**Separating a family of weak Kurepa Hypotheses and the Transversal Hypothesis**

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**Abstract**

We investigate principles which fit between the Kurepa Hypothesis and the weak Kurepa Hypothesis.

## Introduction

The Kurepa Hypothesis (KH) states that there exists a tree $T$ such that $T$ is of height $\omega_1$, all levels of $T$ are at most of size $\omega$ and $T$ has at least $\omega_2$-many cofinal branches. The weak Kurepa Hypothesis (wKH) states that there exists a tree $T$ such that $T$ is of height $\omega_1$, all levels of $T$ are at most of size $\underline{\omega_1}$ and has at least $\omega_2$-many cofinal branches. Hence, KH requires a tree whose levels are all thinner than wKH would do. It is known that, without loss of generality, we may assume each tree, if any, is a downward closed subtree of the complete binary tree ${}^{<\omega_1}2$ of height $\omega_1$. In [M], we introduced and investigated principles which fit between KH and wKH based on pp. 110-111 of [W]. This note continues our previous work [M]. We record the following progress.

(1) We introduce two additional principles. They are the club-weak Kurepa Hypothesis relative to stationary subsets $F$ of $\omega_1$, denoted by club-wKH($F$) and the ($*$)-wKH. We consider their implications and how they fit between KH and wKH. Among others, we record

- KH implies club-wKH. ([M])
- For any stationary subset $F$ of $\omega_1$, club-wKH implies club-wKH($F$).
- For any stationary subset $F$ of $\omega_1$, club-wKH($F$) imlies ($*$)-wKH.
- ($*$)-wKH implies $\tilde{\tilde{\diamondsuit}}$ of [W].

(2) We introduce a notion of forcing $R$ which adds a family $\mathcal{F}$ of almost disjoint functions from $\omega_1$ into $\omega$ with $|\mathcal{F}| = \omega_2$. We separate club-wKH of [M] and the Transversal Hypothesis (TH) by combining the Levy collapse and this $R$. Since KH implies club-wKH, this slightly improves [B], where KH and TH are separated. Let ZFC denote the Zermelo Frankel set theory with the Axiom of Choice. We write Con(ZFC + statements) to indicate that the theory ZFC together with the extra statements is consistent. We have the following.

- The club-wKH implies TH. ([M])
- Con(ZFC + there exists a strongly inaccessible cardinal) iff Con(ZFC + $\neg$club-wKH + TH).

(3) We also separate club-wKH($F$) and ($*$)-wKH by the Levy collapse. Since club-wKH($F$) and ($*$)-wKH fit inbetween club-wKH and stat-wKH, this slightly improves [M], where club-wKH and stat-wKH are separated. We claim

- $\tilde{\tilde{\diamondsuit}}$ implies stat-wKH. ([M])
- Con(ZFC + there exists a strongly inaccessible cardinal) iff Con(ZFC + $\neg$club-wKH + stat-wKH). ([M])
- Con(ZFC + there exists a strongly inaccessible cardinal) iff Con(ZFC + for all stationary subsets $F$ of $\omega_1$, $\neg$club-wKH($F$) + ($*$)-wKH).

We are interested in separating as much principles as possible. So far, we are separating clubs and stationary sets, so to speak. Other separations would require new ideas and techniques, if ever possible.

## §1. **A list of Definitions**

Let us recall a weak form of KH from [M].

**1.1 Definition.** The *club-weak Kurepa Hypothesis (club-wKH)* holds, if there exist $\langle b_\beta \mid \beta < \omega_2\rangle$, $\langle C_\beta \mid \beta < \omega_2\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ such that

- $b_\beta$ is a member of ${}^{\omega_1}2$ and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- $C_\beta$ is a club in $\omega_1$.
- $S_\alpha$ is a countable subset of ${}^{\alpha}2$.
- If $\alpha \in C_\beta$, then $b_\beta\lceil\alpha \in S_\alpha$.

