Formulas with only one variable in Lewis logic **S4** [1]

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Abstract. Here we treat modal formulas with only one propositional variable p in Lewis logic **S4**. The quotient set of the set of formulas modulo the provability of **S4** is Boolean with respect to the derivation of **S4** (cf. Chagrov and Zakharyaschev [CZ97]). We give an inductive construction of the representatives of the quotient set of the set of formulas with only one variable p and with a finite number of occurrences of $\Box$.

1 Preliminaries

We use lower case Latin letters p, q for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\bot$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). By $\mathbf{S}(p)$, we mean the set of formulas constructed from p by using $\wedge$, $\vee$, $\supset$ and $\Box$. For a finite set $\mathbf{S}$, $\#(\mathbf{S})$ denotes the number of elements in $\mathbf{S}$. Let **ENU** be an enumeration of the formulas in $\mathbf{S}(p)$. For a non-empty finite set $\mathbf{S}$ of formulas, the expressions

$$\bigwedge \mathbf{S} \qquad \text{and} \qquad \bigvee \mathbf{S}$$

denote the formulas

$$(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_n) \qquad \text{and} \qquad (\cdots((A_1 \vee A_2) \vee A_3) \cdots \vee A_n),$$

respectively, where $\{A_1, \cdots, A_n\} = \mathbf{S}$ and A_i occurs earlier than A_{i+1} in **ENU**. Also the expressions

$$\bigwedge \emptyset \qquad \text{and} \qquad \bigvee \emptyset$$

denote the formulas $p \supset p$ and $\Box p$, respectively.

By **S4**, we mean the smallest set of formulas containing all the tautologies and the axioms
$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$,
$T : \Box p \supset p$,
$4 : \Box p \supset \Box\Box p$
and closed under modus ponens, substitution and necessitation.

We introduce a sequent system for **S4** following Ohnishi and Matsumoto [OM57]. We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of formulas. The expressions $\Box\Gamma$ and $\Gamma^{\Box}$ denote the sets $\{\Box A \mid A \in \Gamma\}$ and $\{\Box A \mid \Box A \in \Gamma\}$, respectively. By a sequent, we mean the expression $(\Gamma \to \Delta)$. We often write $\Gamma \to \Delta$ instead of the expression with the parenthesis. By **SEQ**, we mean the set of the sequents. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

We put

$$f(\Gamma \to \Delta) = \begin{cases} \bigwedge \Gamma \supset \bigvee \Delta & \text{if } \Gamma \neq \emptyset \\ \bigvee \Delta & \text{if } \Gamma = \emptyset, \end{cases}$$

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and for a set $\mathcal{S}$ of sequents,

$$f(\mathcal{S}) = \{f(X) \mid X \in \mathcal{S}\}.$$

By **GS4**, we mean the system defined by the following axioms and inference rules in the usual way.

Axioms of S4:

$$A \to A$$

$$\bot \to$$

Inference rules of S4:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to w)$$

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}(cut)$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Pi \to \Lambda}{A \supset B, \Gamma, \Pi \to \Delta, \Lambda}(\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to\supset)$$

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box)$$

Lemma 1.1([OM57]).
(1) $\Gamma \to \Delta \in$ **GS4** *if and only if* $f(\Gamma \to \Delta) \in$ **S4**.
(2) *If* $\Gamma \to \Delta \in$ **GS4**, *then there exists a cut-free proof figure for* $\Gamma \to \Delta$ *in* **GS4** .

By the lemma above, we can identify **GS4** with **S4**. So, if there is no confusion, we use the sequent system **GS4** instead of **S4**.

Definition 1.2. The depth $d(A)$ of a formula $A \in \mathbf{S}(p)$ is defined inductively as follows:
(1) $d(p) = 0$,
(2) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
(3) $d(\Box B) = d(B) + 1$.

We put $\mathbf{S}^n(p) = \{A \in \mathbf{S}(p) \mid d(A) \leq n\}$.

2 Main results

For formulas A and B, we use the expression $A \equiv B$ instead of $(A \supset B) \wedge (B \supset A) \in$ **S4**. We note that $\equiv$ is an equivalence relation on a set **S** of formulas. We write $[A] \leq [B]$ if there exist $A' \in [A]$ and $B' \in [B]$ such that $A' \supset B' \in$ **S4**. Our main purpose is to give a concrete representative of each equivalence class of $\mathbf{S}^n(p)$ in an inductive way and elucidate the structure $\langle \mathbf{S}^n(p), \leq \rangle$.

Definition 2.1.

