text{“}\underline{b}_S[G_p](\alpha + 1) = h\big(\bar{b}_T(\alpha)\big)\text{”}.$$

Proof.

$$V[G_P][b_{\dot{T}}] = V\big[\underline{b}_S[G_P]\big][b_{\dot{T}}] \models \text{“}\bar{b}_T = b_{\dot{T}} \text{ is } T\text{−generic over } V\text{”}.$$

Hence $V[G_P][b_{\dot{T}}] \models$ “$\bar{b}_T$ is T-generic over V.

$$V[b_T] = V\big[\tilde{b}_S[b_T]\big]\big[\tilde{b}_{\hat{T}}[b_T]\big] = V\big[\tilde{b}_{\hat{T}}[b_T]\big] \models \text{“}\bar{G}_P = \underline{G}_P\big[\tilde{b}_S[b_T]\big] \text{ is } P\text{−generic over } V\text{”}.$$

and

$$V[b_T] \models \text{“}\bar{b}_{\dot{T}} = \tilde{b}_{\hat{T}}[b_T] = b_T \text{ is } \dot{T}\big[\bar{G}_P[b_T]\big] = T/\underline{b}_S\big[\bar{G}_P[b_T]\big] = T/\tilde{b}_S[b_T]\text{−generic over } V\big[\tilde{b}_S[b_T]\big]\text{”}.$$

and

$$V\big[\tilde{b}_S[b_T]\big] = V\Big[\underline{G}_P\big[\tilde{b}_S[b_T]\big]\Big] = V\big[\bar{G}_P[b_T]\big]\text{”}.$$

Hence $V[b_T] \models$ “$\bar{G}_P * \bar{b}_{\dot{T}}$ is $P * \dot{T}$-generic over V”.

Next, we first observe

$$V[G_P][b_{\dot{T}}] \models \text{“}\tilde{b}_S[b_{\dot{T}}] = \underline{b}_S[G_P]\text{”}.$$

This is because $b_{\dot{T}}$ is $T/\underline{b}_S[G_P]$-generic over $V[G_P]$. In particular, $b_{\dot{T}} \subseteq T/\underline{b}_S[G_P]$. Hence

$$\{h(y) \mid y \in b_{\dot{T}}\} \subseteq \underline{b}_S[G_P].$$

On the other hand,

$$\tilde{b}_S[b_{\dot{T}}] = \{x \in S \mid x \leq_S h(y), y \in b_{\dot{T}}\}.$$

Both $\tilde{b}_S[b_{\dot{T}}]$ and $\underline{b}_S[G_P]$ are S-generic over V. Hence $\tilde{b}_S[b_{\dot{T}}] \subseteq \underline{b}_S[G_P]$ implies they must be identical.

Now,

$$V[G_P][b_{\dot{T}}] \models \text{“}\bar{G}_P\big[\bar{b}_T[G_P * b_{\dot{T}}]\big] = \bar{G}_P[b_{\dot{T}}] = \underline{G}_P\big[\tilde{b}_S[b_{\dot{T}}]\big] = \underline{G}_P\big[\underline{b}_S[G_P]\big] = G_P\text{”}.$$

$$V[G_P][b_{\dot{T}}] \models \text{“}\bar{b}_{\dot{T}}\big[\bar{b}_T[G_P * b_{\dot{T}}]\big] = \bar{b}_{\dot{T}}[b_{\dot{T}}] = b_{\dot{T}}\text{”}.$$

And

$$V[b_T] \models \text{“}\bar{b}_T\big[\bar{G}_P[b_T] * \bar{b}_{\dot{T}}[b_T]\big] = \bar{b}_{\dot{T}}[b_T] = b_T\text{”}.$$

Finally, for $\alpha \geq 1$

$$V[G_P][b_{\dot{T}}] \models \text{“}\underline{b}_S[G_P](\alpha+1) = \tilde{b}_S[b_{\dot{T}}](\alpha+1) = h\big(b_{\dot{T}}(\alpha)\big) = h\big(\bar{b}_T(\alpha)\big)\text{”}.$$

$\square$

3.2 Lemma. *Let (T_n, h_n) $(n = 1, 2, \cdots)$ be given such that*

$$(T_n, h_n, T_{n+1})$$

are steps. Then we may construct an ω-stage iterated forcing $\langle P_n \mid n < \omega\rangle$ as follows;

(0) $P_0 = \{\emptyset\}$.