We may relativize club-wKH with respect to any stationary subset of $\omega_1$ as follows:

**1.2 Definition.** Let $F$ be a stationary subset of $\omega_1$. The *club-weak Kurepa Hypothesis(F)* $\big($*club-wKH(F)*$\big)$ holds, if there exist $\langle b_\beta \mid \beta < \omega_2\rangle$, $\langle C_\beta \mid \beta < \omega_2\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ such that

- $b_\beta$ is a member of ${}^{\omega_1}2$ and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- $C_\beta$ is a club in $\omega_1$.
- $S_\alpha$ is a countable subset of ${}^{\alpha}2$.
- If $\alpha \in \underline{F\cap}C_\beta$, then $b_\beta\lceil\alpha \in S_\alpha$.

We may further weaken club-wKH($F$) as follows:

**1.3 Definition.** The *(∗)-weak Kurepa hypothesis* $\big($*(∗)-wKH*$\big)$ holds, if there exist $\langle b_\beta \mid \beta < \omega_2\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ such that the following is stationary in $[\omega_2]^\omega$.

$$\{X \in [\omega_2]^\omega \mid \forall\, \beta \in X\ b_\beta\lceil(X \cap \omega_1) \in S_{(X\cap\omega_1)}\}$$

The following is somewhat weaker than (∗)-wKH and equivalent (see [M]) to the one introduced in [W].

**1.4 Definition.** $\tilde{\tilde{\diamondsuit}}$ holds, if there exist $\langle b_\beta \mid \beta < \omega_2\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ such that the following is stationary in $[\omega_2]^\omega$.

$$\{X \in [\omega_2]^\omega \mid \underline{\exists\, B \subseteq X\ \bigcup B = \bigcup X\, \forall\, \beta \in B}\ b_\beta\lceil(X \cap \omega_1) \in S_{(X\cap\omega_1)}\}$$

Let us recall the transversal hypothesis and consider a weak form of it.

**1.5 Definition.** The *Transversal Hypothesis (TH)* holds, if there exists a family $\mathcal{F}$ of *almost disjoint functions from* $\omega_1$ *into* $\omega$ *with* $|\,\mathcal{F}\,| = \omega_2$. Namely, if $\mathcal{F} = \{f_\beta \mid \beta < \omega_2\}$, then

- $f_\beta$ is a member of ${}^{\omega_1}\omega$.
- If $\beta_1 \neq \beta_2$, then there exists $\alpha_{\beta_1\beta_2} < \omega_1$ such that for all $\alpha \in [\alpha_{\beta_1\beta_2}, \omega_1)$, we have $f_{\beta_1}(\alpha) \neq f_{\beta_2}(\alpha)$.

We may consider a weak form of TH which is sort of relativized to any given stationary subset of $\omega_1$.

**1.6 Defininition.** Let $F$ be a stationary subset of $\omega_1$. The *Transvesal Hypothesis (F)* $\big($*TH(F)*$\big)$ holds, if there exist $\langle f_\beta \mid \beta < \omega_2\rangle$ and $\langle C_{\beta_1\beta_2} \mid \beta_1, \beta_2 < \omega_2\rangle$ such that

- $f_\beta$ is a member of ${}^{\omega_1}\omega$.
- If $\beta_1 \neq \beta_2$, then $C_{\beta_1\beta_2}$ is a club in $\omega_1$ such that for any $\alpha \in F \cap C_{\beta_1\beta_2}$, we have $f_{\beta_1}(\alpha) \neq f_{\beta_2}(\alpha)$.

We recap two forms of the Chang's Conjecture.

**1.7 Definition.** The *Chang's Conjecture (CC)* holds, if for all sufficiently large regular cardinals $\theta$ and all $X \in [H_\theta]^\omega$, there exist elementary substructures $N$ of $H_\theta$ such that $X \subset N$, $N \cap \omega_1 < \omega_1$ and $|N \cap \omega_2| = \omega_1$.

The following is somewhat stronger than CC.

**1.8 Definition.** The *Strong Chang's Congecture (SCC)* holds, if for all sufficiently large regular cardinals $\theta$ and all countable elementary substructures $N$ of $H_\theta$, there exist countable elementary substructures $M$ of $H_\theta$ such that $N \subset M$, $N \cap \omega_1 = M \cap \omega_1$ and $N \cap \omega_2 \neq M \cap \omega_2$.

## §2. **Easy Implications**

We record implications among these weak forms of KH.

**2.1 Proposition.** *Let $F$ be any stationary subset of $\omega_1$. Then club-wKH implies club-wKH($F$).*

*Proof.* Let $\langle b_\beta \mid \beta < \omega_2\rangle$, $\langle C_\beta \mid \beta < \omega_2\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ be as in club-wKH. Then it is easy to see that they work.