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma, \qquad \mathbf{suc}(\Gamma \to \Delta) = \Delta, \qquad \mathbf{R}(\Gamma \to \Delta) = \{\Sigma \to \Lambda \mid \Gamma^{\Box} \cap \Lambda^{\Box} \neq \emptyset\}.$$

Definition 2.2. The sets $\mathcal{S}_n, \mathcal{G}_n, \mathcal{G}_n^*$ $(n = 0, 1, 2. \cdots)$ of sequents, and the mappings
$\mathcal{S}^+ : \bigcup_{k=1}^{\infty} \mathcal{S}_k \to \mathbf{SEQ}$,
$\mathcal{T} : \bigcup_{k=1}^{\infty} \mathcal{S}_k \to \mathbf{SEQ}$,
$\mathcal{S} : \bigcup_{k=1}^{\infty} \mathcal{S}_k \to \mathbf{SEQ}$,

$^* : \bigcup_{k=2}^{\infty} \bigcup_{X \in \mathcal{S}_k} \mathcal{S}^+(X) \to \mathbf{SEQ}$
are defined inductively as follows:

$\mathcal{S}_0 = \{\to p\}$, $\mathcal{G}_0 = \mathcal{G}_0^* = \emptyset$,

$\mathcal{S}_1 = \{(p \to \Box p), (\to p)\}$, $\mathcal{G}_1 = \mathcal{G}_1^* = \emptyset$,

Suppose $n \geq 1$. For $X \in \mathcal{S}_n$,

$\mathcal{S}^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = f(\mathcal{S}_n), \Gamma \cap \Delta = \emptyset, f(X) \in \Delta\}$,

$\mathcal{T}(X) = \{Y \in \mathcal{S}^+(X) \mid Y \in \mathbf{S4}\}$,

$\mathcal{S}(X) = \mathcal{S}^+(X) - \mathcal{T}(X)$,

for $Z \in \mathcal{S}^+(X)$, $Z^* = \begin{cases} (\Box\Gamma, \mathbf{ant}(Y) \to \mathbf{suc}(Y), \Box Y, \Box X & \text{if } Z = (\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box X, \Box Y) \\ Z & \text{otherwise.} \end{cases}$

And

$\mathcal{S}_{n+1} = \bigcup_{X \in \mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*)} \mathcal{S}(X)$,

$\mathcal{G}_{n+1} = \{X \in \mathcal{S}_{n+1} \mid \mathbf{R}(X) \cap \mathcal{S}_{n+1} = \mathcal{S}_{n+1} - \{X\}\}$,

$\mathcal{G}_{n+1}^* = \{X \in \mathcal{S}_{n+1} \mid X \neq X^*, X^* \in \mathcal{S}_{n+1}, \mathbf{R}(X) \cap \mathcal{S}_{n+1} = \mathcal{S}_{n+1} - \{X, X^*\}\}$.

Definition 2.3.

$$\mathbf{G}_n = \mathcal{S}_n \cup \bigcup_{k=0}^{n-1} (\mathcal{G}_k \cup \mathcal{G}_k^*)$$

The main result in this paper is

Theorem 2.4.
(1) $\mathbf{S}^n(p)/\equiv = \{[\bigwedge f(\mathbf{S})] \mid \mathbf{S} \subseteq \mathbf{G}_n\}$.
(2) *For subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}_n$*,*

(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge f(\mathbf{S}_2)] \leq [\bigwedge f(\mathbf{S}_1)]$,

(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge f(\mathbf{S}_1)] = [\bigwedge f(\mathbf{S}_2)]$.

The theorem above can be shown by using the following Main lemma.

Main lemma 2.5.
(1) For any $X, Y \in \mathbf{G}_n$, $X \neq Y$ implies $f(X) \vee f(Y) \in \mathbf{S4}$.
(2) For any $A \in \mathbf{S}^n(p)$, there exists a subset $\mathbf{S} \subseteq \mathbf{G}_n$ such that $A \equiv \bigwedge f(\mathbf{S})$.
(3) For any $X \in \mathbf{G}_n$, $X \notin \mathbf{S4}$.

Theorem 2.4 (1) follows from Main lemma 2.5(2). We show Theorem 2.4(2) using Main lemma 2.5. Suppose that $\mathbf{S}_1 \not\subseteq \mathbf{S}_2$. Then there exists a sequent $X \in \mathbf{S}_1 - \mathbf{S}_2$. By Main lemma 2.5(1), $f(X) \vee \bigwedge f(\mathbf{S}_2) \in \mathbf{S4}$. By Main lemma 2.5(3), $f(X) \notin \mathbf{S4}$. So, By the following figure, we note that $\bigwedge f(\mathbf{S}_2) \to \bigwedge f(\mathbf{S}_1) \notin \mathbf{S4}$.

$$\dfrac{\to f(X) \vee \bigwedge f(\mathbf{S}_2) \qquad \dfrac{f(X) \to f(X) \qquad \dfrac{\bigwedge f(\mathbf{S}_2) \to \bigwedge f(\mathbf{S}_1) \qquad \bigwedge f(\mathbf{S}_1) \to f(X)}{\bigwedge f(\mathbf{S}_2) \to f(X)}}{f(X) \vee \bigwedge f(\mathbf{S}_2) \to f(X)}}{\to f(X)}$$

From the definition, we immediately have Main Lemma 2.5(3). Main Lemma 2.5(2) will be proved in the next section. Here we show Main Lemma 2.5(1).

Proof of Main Lemma 2.5(1). We use an induction on n.