(1) $P_1 \equiv T_1$ *via the P_1-name $\bar{b}_1$ and the T_1-name $\bar{G}_1$.*

(2) $P_n \equiv T_n$ *via the P_n-name $\bar{b}_n$ and the T_n-name $\bar{G}_n$ for $n \geq 1$.*

(3) $P_{n+1} \equiv P_n * T_{n+1}/\bar{b}_n[G_n] \equiv T_{n+1}$.

Let G_n $(n = 0, 1, 2, \cdots)$ be P_n-generic filters such that $G_{n+1}\lceil n = G_n$ and for $n \geq 1$, let

$$b_n = \bar{b}_n[G_n].$$

Then we have

(4) $b_n(0) = \mathrm{root}_{T_n}$ *and* $b_n(\alpha+1) = h_n\big(b_{n+1}(\alpha)\big)$ *for $n \geq 1$ and $\alpha \geq 1$.*

(5) $G_0 = \{\emptyset\}$ *and* $G_n = \bar{G}_n[b_n]$ *for $n \geq 1$.*

Hence if the T_n further satisfy (At most one), then we have

$$V\big[\langle G_n \mid n < \omega\rangle\big] = V\big[\langle b_n \mid n = 1, 2, \cdots\rangle\big] = V\big[\langle b_n(1) \mid n = 1, 2, \cdots\rangle\big].$$

Proof. Construct $\langle P_n \mid n < \omega\rangle$ by recursion on n. Let

$$P_0 = \{\emptyset\} \text{ and } P_1 \equiv T_1.$$

Suppose $n \geq 1$ and have constructed P_n such that $P_n \equiv T_n$ with $\bar{b}_n$ and $\bar{G}_n$. Then apply 3.1 Lemma to get

$$P_{n+1} \equiv P_n * \dot{T}_{n+1} = P_n * T_{n+1}/\bar{b}_n[G_n] \equiv T_{n+1}.$$

with $\bar{b}_{n+1}$ and $\bar{G}_{n+1}$. For $\alpha \geq 1$, we have

$$V[G_{n+1}] \models \text{“}\bar{b}_n[G_{n+1}\lceil n](\alpha+1) = h_n\big(\bar{b}_{n+1}[G_{n+1}](\alpha)\big)\text{”}.$$

Next, let G_n be P_n-generic and $G_{n+1}\lceil n = G_n$ for all $n < \omega$. Let $b_n = \bar{b}_n[G_n]$ for $n \geq 1$. Then for $\alpha \geq 1$,

$$b_n(\alpha+1) = h_n\big(b_{n+1}(\alpha)\big).$$

and

$$G_n = \bar{G}_n\big[\bar{b}_n[G_n]\big] = \bar{G}_n[b_n].$$

Therefore

$$V[\langle G_n \mid n < \omega\rangle] = V[\langle b_n \mid n = 1, 2, \cdots\rangle].$$

Finally, if (At most one) gets satisfied for all T_n, then by 2.6 Lemma, we have

$$V[\langle b_n \mid n = 1, 2, \cdots\rangle] = V[\langle b_n(1) \mid n = 1, 2, \cdots\rangle].$$

Hence

$$V[\langle G_n \mid n < \omega\rangle] = V[\langle b_n(1) \mid n = 1, 2, \cdots\rangle].$$

□

§4. **Diamond**

We prepare a suitable form of $\diamondsuit$-sequence. Let us begin with a recap.

4.1 Definition. We denote $\diamondsuit$ to mean that there exists $\langle A_\alpha \mid \alpha < \omega_1\rangle$ such that $A_\alpha \subseteq \alpha$ and for any $A \subseteq \omega_1$, we demand $\{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary.

We make use of reformulations.

4.2 Lemma. *Let us assume $\diamondsuit$. Then*

(1) There exists $\langle f_\alpha \mid \alpha < \omega_1\rangle$ such that $f_\alpha : \alpha \longrightarrow H_{\omega_1}$ and for any $f : \omega_1 \longrightarrow H_{\omega_1}$, we have $\{\alpha < \omega_1 \mid f\lceil\alpha = f_\alpha\}$ is stationary.

(2) Let $\langle Z_\alpha \mid \alpha < \omega_1\rangle$ be any continuously $\subseteq$-increasing countable subsets of ${}^{<\omega_1}\omega$ with $\bigcup\{Z_\alpha \mid \alpha < \omega_1\} = {}^{<\omega_1}\omega$. Then there exists $\langle B_\alpha \mid \alpha < \omega_1\rangle$ such that $B_\alpha \subseteq Z_\alpha$ and for any $B \subseteq {}^{<\omega_1}\omega$, we have $\{\alpha < \omega_1 \mid B \cap Z_\alpha = B_\alpha\}$ is stationary.