$\square$

Notice that club-wKH iff club-wKH($\omega_1$).

**2.2 Proposition.** *Let $F$ be any stationary subset of $\omega_1$. Then club-wKH($F$) implies ($*$)-wKH.*

*Proof.* Let $\langle b_\beta \mid \beta < \omega_2\rangle$, $\langle C_\beta \mid \beta < \omega_2\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ be as in club-wKH($F$). Let $\varphi : {}^{<\omega}\omega_2 \longrightarrow \omega_2$. Let $\theta$ be a sufficiently large regular cardinal and take a countable elementary substructure $N$ of $H_\theta$ such that $N \cap \omega_1 \in F$ and $\langle C_\beta \mid \beta < \omega_2\rangle, \varphi \in N$. Let $\delta = N \cap \omega_1$.

For any $\beta \in N \cap \omega_2$, we have $C_\beta \in N$ and so $\delta \in F \cap C_\beta$. Hence $b_\beta\lceil\delta \in S_\delta$. Since $N \cap \omega_2$ is $\varphi$-closed, we are done.

$\square$

It is clear by definition that ($*$)-wKH implies $\tilde{\tilde{\diamondsuit}}$. We know ([M]) that club-wKH implies TH($\omega_1$). It is not trival but TH iff TH($\omega_1$) (see [M]). It is well-known that TH negates CC.

**2.3 Proposition.** *Let $F$ be a stationary subset of $\omega_1$. Then club-wKH($F$) implies TH($F$).*

*Proof.* It is straightforward. Let $\langle b_\beta \mid \beta < \omega_2\rangle$, $\langle C_\beta \mid \beta < \omega_2\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ be as in club-wKH($F$). Let $S_\alpha = \{a^\alpha_m \mid m < \omega\}$. Define $f_\beta : \omega_1 \longrightarrow \omega$ so that $f_\beta(\alpha) = m$, if $b_\beta\lceil\alpha \in S_\alpha$ and $m$ is the least with $b_\beta\lceil\alpha = a^\alpha_m$.

Let $\beta_1$ and $\beta_2$ be given two different elements of $\omega_2$. Let $\alpha^* < \omega_1$ with $b_{\beta_1}\lceil\alpha^* \neq b_{\beta_2}\lceil\alpha^*$. Let

$$C_{\beta_1\beta_2} = C_{\beta_1} \cap C_{\beta_2} \cap \{\alpha < \omega_1 \mid \alpha^* \leq \alpha\}.$$

Let $\alpha \in F \cap C_{\beta_1\beta_2}$. Then $b_{\beta_1}\lceil\alpha \in S_\alpha$, $b_{\beta_2}\lceil\alpha \in S_\alpha$ and $b_{\beta_1}\lceil\alpha \neq b_{\beta_2}\lceil\alpha$. Hence $f_{\beta_1}(\alpha) \neq f_{\beta_2}(\alpha)$.

$\square$

**2.4 Proposition.** *Let $F$ be any stationary subset of $\omega_1$. Then SCC negates TH($F$).*

*Proof.* By Contradiction. Let $\langle f_\beta \mid \beta < \omega_2\rangle$ and $\langle C_{\beta_1\beta_2} \mid \beta_1, \beta_2 < \omega_2\rangle$ be as in TH($F$). Let $\theta$ be a sufficiently large regular cardinal and $N$ be an elementary substructure of $H_\theta$ such that $N \cap \omega_1 \in F$, $|N \cap \omega_2| = \omega_1$ and $\langle C_{\beta_1\beta_2} \mid \beta_1, \beta_2 < \omega_2\rangle \in N$. This is possible by building a continuously increasing chain of countable elementary substructures by SCC. Let $\delta = N \cap \omega_1$. We may observe $\langle \beta \mapsto f_\beta(\delta) \mid \beta \in N \cap \omega_2\rangle$ is one-to-one as follows:

Let $\beta_1, \beta_2 \in N \cap \omega_2$ with $\beta_1 \neq \beta_2$. Then $C_{\beta_1 \beta_2} \in N$ and so $\delta \in F \cap C_{\beta_1 \beta_2}$. So $f_{\beta_1}(\delta) \neq f_{\beta_2}(\delta)$. Hence $\{f_\beta(\delta) \mid \beta \in N \cap \omega_2\}$ is of size $\omega_1$ yet is a subset of $\omega$. This is a contradiction.