Basis($n \in \{0, 1\}$): If $n = 0$, then the theorem is clear since $\mathbf{G}_n$ has only one element. If $n = 1$, then $\mathbf{G}_n = \{(p \to \Box p), (\to p)\}$, and so the theorem follows from $\to p \supset \Box p, p \in \mathbf{S4}$.

Induction step($n \geq 2$): We divide into the following two cases.

The case that $\{X, Y\} \subseteq \mathcal{S}_n$. There exist $X_0, Y_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)$ such that $X \in \mathcal{S}(X_0)$ and $Y \in \mathcal{S}(Y_0)$. So, there exist sets $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ of formulas such that

(1) $X = (\Box\Gamma_1, \mathbf{ant}(X_0) \to \mathbf{suc}(X_0), \Box\Delta_1)$,
(2) $Y = (\Box\Gamma_2, \mathbf{ant}(Y_0) \to \mathbf{suc}(Y_0), \Box\Delta_2)$,
(3) $\Gamma_i \cup \Delta_i = f(\mathcal{S}_n)$ $(i = 1, 2)$,
(4) $\Gamma_i \cap \Delta_i = \emptyset$ $(i = 1, 2)$,
(5) $f(X_0) \in \Delta_1, f(Y_0) \in \Delta_2$.

If $X_0 \neq Y_0$, then by the induction hypothesis, $f(X_0) \vee f(Y_0) \in \mathbf{S4}$, and so, we obtain the theorem. Suppose that $X_0 = Y_0$. Then by $X \neq Y$, we have either $\Gamma_1 \neq \Gamma_2$ or $\Delta_1 \neq \Delta_2$, and using (3) and (4), we have both. Without loss of generality, we can suppose that $\Gamma_1 \not\subseteq \Gamma_2$. So, there exists a formula $A \in \Gamma_1 - \Gamma_2$, that is $A \in \Gamma_1 \cap \Delta_2$. So, we have $\Box\Gamma_1 \to \Box\Delta_2 \in \mathbf{S4}$. Hence $\to f(X), f(Y) \in \mathbf{S4}$(see the following figure).

$$\dfrac{\dfrac{\to \Box\Gamma_1, f(X) \quad \Box\Gamma_1 \to, \Box\Delta_2}{\to f(X), \Box\Delta_2} \quad \Box\Delta_2 \to f(Y)}{\to f(X), f(Y)}$$

The case that $\{X, Y\} \not\subseteq \mathcal{S}_n$. There exists $Z \in \{X, Y\} - \mathcal{S}_n$. Without loss of generality, we can suppose that $Z = Y \notin \mathcal{S}_n$.

If $X \notin \mathcal{S}_n$, then $X, Y \in \bigcup_{k=0}^{n-1}(\mathcal{G}_k \cup \mathcal{G}_k^*)$, and so, $X, Y \in \mathbf{G}_{n-1}$. By the induction hypothesis, we obtain the theorem.

Suppose that $X \in \mathcal{S}_n$. Then there exist $X_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)$ such that $X \in \mathcal{S}(X_0)$. We note that $Y \neq X_0$. By the induction hypothesis, we have $f(X_0) \vee f(Y) \in \mathbf{S4}$. Hence $\to f(X), f(Y) \in \mathbf{S4}$(see the following figure).

$$\dfrac{\to f(X_0) \vee f(Y) \quad \dfrac{f(X_0) \to f(X), f(Y) \quad f(Y) \to f(X), f(Y)}{f(X_0) \vee f(Y) \to f(X), f(Y)}}{\to f(X), f(Y)}$$

$\dashv$

3 Proof of Main lemma 2.5(2)

To prove Main lemma 2.5(2), we need some lemmas.

Lemma 3.1. For any subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_n$,
(1) $\bigwedge \mathbf{S}_1 \wedge \bigwedge \mathbf{S}_2 \equiv \bigwedge(\mathbf{S}_1 \cup \mathbf{S}_2)$,
(2) $\bigwedge \mathbf{S}_1 \vee \bigwedge \mathbf{S}_2 \equiv \bigwedge(\mathbf{S}_1 \cap \mathbf{S}_2)$.

Proof. (1) is clear. We show (2). Let A be in $\mathbf{S}_1$. Then by Main lemma 2.5(1), we have $A \vee B \in \mathbf{S4}$ for any $B \in \mathbf{S}_2 - \{A\}$. So, if $A \in \mathbf{S}_2$, then

$$\begin{aligned} A \vee \bigwedge \mathbf{S}_2 &\equiv (A \vee A) \wedge (A \vee \bigwedge(\mathbf{S}_2 - \{A\}) \\ &\equiv A \wedge (A \vee \bigwedge(\mathbf{S}_2 - \{A\}) \\ &\equiv A; \end{aligned}$$

if not,

$$\begin{aligned} A \vee \bigwedge \mathbf{S}_2 &\equiv \bigwedge\{A \vee B \mid B \in \mathbf{S}_2\} \\ &\equiv p \supset p. \end{aligned}$$