Proof. For (1): Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. Fix a bijection $\pi : \omega_1 \longrightarrow \omega_1 \times H_{\omega_1}$. Define $f_\alpha : \alpha \longrightarrow H_{\omega_1}$ such that

$$f_\alpha = \begin{cases} \pi\text{“}A_\alpha, & \text{if } \pi\text{“}A_\alpha : \alpha \longrightarrow H_{\omega_1} \\ \text{any function}, & \text{otherwise.} \end{cases}$$

We claim this $\langle f_\alpha \mid \alpha < \omega_1\rangle$ works. To this end, let

$$f : \omega_1 \longrightarrow H_{\omega_1}.$$

Since $f \subset \omega_1 \times H_{\omega_1}$, we may define $A \subset \omega_1$ so that

$$\pi``A = f.$$

Let

$$C = \{\alpha < \omega_1 \mid \pi``(A \cap \alpha) = f\lceil\alpha\}.$$

Then this C is a club. Let

$$S = \{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}.$$

Then this S is stationary. Since

$$S \cap C \subseteq \{\alpha < \omega_1 \mid f_\alpha = f\lceil\alpha\}.$$

holds, we are done.

For (2): Let $\langle f_\alpha \mid \alpha < \omega_1\rangle$ be as in (1). Fix any continuously $\subseteq$-increasing countable subsets Z_α of ${}^{<\omega_1}\omega$ such that $\bigcup\{Z_\alpha \mid \alpha < \omega_1\} = {}^{<\omega_1}\omega$. Let us define $B_\alpha \subseteq Z_\alpha$ by

$$B_\alpha = (f_\alpha``\alpha) \cap Z_\alpha.$$

We claim this $\langle B_\alpha \mid \alpha < \omega_1\rangle$ works. To this end, let

$$B \subseteq {}^{<\omega_1}\omega.$$

Let $f : \omega_1 \longrightarrow B$ be an enumeration. Let

$$C = \{\alpha < \omega_1 \mid f``\alpha = B \cap Z_\alpha\}.$$

Then this C is a club. Let

$$S = \{\alpha < \omega_1 \mid f\lceil\alpha = f_\alpha\}.$$

Then this S is stationary. Since

$$C \cap S \subseteq \{\alpha < \omega_1 \mid B \cap Z_\alpha = B_\alpha\}.$$

holds, we are done.

□

4.3 Lemma. *($\diamondsuit$) Let P be any p.o. set such that P has the c.c.c, is σ-Baire and $P \subseteq H_{\omega_1}$ and so $|P| \leq \omega_1$. Let Z_α be as above. Then we may construct $\langle \mathcal{A}_\alpha \mid \alpha < \omega_1\rangle$ such that $\mathcal{A}_\alpha = \{a^n_\alpha \mid n < \omega\}$, $a^n_\alpha \subseteq Z_\alpha$ and $\Vdash_P$ "for any $\dot{A} \subseteq {}^{<\omega_1}\omega$, we have $\{\alpha < \omega_1 \mid \dot{A} \cap Z_\alpha \in \mathcal{A}_\alpha\}$ is stationary".*

Proof. (Step 1) ([K]) Since P has the c.c.c. and $|P| \leq \omega_1$, $\diamondsuit$ implies $\Vdash_P$ "$\diamondsuit$".

Proof. By $\diamondsuit$, we have a fixed $\langle f_\alpha \mid \alpha < \omega_1\rangle$ such that for any $f : \omega_1 \longrightarrow H_{\omega_1}$,

$$\{\alpha < \omega_1 \mid f\lceil\alpha = f_\alpha\}$$

is stationary. Let us define $\dot{A}_\alpha$ by $\Vdash_P$ "$\dot{A}_\alpha = \{\xi < \alpha \mid f_\alpha(\xi) \cap G_P \neq \emptyset\}$".