□

## §3. **Not club-wKH and TH**

It is mentioned on p. 211 in [Ka] that an unpublished work of [B] shows that KH implies TH but that the converse does not hold. (Note: The weak Kurepa Hypothesis (wKH) in [Ka] is different from our weak Kurepa Hypothesis. The wKH in [Ka] states that there exists a family $\mathcal{F}$ of almost disjoint functions from $\omega_1$ into $\omega$ with $|\mathcal{F}| = \omega_2$. So it is our TH. On the other hand, our wKH states that there exists a downward closed substree $T$ of ${}^{<\omega_1}2$ such that $T$ is of size $\omega_1$ and has at least $\omega_2$-many cofinal branches as in p. 111 of [W].)

We reformulate [B] and separate club-wKH and TH. Namely, club-wKH implies TH but the converse does not hold. Note that KH implies club-wKH and the converse does not hold ([M]).

**3.1 Theorem.** *Let $\kappa$ be a strongly inaccessible cardinal. Then there exists a notion of forcing $P$ such that $P$ is $\sigma$-closed, has the $\kappa$-c.c. and so preserves both $\omega_1$ and $\kappa$ to be cardinals. If we extend the ground model $V$ via this $P$, then in the generic extensions $V[P]$, the cardinals in $V$ which are strictly greater than $\omega_1$ and strictly less than $\kappa$ are all collapsed to be of size $\omega_1$, $\kappa$ is the new $\omega_2$, club-wKH does not hold yet TH holds.*

*Proof.* We just out-line here. Details are provided in the next sections.

(Out-line) Let $\mathrm{Lv}(\kappa, \omega_1)$ denote the Levy collapse which is $\sigma$-closed, has the $\kappa$-c.c. and the cardinals which are strictly greater than $\omega_1$ and strictly less than $\kappa$ are all collapsed to be of size $\omega_1$. Let $R$ denote a notion of forcing which adds a family of almost disjoint functions $\langle g_\beta \mid \beta < \kappa \rangle$ from $\omega_1$ into $\omega$. Let $P$ be the forcing product $\mathrm{Lv}(\kappa, \omega_1) \times R$. We claim this $P$ works. Both $\mathrm{Lv}(\kappa, \omega_1)$ and $R$ satisfy a stronger form of $\kappa$-c.c. By absoluteness, we may view $P$ as two stage iterations $R * \mathrm{Lv}(\kappa, \omega_1)$ and $\mathrm{Lv}(\kappa, \omega_1) * R$ which are all forcing equivalent, $\sigma$-closed, have the $\kappa$-c.c.

To see TH in $V[P]$, we view

$$V[P] = V[\mathrm{Lv}(\kappa, \omega_1)][R].$$

To see club-wKH gets negated in $V[P]$, let $\langle b_\beta \mid \beta < \kappa \rangle$, $\langle C_\beta \mid \beta < \kappa \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be a possible combination to club-wKH in $V[P]$. Then by the $\kappa$-c.c, there exists $\omega_1 \leq \xi < \kappa$ such that we may factor

$$V[P] = V[\mathrm{Lv}(\xi, \omega_1)][R_\xi][R^\xi][\mathrm{Lv}\big([\xi, \kappa), \omega_1\big)]$$

and

$$\langle S_\alpha \mid \alpha < \omega_1 \rangle \in V[\mathrm{Lv}(\xi, \omega_1)][R_\xi].$$

The p.o. sets $R_\xi$ and $R^\xi$ are defined and studied in detail later.

In $V[\mathrm{Lv}(\xi, \omega_1)][R_\xi]$, $\kappa$ remains strongly inaccessible and $R^\xi \times \mathrm{Lv}\big([\xi, \kappa), \omega_1\big)$ is $E_\xi$-complete and has the $\kappa$-c.c. for some $E_\xi \subset [\xi]^\omega$ stationary. Therefore we may modify the Silver's original argument (see [Si] or [M]) in this context to conclude all $b_\beta \in {}^{\omega_1}2$ are in this intermediate stage. But this would be a contradiction.

This out-lines the proof.