Hence

$$\begin{aligned} \bigwedge \mathbf{S}_1 \vee \bigwedge \mathbf{S}_2 &\equiv \bigwedge\{A \vee \bigwedge \mathbf{S}_2 \mid A \in \mathbf{S}_1\} \\ &\equiv \bigwedge\{A \vee \bigwedge \mathbf{S}_2 \mid A \in \mathbf{S}_1 - \mathbf{S}_2\} \wedge \bigwedge\{A \vee \bigwedge \mathbf{S}_2 \mid A \in \mathbf{S}_1 \cap \mathbf{S}_2\} \\ &\equiv (p \supset p) \wedge \bigwedge\{A \vee (A \wedge \bigwedge(\mathbf{S}_2 - \{A\}) \mid A \in \mathbf{S}_1 \cap \mathbf{S}_2\} \\ &\equiv \bigwedge(\mathbf{S}_1 \cap \mathbf{S}_2). \end{aligned}$$

$\dashv$

Lemma 3.2. Let A be a formula and let $\Sigma, \Gamma, \Gamma_1, \Delta, \Delta_1$ be set of formulas. Then for any subset $\Sigma' \subseteq \Sigma$,

$$\Box\Sigma', \{f(\Box\Gamma, \Box\Phi, \Gamma_1 \to \Delta_1, \Box\Psi, \Box\Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

Proof. We put
$$\mathbf{S} = \{f(\Box\Gamma, \Box\Phi, \Gamma_1 \to \Delta_1, \Box\Psi, \Box\Delta) \mid \Phi \cup \Psi = \Sigma, \Phi \cap \Psi = \emptyset\}$$
and prove
$$\Box\Sigma', \mathbf{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$

We use an induction on $\#(\Sigma - \Sigma')$.

Basis($\Sigma' = \Sigma$): We note that
$$f(\Box\Gamma, \Box\Sigma, \Gamma_1 \to \Delta_1, \Box\Delta) \in \mathbf{S}$$
and
$$\Box\Sigma, f(\Box\Gamma, \Box\Sigma, \Gamma_1 \to \Delta_1, \Box\Delta), \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$
Using weakening rules, we obtain the lemma.

Induction step($\Sigma' \neq \Sigma$)): By the induction hypothesis, for any $A \in \Sigma - \Sigma'$,
$$\Box(\Sigma' \cup \{A\}), \mathbf{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$
Using $(\vee \to)$, possibly several times,
$$\Box\Sigma', \bigvee(\Box(\Sigma - \Sigma')), \mathbf{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$
Using $(\vee \to)$, possibly several times,
$$\Box\Sigma', \bigvee(\Delta_1 \cup \Box\Delta \cup \Box(\Sigma - \Sigma')), \mathbf{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$
Using $(\supset\to)$, possibly several times,
$$\Box\Sigma', f(\Box\Gamma, \Gamma_1, \Box\Sigma' \to \Delta_1, \Box\Delta, \Box(\Sigma - \Sigma')), \mathbf{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$
We note that
$$f(\Box\Gamma, \Gamma_1, \Box\Sigma' \to \Delta_1, \Box\Delta, \Box(\Sigma - \Sigma')) \in \mathbf{S},$$
and so,
$$\Box\Sigma', \mathbf{S}, \Box\Gamma, \Gamma_1 \to \Delta_1, \Box\Delta \in \mathbf{S4}.$$
⊣

Corollary 3.3. Let X be a sequent in $\mathcal{S}_n$ $(n \geq 1)$. Then
(1) $f(\mathcal{S}^+(X)) \to f(X) \in \mathbf{S4}$,
(2) $\{f(Z) \mid Z \in \mathcal{S}^+(X), \Box f(Y) \in \mathbf{suc}(Z)\} \to f(X), \Box f(Y) \in \mathbf{S4}$.

Lemma 3.4. Let X and Y be sequents in $\mathcal{S}_n$ $(n \geq 1)$. Then
$$Y \in \mathbf{R}(X) \text{ implies } (\to f(X), \Box f(Y)) \in \mathbf{S4}.$$

Proof. Since $Y \in \mathbf{R}(X)$, we have $(\mathbf{ant}(X))^{\Box} \cap (\mathbf{suc}(Y))^{\Box} \neq \emptyset$. So, there exists a formula $\Box A \in \mathbf{ant}(X))^{\Box} \cap (\mathbf{suc}(Y))^{\Box}$. So,
$$\Box A \to \mathbf{suc}(Y) \in \mathbf{S4}.$$
Hence
$$\Box A \to f(Y) \in \mathbf{S4}.$$
Using $(\to \Box)$,
$$\Box A \to \Box f(Y) \in \mathbf{S4}.$$
Using weakening rule, possibly several times,
$$\mathbf{ant}(X) \to \mathbf{suc}(X), \Box f(Y) \in \mathbf{S4}.$$
Hence
$$\to f(X), \Box f(Y) \in \mathbf{S4}.$$