We claim $\Vdash_P$ "$\langle \dot{A}_\alpha \mid \alpha < \omega_1\rangle$ is a $\diamondsuit$-sequence". To this end, let $\dot{A}$ be a P-name such that $\Vdash_P$ "$\dot{A} \subseteq \omega_1$". We represent $\dot{A}$ as a sequence $\langle A_\xi \mid \xi < \omega_1\rangle$ such that $\Vdash_P$ "$\xi \in \dot{A}$ iff $A_\xi \cap G_P \neq \emptyset$" ($\xi < \omega_1$). Since P has the c.c.c, we may assume A_ξ is a countable subset of $P \subseteq H_{\omega_1}$ and so $A_\xi \in H_{\omega_1}$. Let $f : \omega_1 \longrightarrow H_{\omega_1}$ be defined by $f(\xi) = A_\xi$. Then

$$S = \{\alpha < \omega_1 \mid f\lceil\alpha = f_\alpha\}$$

is stationary. Since P has the c.c.c, this S remains stationary in V^P. For $\alpha \in S$, we have $\Vdash_P$ "$\dot{A} \cap \alpha = \{\xi < \alpha \mid f(\xi) \cap G_P \neq \emptyset\} = \{\xi < \alpha \mid f_\alpha(\xi) \cap G_P \neq \emptyset\}$". Hence $\Vdash_P$ "$S \subseteq \{\alpha < \omega_1 \mid \dot{A} \cap \alpha = \dot{A}_\alpha\}$" and so we are done.

(Step 2) Fix any Z_α's. Since P has the c.c.c. and is σ-Baire, they have the same properties in V^P. Apply 4.2 Lemma (2) in V^P. In V^P, we have $\langle \dot{B}_\alpha \mid \alpha < \omega_1 \rangle$ such that $\dot{B}_\alpha \subseteq Z_\alpha$ and for any $\dot{B} \subseteq {}^{<\omega_1}\omega$, it holds that $\{\alpha < \omega_1 \mid \dot{B} \cap Z_\alpha = \dot{B}_\alpha\}$ is stationary.

Now in V, let

$$\mathcal{A}_\alpha = \{a \subset Z_\alpha \mid 0 < ||a = \dot{B}_\alpha||\}.$$

Then by c.c.c, $\mathcal{A}_\alpha$ is countable.

We claim $\Vdash_P$ "$\langle \mathcal{A}_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence". To this end, let $\Vdash_P$ "$\dot{B} \subseteq {}^{<\omega_1}\omega$". Since P is σ-Baire, we have $\Vdash_P$ "$\{\alpha < \omega_1 \mid \dot{B} \cap Z_\alpha = \dot{B}_\alpha\} \subseteq \{\alpha < \omega_1 \mid \dot{B} \cap Z_\alpha \in \mathcal{A}_\alpha\}$". Hence we are done.

□

We use this last type of $\diamondsuit$ to construct a step (S, h, T).

§5. **Construction**

This section is a remake of [DJ]. We make use of $\diamondsuit$ rather than starting in the constructible universe. We consider Souslin trees which are normal and subtrees of ${}^{<\omega_1}\omega$.

5.1 Lemma. *($\diamondsuit$) Let $S \subset {}^{<\omega_1}\omega$ be a Souslin tree and $\langle \mathcal{A}_\alpha \mid \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence such that*

- *We have a fixed continuously $\subseteq$-increasing countable subsets $\langle Z_\alpha \mid \alpha < \omega_1 \rangle$ of ${}^{<\omega_1}\omega$ such that*

$$\bigcup \{Z_\alpha \mid \alpha < \omega_1\} = {}^{<\omega_1}\omega.$$

- *$\mathcal{A}_\alpha = \{a^n_\alpha \mid n < \omega\}$ and $a^n_\alpha \subseteq Z_\alpha$.*
- *$\Vdash_S$ "For any $\dot{A} \subseteq {}^{<\omega_1}\omega$, $\{\alpha < \omega_1 \mid \dot{A} \cap Z_\alpha \in \mathcal{A}_\alpha\}$ is stationary".*

Then we have a map $\langle x \mapsto t_x \mid x \in S \rangle$ such that

(1) t_x is a normal subtree of ${}^{<\omega_1}\omega$, the heigth of t_x is $|x|$, the height of x in S. (Height)

(2) If $x_1 <_S x_2$, then t_{x_1} gets end-extended to t_{x_2}. (Coherence)

(3) • $t_\emptyset = \emptyset$ and for $\langle i \rangle \in S$, $t_{\langle i \rangle} = \{\emptyset\}$.
- *If $\langle i, j \rangle \neq \langle i', j' \rangle$ in S, then $(t_{\langle i,j \rangle})_1 \cap (t_{\langle i',j' \rangle})_1 = \emptyset$.*
- *If $|x| \geq 1$ and $x^\frown\langle i \rangle \neq x^\frown\langle i' \rangle$ in S, then $(t_{x^\frown\langle i \rangle})_{|x|} \cap (t_{x^\frown\langle i' \rangle})_{|x|} = \emptyset$. (Forking)*

(4) If $|x| = \alpha$ is a limit and $a^n_\alpha \subseteq t_x$ is a maximal antichain in t_x, then for any $i < \omega$ with $x^\frown\langle i \rangle \in S$, a^n_α remains a maximal antichain in $t_{x^\frown\langle i \rangle}$. (Diamond)

Proof. Let us partition ω into $B_{\langle i,j \rangle}$'s so that each $B_{\langle i,j \rangle}$ is infinite for $i, j \in \omega$. So we have fixed a pairwise disjoint union.

$$\omega = \bigcup \{B_{\langle i,j \rangle} \mid i, j \in \omega\}.$$

We define $\langle x \mapsto t_x \mid x \in S \rangle$ by recursion on $|x|$.

(<u>$|x| = 0$ and $|x| = 0 + 1$</u>)

Let

$$t_\emptyset = \emptyset, \quad t_{\langle i \rangle} = \{\emptyset\} \text{ (for } \langle i \rangle \in S_1).$$

(<u>$|x^\frown\langle i, j \rangle|$ is a successor $+1$</u>)

Suppose t_x and $t_{x^\frown\langle i \rangle}$ have been constructed. Let $|x| = \alpha$. So $x^\frown\langle i \rangle \in S_{\alpha+1}$ holds.

Now let

$$t_{x^\frown\langle i,j \rangle} = t_{x^\frown\langle i \rangle} \cup \{y^\frown\langle k \rangle \mid y \in (t_{x^\frown\langle i \rangle})_\alpha, k \in B_{\langle i,j \rangle}\}.$$

(<u>$|x^\frown\langle i \rangle|$ is a limit $+1$</u>)

Supoose $\alpha = |x|$ is a limit and t_x have been constructed.

For each $i < \omega$ with $x^\frown\langle i\rangle \in S_{\alpha+1}$, we construct

$$B^{x^\frown\langle i\rangle} = \{b_l^{x^\frown\langle i\rangle} \mid l < \omega\}$$

and set

$$t_{x^\frown\langle i\rangle} = t_x \cup B^{x^\frown\langle i\rangle}.$$

Where

- Each member of $B^{x^\frown\langle i\rangle}$ is identified with a path through t_x and in ${}^{\alpha}\omega$.
- If $x^\frown\langle i\rangle \neq x^\frown\langle i'\rangle$ in $S_{\alpha+1}$, then $B^{x^\frown\langle i\rangle} \cap B^{x^\frown\langle i'\rangle} = \emptyset$.
- For each $\sigma \in t_x$, there exists $l < \omega$ such that $\sigma \subset b_l^{x^\frown\langle i\rangle}$. (Dense)
- For each maximal antichain a_α^n in t_x and each $l < \omega$, there exists $\sigma \in a_\alpha^n$ such that $\sigma \subset b_l^{x^\frown\langle i\rangle}$. (Diamond)

Hence a_α^n remains maximal in $t_{x^\frown\langle i\rangle}$.

More details to follow. Let

$$t_x = \{\sigma_l \mid l < \omega\}.$$

We assume that $\mathrm{suc}_S(x) = \{x^\frown\langle i\rangle \mid i < \omega\}$ for a simpler notation. We construct

$$\left\langle B^{x^\frown\langle 0\rangle}\lceil l+1, \cdots, B^{x^\frown\langle l\rangle}\lceil l+1\right\rangle$$

by recursion on l so that

- $\sigma_l \subset b_l^{x^\frown\langle i\rangle}$ and for all $\beta < \alpha$, we demand $b_l^{x^\frown\langle i\rangle}\lceil\beta \in t_x$.
- If $(l, i) \neq (l', i')$, then $b_l^{x^\frown\langle i\rangle} \neq b_{l'}^{x^\frown\langle i'\rangle}$.
- If a_α^n is a maximal antichain in t_x, then there exists $\sigma \in a_\alpha^n$ with $\sigma \subset b_l^{x^\frown\langle i\rangle}$.