$\dashv$

Lemma 3.5. Let X be a sequent in $\mathcal{G}_n^*$ $(n \geq 1)$. Then

$$\Box f(X^*) \to f(X) \in \mathbf{S4}.$$

Proof. We note $\mathcal{G}_1^* = \mathcal{G}_2^* = \emptyset$. So, we can assume that $n \geq 3$. By $X \in \mathcal{G}_n^*$, we have $X^* \in \mathcal{S}_n$ and $X \neq X^*$. Also there exist $X_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)$ and $Y_0 \in \mathcal{S}_{n-1}$ such that

$$X = (\Box f(\mathcal{S}_{n-1} - \{X_0, Y_0\}), \mathbf{ant}(X_0) \to \mathbf{suc}(X_0), \Box f(X_0), \Box f(Y_0)).$$

By $Y_0 \in \mathcal{S}_{n-1}$ and Corollary 3.3,

$$f(\mathcal{S}^+(Y_0)) \to f(Y_0) \in \mathbf{S4}.$$

Hence

$$f(\mathcal{S}(Y_0)) \to f(Y_0) \in \mathbf{S4}.$$

By $X^* \neq X$, we have $X_0 \neq Y_0$, and so, $\mathcal{S}(Y_0) \subseteq \mathcal{S}_n - \{X\}$. Hence

$$\Box f(\mathcal{S}_n - \{X\}) \to f(Y_0) \in \mathbf{S4}.$$

Hence

$$\Box f(\mathcal{S}_n - \{X\}) \to \Box f(Y_0) \in \mathbf{S4}.$$

Since $\Box f(Y_0) \to f(X) \in \mathbf{S4}$,

$$\Box f(\mathcal{S}_n - \{X\}) \to f(X) \in \mathbf{S4},$$

that is,

$$\Box f(\mathcal{S}_n - \{X, X^*\}), \Box f(X^*) \to f(X) \in \mathbf{S4}.$$

Also by $X \in \mathcal{G}_n^*$, we have $\mathbf{R}(X) \cap \mathcal{S}_n = \mathcal{S}_n - \{X, X^*\}$, and so,

$$\Box f(\mathcal{S}_n \cap \mathbf{R}(X)), \Box f(X^*) \to f(X) \in \mathbf{S4}.$$

Hence

$$\{f(X) \vee \Box f(Y) \mid Y \in \mathcal{S}_n \cap \mathbf{R}(X)\}, \Box f(X^*) \to f(X) \in \mathbf{S4}.$$

By Lemma 3.4, $f(X) \vee \Box f(Y) \in \mathbf{S4}$ for any $Y \in \mathcal{S}_n \cap \mathbf{R}(X)$, and so,

$$\Box f(X^*) \to f(X) \in \mathbf{S4}.$$

$\dashv$

Lemma 3.6. Let X be a sequent in $\mathcal{G}_n^*$ $(n \geq 1)$. Then

$$X^* \in \mathcal{G}_n^*.$$

Proof. We note $\mathcal{G}_1^* = \mathcal{G}_2^* = \emptyset$. So, we can assume that $n \geq 3$. By $X \in \mathcal{G}_n^*$, there exist a sequent $X_0 \in \mathcal{S}_{n-1}$ such that $X \in \mathcal{S}(X_0)$. It is not hard to see that $X^* \in \mathcal{S}^+(Y_0)$ for some $Y_0 \in \mathcal{S}_{n-1}$.

By Lemma 3.5,

$$\Box f(X^*) \to f(X) \in \mathbf{S4}.$$

Since $X \notin \mathbf{S4}$,

$$\to \Box f(X^*) \notin \mathbf{S4},$$

and so,

$$\to f(X^*) \notin \mathbf{S4}.$$

Hence $X^* \in \mathcal{S}(Y_0)$. $\dashv$

For $X \in \mathcal{S}_n$ $(n \geq 1)$, we put

$$F(X) = \begin{cases} p \supset p & \text{if } X \notin \mathcal{G}_n \cup \mathcal{G}_n^* \\ f(X) & \text{if } X \in \mathcal{G}_n \\ f(X) \wedge f(X^*) & \text{if } X \in \mathcal{G}_n^* \end{cases}$$

Lemma 3.7. Let X be a sequent in $\mathcal{S}_n$ $(n \geq 1)$ and let Σ be a subset of $(\mathbf{ant}(X))^\Box$.

$$\Sigma, F(X), \{f(Y) \mid Y \in \mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*), (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \to \Box f(X) \in \mathbf{S4}.$$

Proof. We use an induction on $\omega n + \#((\mathbf{ant}(X))^\Box - \Sigma)$.
Basis$(n = 1)$: We note that

$\mathcal{S}_1 = \{(p \to \Box p), (\to p)\}, \qquad \mathcal{G}_1 = \mathcal{G}_1^* = \emptyset,$
$\{f(Y) \mid Y \in \mathcal{S}_1, (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(\to p))^\Box\} = \{f(\to p), f(p \to \Box p)\},$
$\{f(Y) \mid Y \in \mathcal{S}_1, (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(p \to \Box p))^\Box\} = \{f(\to p), f(p \to \Box p)\}.$