$(l = 0)$ Want $\left\langle B^{x^\frown\langle 0\rangle}\lceil 1\right\rangle = \left\langle\{b_0^{x^\frown\langle 0\rangle}\}\right\rangle$ so that

- $\sigma_0 \subset b_0^{x^\frown\langle 0\rangle} \in {}^{\alpha}\omega$.
- For each maximal antichain a_n^α in t_x, there exists $\sigma \in a_\alpha^n$ with $\sigma \subset b_0^{x^\frown\langle 0\rangle}$.

This is carried out by 1.2 Proposition.

$(l+1)$ Suppose we have constructed

$$\left\langle B^{x^\frown\langle 0\rangle}\lceil l+1, \cdots, B^{x^\frown\langle l\rangle}\lceil l+1\right\rangle$$

Want

$$\left\langle B^{x^\frown\langle 0\rangle}\lceil l+2, \cdots, B^{x^\frown\langle l\rangle}\lceil l+2, B^{x^\frown\langle l+1\rangle}\lceil l+2\right\rangle$$

Namely,

$$b_{l+1}^{x^\frown\langle 0\rangle}, \cdots, b_{l+1}^{x^\frown\langle l\rangle} \text{ and } b_0^{x^\frown\langle l+1\rangle}, \cdots, b_{l+1}^{x^\frown\langle l+1\rangle}.$$

This construction is done by a repeated use of 1.2 Proposition.

<u>($|x|$ is a limit)</u>

Suppose α is a limit and $x \in S_\alpha$. Suppose we have constructed $t_{x\lceil\beta}$ for all $\beta < \alpha$. Then let

$$t_x = \bigcup\{t_{x\lceil\beta} \mid \beta < \alpha\}.$$

This completes the construction. It is straightforward to check that this $\langle x \mapsto t_x \mid x \in S\rangle$ works.

□

5.2 Lemma. *($\diamondsuit$) Let $\langle x \mapsto t_x \mid x \in S\rangle$ be as above. Let*

$$T = \bigcup\{t_x \mid x \in S\}.$$

Then T is a normal subtree of ${}^{<\omega_1}\omega$ such that the height of T is ω_1. For each $y \in T$ with $1 \leq |y|$, there exists a unique $x \in S$ such that

- $|x| = |y| + 1$.
- $y \in t_x$.

Proof. We first mention that T is a normal subtree of ${}^{<\omega_1}\omega$ such that the height is ω_1. This is because each t_x is a normal subtree of ${}^{<\omega_1}\omega$ and so their union T is a downward-closed subset and satisfies (ω-many successors). The t_x enjoy (Coherence) and the heights of $t_{x'}$ ($x <_S x'$) get higher, so T is (Dense) and the height of T is ω_1.

Let $y \in T$ with $|y| \geq 1$. We show the existence and uniqueness of x as claimed.

(Existence) Then there exists $x' \in S$ with $y \in t_{x'}$. Notice that $|y| < \mathrm{ht}(t_{x'}) = |x'|$ and so $|y| + 1 \leq |x'|$ holds. Let $x = x'\lceil|y| + 1$. Then t_x is an initial segment of $t_{x'}$ and so $y \in (t_{x'})_{|y|} = (t_x)_{|y|}$. Hence this x works.

(Uniqueness) Suppose $|x| = |x'| = |y| + 1$ and $y \in t_x \cap t_{x'}$. Want to show $x = x'$. Suppose $x \neq x'$ to the contrary. We have two cases.

Case 1. $x\lceil|y| = x'\lceil|y|$: Since we assume $1 \leq |y|$, we may apply (Forking). By (Forking), we have $y \in (t_x)_{|y|} \cap (t_{x'})_{|y|} = \emptyset$. This is a contradiction.

Case 2. $x\lceil|y| \neq x'\lceil|y|$: Let $\beta < |y|$ be such that $x\lceil\beta = x'\lceil\beta$ and $x(\beta) \neq x'(\beta)$.

Subcase 2.1 $\beta \geq 1$: By (Forking), we have $y\lceil\beta \in (t_{x\lceil\beta+1})_\beta \cap (t_{x'\lceil\beta+1})_\beta = \emptyset$. This is a contradiction.

Subcase 2.2 $\beta = 0$: Since $x\lceil 2 \neq x'\lceil 2$, we have $y\lceil 1 \in (t_{x\lceil 2})_1 \cap (t_{x'\lceil 2})_1 = \emptyset$. This is a contradiction.