It is not hard to see that

$$f(\to p), f(p \to \Box p) \to \Box f(\to p) \in \mathbf{S4},$$

$$f(\to p), f(p \to \Box p) \to \Box f(p \to \Box p) \in \mathbf{S4}.$$

Induction step$(n \geq 2)$: We put

$$\Phi = \{f(Y) \mid Y \in \mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*), (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\}.$$

By $n \geq 2$, there exists a sequent $X_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)$ such that $X \in \mathcal{S}(X_0)$. By the induction hypothesis,

$$\{f(Y_0) \mid Y_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*), (\mathbf{ant}(Y_0))^\Box \subseteq (\mathbf{ant}(X_0))^\Box\} \to \Box f(X_0) \in \mathbf{S4}.$$

Since $\Box f(X_0) \to \Box f(X) \in \mathbf{S4}$,

$$\{f(Y_0) \mid Y_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*), (\mathbf{ant}(Y_0))^\Box \subseteq (\mathbf{ant}(X_0))^\Box\} \to \Box f(X) \in \mathbf{S4}.$$

So,

$$\Sigma, \Phi, \{f(Y_0) \mid Y_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)\} \to \Box f(X) \in \mathbf{S4}.$$

On the other hand, by the induction hypothesis, for any $A \in (\Box f(\mathcal{S}_{n-1}) - \Sigma)$, $(\Sigma, F(X), \Phi, A \to \Box f(X)) \in \mathbf{S4}$, and so,

$$\Sigma, F(X), \Phi, \bigvee(\Box f(\mathcal{S}_{n-1}) - \Sigma) \to \Box f(X) \in \mathbf{S4}.$$

Using $(\vee \to)$,

$$\Sigma, F(X), \Phi, \{f(Y_0) \mid Y_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)\} \vee \bigvee(\Box f(\mathcal{S}_{n-1}) - \Sigma) \to \Box f(X) \in \mathbf{S4}.$$

So,

$$\Sigma, F(X), \Phi, \{f(Y_0) \vee \bigvee(\Box f(\mathcal{S}_{n-1}) - \Sigma) \mid Y_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)\} \to \Box f(X) \in \mathbf{S4}.$$

So,

$$\Sigma, F(X), \Phi, \{f(Y_0) \vee \bigvee(\mathbf{suc}(Y))^\Box) \mid Y_0 \in \mathcal{S}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*), Y \in \mathcal{S}(Y_0), (\mathbf{ant}(Y))^\Box = \Sigma\} \to \Box f(X) \in \mathbf{S4}.$$

For any Y, $(\mathbf{ant}(Y))^\Box = \Sigma$ implies $\Sigma \to \bigwedge \mathbf{ant}(Y) \in \mathbf{S4}$, so using $(\supset\to)$,

$$\Sigma, F(X), \Phi, \{f(Y) \mid Y \in \mathcal{S}_n, (\mathbf{ant}(Y))^\Box = \Sigma\} \to \Box f(X) \in \mathbf{S4}.$$

Hence

$$\Sigma, F(X), \Phi, \{f(Y) \mid Y \in \mathcal{S}_n, (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \to \Box f(X) \in \mathbf{S4}.$$

Hence

$$\Sigma, F(X), \Phi, \{f(Y) \mid Y \in \mathcal{G}_n \cup \mathcal{G}_n^*, (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \to \Box f(X) \in \mathbf{S4}. \cdots\cdots (*)$$

We show the case that $X \notin \mathcal{G}_n \cup \mathcal{G}_n^*$. By $(*)$,

$$\Sigma, \Phi, \{f(Y) \vee \Box f(X) \mid Y \in \mathcal{G}_n \cup \mathcal{G}_n^*, (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \to \Box f(X) \in \mathbf{S4}.$$

Let Y be a sequent in $\mathcal{G}_n$. Then $\mathbf{R}(Y) \cap \mathcal{S}_n = \mathcal{S}_n - \{Y\}$. Since $X \notin \mathcal{G}_n \cup \mathcal{G}_n^*$, we have $X \neq Y$, and so, $X \in \mathbf{R}(Y)$. Using Lemma 3.4, we have $f(Y) \vee \Box f(X) \in \mathbf{S4}$. Hence

$$\Sigma, \Phi, \{f(Y) \vee \Box f(X) \mid Y \in \mathcal{G}_n^*, (\mathbf{ant}(Y))^\Box = (\mathbf{ant}(X))^\Box\} \to \Box f(X) \in \mathbf{S4}.$$

Let Y be a sequent in $\mathcal{G}_n^*$. Then $\mathbf{R}(Y) \cap \mathcal{S}_n = \mathcal{S}_n - \{Y, Y^*\}$. By Lemma 3.6, $Y^* \in \mathcal{G}_n^*$. So, since $X \notin \mathcal{G}_n \cup \mathcal{G}_n^*$, we have $X \neq Y$ and $X \neq Y^*$, and so, $X \in \mathbf{R}(Y)$. Using Lemma 3.4, we have $f(Y) \vee \Box f(X) \in \mathbf{S4}$. Hence

$$\Sigma, \Phi \to \Box f(X) \in \mathbf{S4}.$$

We show the case that $X \in \mathcal{G}_n$. By $(*)$, it is sufficient to show

$$\{f(Y) \mid Y \in \mathcal{G}_n \cup \mathcal{G}_n^*, (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \subseteq \{f(X)\}.$$