□

5.3 Lemma. *Define $h : T \longrightarrow S$ by $h(\mathrm{root}_T) = \mathrm{root_S} = \emptyset$ and for $y \in T$ with $|y| \geq 1$, let $h(y) = x$ so that*

- $|h(y)| = |y| + 1$ *for* $|y| \geq 1$.
- $y \in t_{h(y)}$.

Then (S, h, T) is a step such that both S and T satisfy (At most one) .

Proof. Define $S \longleftarrow T : h$ by

$$h(y) = \begin{cases} x, & \text{if } |y| \geq 1 \text{ and } |x| = |y| + 1, y \in t_x \\ \emptyset \ (= \mathrm{root}_S), & \text{if } y = \emptyset \ (= \mathrm{root}_T) \end{cases}$$

(Order-preserving) Let $y_1 \leq_T y_2$. If $y_1 = \emptyset$, then $\emptyset = h(y_1) \leq_S h(y_2)$. If $|y_1| \geq 1$, then let $h(y_1) = x_1$, $h(y_2) = x_2$, $|y_1| = \alpha_1$ and $|y_2| = \alpha_2$. Since $y_1 \leq_T y_2 \in t_{x_2}$, we have $y_1 \in (t_{x_2})_{\alpha_1} = (t_{x_2\lceil\alpha_1+1})_{\alpha_1}$ by (Coherence). Hence $x_1 = x_2\lceil\alpha_1 + 1$. So $x_1 \leq_S x_2$ holds.

(Reduction) Let $h(y) \leq_S x$. Want y' such that $y \leq_T y'$ with $x \leq_S h(y')$. To this end, take x' such that $x <_S x'$ and $|x'| = \alpha' + 1$. Then $t_{h(y)}$ gets end-extended by $t_{x'}$. Let us choose $y' \in (t_{x'})_{\alpha'}$ with $y <_T y'$. Then $x <_S x' = h(y')$ and so this y' works.

(Ahead) This is clear by definition.

We conclude (S, h, T) is a step. We have observed that T is a normal subtree of ${}^{<\omega_1}\omega$. In particular, both S and T satisfy (At most one).

□

5.4 Lemma. *Let (S,h,T) be the step as above. Then $\Vdash_S$ "$T/b_S = \{y \in T \mid h(y) \in b_S\} = \bigcup\{t_x \mid x \in b_S\}$ is a Souslin tree" and so the normal tree $T \equiv S * (T/b_S)$ is a Souslin tree.*

Proof. Let b_S be any generic cofinal path through the Souslin tree S over V. We argue in the generic extension $V[S]$ for the rest.

We first observe $T/b_S = \bigcup\{t_x \mid x \in b_S\}$. Let us simply denote $T' = \bigcup\{t_x \mid x \in b_S\}$. Let $y \in T/b_S$. Then $y \in T$ with $h(y) = x \in b_S$. If $y = \emptyset$, then $y \in T'$. If $|y| \geq 1$, then $y \in t_x$ with $x \in b_S$. Hence $y \in T'$.

Conversely, let $y \in t_x$ for some $x \in b_S$. If $y = \emptyset$, then $h(y) = \emptyset \in b_S$. Hence $y \in T/b_S$. If $|y| \geq 1$, then we may assume that $|x| = |y| + 1$ by considering an initial segment of x. This is possible, since b_S is downward-closed. Then $h(y) = x \in b_S$. So $y \in T/b_S$.

(C.C.C.) We show that $T/b_S = T'$ is a Souslin tree. However, it is straightforward to see that T' is a subtree of ${}^{<\omega_1}\omega$ such that T' satisfies (Dense), (ω-many successors), (At most one) and is of height ω_1.

Let A be any maximal antichain of T'. We want to show A is countable. Since $\langle \mathcal{A}_\alpha \mid \alpha < \omega_1\rangle$ is the specified $\diamondsuit$-sequence, we have

$$E = \{\alpha < \omega_1 \mid A \cap Z_\alpha \in \mathcal{A}_\alpha\}$$

is stationary. For all limit α, since $T'\lceil\alpha+1 = t_{b_S(\alpha+1)}$, we have T'_α are all countable. Hence

$$C = \{\alpha < \omega_1 \mid A \cap (T'\lceil\alpha) \text{ is maximal in } T'\lceil\alpha\}$$

$$D = \{\alpha < \omega_1 \mid T' \cap Z_\alpha = T'\lceil\alpha\}$$

are clubs.