Suppose that $Y \in \mathcal{G}_n \cup \mathcal{G}_n^*$ and $(\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box$. Then $(\mathbf{ant}(Y))^\Box \cap (\mathbf{suc}(X))^\Box = \emptyset$. So, $X \notin \mathbf{R}(Y)$. On the other hand, by $Y \in \mathcal{G}_n \cup \mathcal{G}_n^*$, $\mathcal{S}_n - \mathbf{R}(Y) \subseteq \{Y, Y^*\}$. So, $X \in \{Y, Y^*\}$. If $X = Y$, then we obtain the lemma.

Suppose $X = Y^*$. Then $X^* = (Y^*)^* = Y$. By $X \in \mathcal{G}_n$, it is not hard to see $X^* \notin \mathbf{R}(X)$, so $X^* = X$ or $X^* \notin \mathcal{S}_n$. By $Y = X^* \in \mathcal{S}_n$, we have $X^* = X$, and so, we obtain the lemma.

The case that $X \in \mathcal{G}_n^*$ can be shown similarly to the proof of the above case, that is, we can show

$$\{f(Y) \mid Y \in \mathcal{G}_n \cup \mathcal{G}_n^*, (\mathbf{ant}(Y))^\Box \subseteq (\mathbf{ant}(X))^\Box\} \subseteq \{f(X), f(X^*)\}.$$

⊣

Lemma 3.8. Let X be a sequent in $\mathcal{S}_n$ $(n \geq 1)$.

$$\Box f(X) \equiv F(X) \wedge \bigwedge\{f(Y) \mid Y \in \mathcal{S}_{n+1}, \Box f(X) \in \mathbf{suc}(Y)\}.$$

Proof. By Lemma 3.5,

$$\Box f(X) \to F(X) \wedge \bigwedge\{f(Y) \mid Y \in \mathcal{S}_{n+1}, \Box f(X) \in \mathbf{suc}(Y)\} \in \mathbf{S4}.$$

We show the converse. By Lemma 3.7,

$$F(X), \bigwedge(f(\mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*))) \to \Box f(X) \in \mathbf{S4}. \cdots\cdots (*)$$

By Corollary 3.3, for any $Y_0 \in \mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*)$,

$$\{f(Y) \mid Y \in \mathcal{S}^+(Y_0), \Box f(X) \in \mathbf{suc}(Y)\} \to f(Y_0), \Box f(X) \in \mathbf{S4}.$$

Hence

$$\{f(Y) \mid Y \in \mathcal{S}^+(Y_0), Y_0 \in \mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*), \Box f(X) \in \mathbf{suc}(Y)\} \to \bigwedge(f(\mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*))), \Box f(X) \in \mathbf{S4}.$$

Using $(*)$,

$$F(X), \{f(Y) \mid Y \in \mathcal{S}^+(Y_0), Y_0 \in \mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*), \Box f(X) \in \mathbf{suc}(Y)\} \to \Box f(X) \in \mathbf{S4}.$$

Hence

$$F(X), \{f(Y) \mid Y \in \mathcal{S}_{n+1}, \Box f(X) \in \mathbf{suc}(Y)\} \to \Box f(X) \in \mathbf{S4}.$$

⊣

Lemma 3.9. Let X be a sequent in $\mathcal{S}_n - (\mathcal{G}_n \cup \mathcal{G}_n^*)$ $(n \geq 1)$. Then

$$f(X) \equiv \bigwedge f(\mathcal{S}(X)).$$

Proof. We show the case that $n = 1$. We note that $\mathcal{S}_1 = \{(p \to \Box p), (\to p)\}$. Also

$$\bigwedge f(\mathcal{S}(\to p)) = \bigwedge \{f(\to p, \Box p, \Box(p \supset \Box p)), f(\Box(p \supset \Box p) \to p, \Box p)\} \equiv p = f(\to p).$$

$$\bigwedge f(\mathcal{S}(p \to \Box p)) = f(p \to \Box p, \Box(p \supset \Box p)) \equiv f(p \to \Box p)$$

Suppose that $n \geq 2$. Clearly,

$$f(X) \to \bigwedge f(\mathcal{S}(X)) \in \mathbf{S4}.$$

By Corollary 3.3,

$$\bigwedge f(\mathcal{S}(X)) \to f(X) \in \mathbf{S4}.$$

⊣

Lemma 3.10. For $n \geq 1$, there exists a subset $\mathbf{S}$ of $\mathbf{G}_n$ such that

$$p \equiv \bigwedge f(\mathbf{S}) \qquad \text{and} \qquad p \supset \Box p \equiv \bigwedge f(\mathbf{G}_n - \mathbf{S}).$$

Proof. We use an induction on n.
Basis($n = 1$): We note that $\{\to p\} \subseteq \mathbf{G}_1$, $\{p \to \Box p\} = \mathbf{G}_1 - \{\to p\}$,

$$p \equiv \bigwedge f(\{\to p\}) \qquad \text{and} \qquad p \supset \Box p \equiv \bigwedge f(\{p \to \Box p\}) \equiv \bigwedge f(\mathbf{G}_1 - \{\to p\}).$$

Induction step($n > 1$): By the induction hypothesis, there exists $\mathbf{S}(\subseteq \mathbf{G}_{n-1})$ such that

$$p \equiv \bigwedge f(\mathbf{S}) \qquad \text{and} \qquad p \supset \Box p \equiv \bigwedge f(\mathbf{G}_{n-1} - \mathbf{S}).$$

Using Lemma 3.9,

$$p \equiv \bigwedge f(\mathbf{S} \cap (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)) \wedge \bigwedge \bigcup_{X \in \mathbf{S} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)} f(\mathcal{S}(X))$$

and

$$p \supset \Box p \equiv \bigwedge f((\mathbf{G}_{n-1} - \mathbf{S}) \cap (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)) \wedge \bigwedge \bigcup_{X \in (\mathbf{G}_{n-1} - \mathbf{S}) - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)} f(\mathcal{S}(X))$$

We note that $\mathcal{S}(X) \subseteq \mathbf{G}_n$ for any $X \in \mathbf{G}_{n-1} - (\mathcal{G}_{n-1} \cup \mathcal{G}_{n-1}^*)$. Hence we obtain the lemma. ⊣

Corollary 3.11. For $n \geq 1$,

$$\Box p \equiv \bigwedge f(\mathbf{G}_n).$$

Proof. By Lemma 3.11 and $\Box p \equiv (p \supset \Box p) \wedge p$. ⊣

Lemma 3.12. For $n \geq 1$ and a subset $\mathbf{S}$ of $\mathbf{G}_n$

$$\bigwedge f(\mathbf{S}) \supset \Box p \equiv \bigwedge f(\mathbf{G}_n - \mathbf{S}).$$

Proof. By Corollary 3.11,

$$\bigwedge f(\mathbf{G}_n - \mathbf{S}) \to \bigwedge f(\mathbf{S}) \supset \Box p \in \mathbf{S4}.$$

By Main lemma 2.5(1),

$$\to \bigwedge f(\mathbf{S}), \bigwedge f(\mathbf{G}_n - \mathbf{S}) \in \mathbf{S4}.$$

It is not hard to see

$$\Box p \to f(\mathbf{G}_n - \mathbf{S}) \in \mathbf{S4}.$$

Hence

$$\bigwedge f(\mathbf{S}) \supset \Box p \to \bigwedge f(\mathbf{G}_n - \mathbf{S}) \in \mathbf{S4}.$$

⊣

Proof of Main Lemma 2.5(2). We use an induction on A.

Basis: If $n = 0$, then the lemma is clear; if not, from Lemma 3.10.

Induction step:

If $A = B \wedge C$, then by the induction hypothesis, there exist subsets $\mathbf{S}_B$ and $\mathbf{S}_C$ of $\mathbf{G}_n$ such that

$$B \equiv \bigwedge f(\mathbf{S}_B), \quad \text{and} \quad C \equiv \bigwedge f(\mathbf{S}_C).$$

Using Lemma 3.1,

$$B \wedge C \equiv \bigwedge f(\mathbf{S}_B) \wedge \bigwedge f(\mathbf{S}_C) \equiv \bigwedge f(\mathbf{S}_B \cup \mathbf{S}_C).$$

Similarly, if $A = B \vee C$, then

$$B \vee C \equiv \bigwedge f(\mathbf{S}_B \cap \mathbf{S}_C).$$

Also, if $A = B \supset C$, then using Lemma 3.12,

$$B \supset C \equiv (B \supset \Box p) \vee C \equiv \bigwedge f((\mathbf{G}_n - \mathbf{S}_B) \cap \mathbf{S}_C).$$

If $A = \Box B$, then by the induction hypothesis and Lemma 3.8,

$$\Box B \equiv \Box \bigwedge f(\mathbf{S}_B) \equiv \bigwedge \Box f(\mathbf{S}_B)$$

$$\equiv \bigwedge f(\bigcup_{X \in \mathbf{S}_B} \{Y \in \mathcal{S}_{n+1} \mid \Box f(X) \in \mathbf{suc}(Y)\}) \cup f(\mathbf{S}_B \cap (\mathcal{G}_n \cup \mathcal{G}_n^*)) \cup f(\{X^* \mid X \in \mathbf{S}_B \cap \mathcal{G}_n^*\}).$$

⊣

References

[OM57] M. Ohnishi and K. Matsumoto, *Gentzen method in modal calculi*, Osaka Mathematical Journal, 9, 1957, pp. 113–130.

[CZ97] A. Chagrov and M. Zakharyaschev, *Modal Logic*, Oxford University Press, 1997.