Let α be a limit with $\alpha \in E \cap C \cap D$. Then there exists n such that $a^n_\alpha = A \cap Z_\alpha = A \cap T'\lceil\alpha \subseteq t_{b_S(\alpha)}$ is a maximal antichain in $T'\lceil\alpha = t_{b_S(\alpha)}$. By construction a^n_α remains maximal in $t_{b_S(\alpha+1)} = T'\lceil\alpha+1$ regard less of the actual value $b_S(\alpha+1)(\alpha) < \omega$. Hence so does in the whole T'. Therefore $A = A \cap Z_\alpha$ is countable.

Since S has the c.c.c. and $\Vdash_S$"T/b_S has the c.c.c.", so does $T \equiv S * T/b_S$ by 1.7 Proposition. Hence T is a Souslin tree.

□

Here is our main observation.

5.5 Theorem. *($\diamondsuit$) There exists an ω-stage iteration $\langle P_n, \dot{Q}_n \mid n < \omega\rangle$ such that $\Vdash_{P_n}$"$\dot{Q}_n$ is a Souslin tree" and so $\Vdash_{P_n}$"$\dot{Q}_n$ has the c.c.c. and is σ-Baire" and for any $\langle G_n \mid n < \omega\rangle$ such that G_n is P_n-generic over V and $G_{n+1}\lceil n = G_n$, we have*

$$V\big[\langle G_n \mid n < \omega\rangle\big] = V\big[\langle b_n(1) \mid n = 1, 2, \cdots\rangle\big].$$

where b_{n+1} is the $\dot{Q}_n[G_n]$-generic cofinal path over $V[G_n]$ induced by G_{n+1}. In particular, if P_ω is any limit of the P_n, then P_ω is never σ-Baire.

Proof. Construct T_n, h_n $(n = 1, 2, \cdots)$ by recursion so that (T_n, h_n, T_{n+1}) are steps and T_n are Souslin trees. Then by 3.2 Lemma, get $\langle P_n, \dot{Q}_n \mid n < \omega\rangle$ such that if $n \geq 1$, then $\dot{Q}_n$ forces a generic cofinal path b_{n+1} through T_{n+1}/b_n over $V[b_n] = V[G_n]$ and $\dot{Q}_0$ does b_1 through T_1 over $V = V[G_0]$.

□

§6. A Strong Form of $\psi_{\rm AC}$

The combinatorial principle $\psi_{\rm AC}$ related to the size of the reals is found by [W]. We reformulate a stronger $\psi^+_{\rm AC}$ from [M].

6.1 Definition. $\psi^+_{\rm AC}$ holds, if for any sequence of stationary subsets $\langle E_\alpha \mid \alpha < \omega_1\rangle$ of ω_1, there exists γ with $\omega_1 < \gamma < \omega_2$ and a continuously $\subseteq$-increasing sequence of countable subsets $\langle X_\alpha \mid \alpha < \omega_1\rangle$ of γ such that $\bigcup\{X_\alpha \mid \alpha < \omega_1\} = \gamma$ and for all $\alpha < \omega_1$, $\text{o.t.}(X_\alpha) \in E_\alpha$ hold.

To force $\psi^+_{\rm AC}$, we may iteratively force with the following notions of semiproper forcing.

6.2 Definition. Let κ be a measurable cardinal and $\langle E_\alpha \mid \alpha < \omega_1\rangle$ be a sequence of stationary subsets of ω_1. Let $p = \langle X^p_\alpha \mid \alpha \leq \alpha^p\rangle \in P\big(\kappa, \langle E_\alpha \mid \alpha < \omega_1\rangle\big)$, if $p = \emptyset$ and otherwise

(1) $\alpha^p < \omega_1$.

(2) The X^p_α are continuously $\subseteq$-increasing countable subsets of κ.

(3) $\text{o.t.}(X^p_\alpha) \in E_\alpha$.

For $p, q \in P\big(\kappa, \langle E_\alpha \mid \alpha < \omega_1\rangle\big)$, let $q \leq p$, if q end-extends p.

6.3 Proposition. *Let $S = P\big(\kappa, \langle E_\alpha \mid \alpha < \omega_1\rangle\big)$ be as above. Then $(S, \subset)$ is a tree which satisfies (Root), (Dense), (At most one) and is of height ω_1.*

6.4 Question. Formulate a general theory which would encompass the iteration of Souslin trees in this note and the iteration for $\psi^+_{\rm AC}$ as outlined above. Then apply your theory to iterated forcing constructions for the saturation of the non-stationary ideal on ω_1. Do you see any real coded ?

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