□

According to this out-line, the main point is to factor the relevant p.o. sets. This takes a careful treatment of the quotients, in particular.

## §4. **Notions of forcing which are $E$-complete and $F$-complete**

In this section we deal with notions of forcing which are $E$-complete and $F$-complete. Our treatment of $E$-completeness is an example of a more general notion found in [S].

**4.1 Definition.** ([S]) Let $\xi$ be an ordinal with $\omega_1 \leq \xi$ and let $E \subseteq [\xi]^\omega$ be stationary in $[\xi]^\omega$. A notion of forcing $Q$ is *$E$-complete*, if for all sufficiently large regular cardinals $\theta$ and all countable elementary substructures $N$ of $H_\theta$ with $E, Q \in N$ and $\underline{N \cap \xi \in E}$, if $\langle q_n \mid n < \omega\rangle$ is any $(Q, N)$-generic sequence, then there exists $q \in Q$ such that for all $n < \omega$, it holds that $q \leq q_n$ in $Q$.

Note that if $F$ is a stationary subset of $\omega_1$, then $F$ is stationary in $[\omega_1]^{\leq\omega}$. Hence it makes sence that a notion of forcing $Q$ to be $F$-complete.

**4.2 Proposition.** ([S]) *If $Q$ is $E$-compltete, then $Q$ is $\sigma$-Baire and $E$ remains stationary in the generic extensions $V[Q]$.*

*Proof.* Let $\langle D_n \mid n < \omega\rangle$ be a sequence of open dense subsets of $Q$. We need to show that $\bigcap\{D_n \mid n < \omega\}$ is dense in $Q$. To this end, let $q \in Q$ and $\theta$ be a sufficiently large regular cardinal and $N$ be a countable elementary substructure of $H_\theta$ with $E, Q \in N$ and $N \cap \xi \in E$. We may assume $q, \langle D_n \mid n < \omega\rangle \in N$ and so $D_n \in N$ for all $n < \omega$. Let $\langle q_n \mid n < \omega\rangle$ be a $(Q, N)$-generic sequence with $q_0 = q$. Then we have a lower bound $q^* \in Q$ of the $q_n$'s. Since $q_m \in D_n$ for some $m$, we conclude $q^* \in \bigcap\{D_n \mid n < \omega\}$.

To show that $E$ remains stationary in $V[Q]$, let $\dot\varphi$ be a $Q$-name such that $q \Vdash_Q$ "$\dot\varphi : {}^{<\omega}\xi \longrightarrow \xi$". Take a sufficiently large regular cardinal $\theta$ and a countable elementary substructure $N$ of $\theta$ such that $\xi, Q \in N$ and $N \cap \xi \in E$. We may assume that $q, \dot\varphi \in N$. Let $\langle q_n \mid n < \omega\rangle$ be a $(Q, N)$-generic sequence with $q_0 = q$ and $q^*$ be a lower bound of the $q_n$'s. Then $q^* \Vdash_Q$ "$N \cap \xi = N[\dot G_Q] \cap \xi$ is $\dot\varphi$-closed". Hence $E$ remains stationary in $V[Q]$.

□

**4.3 Proposition.** *Let $Q$ be a notion of forcing which is $\sigma$-Baire. If $p \Vdash_Q$ "$\dot b : \omega_1 \longrightarrow 2, \dot b \notin V$", then*

$$\forall\, \alpha < \omega_1 \,\forall\, q \leq p \,\exists\, \beta\ \alpha \leq \beta < \omega_1 \,\exists\, r_1, r_2 \leq q \,\exists\, \sigma_1, \sigma_2 \in {}^\beta 2 \ \text{ such that}$$
$$r_1 \Vdash_Q \text{“}\dot b\lceil\beta = \sigma_1\text{”},$$
$$r_2 \Vdash_Q \text{“}\dot b\lceil\beta = \sigma_2\text{”},$$
$$\sigma_1 \neq \sigma_2.$$

*Proof.* By contradiction. Suppose not and fix $\alpha_0 < \omega_1$ and $q_0 \in Q$ such that $q_0 \leq p$ and that

$$\forall\, \beta\ \alpha_0 \leq \beta < \omega_1 \,\forall\, r_1, r_2 \leq q_0 \,\forall\, \sigma_1, \sigma_2 \in {}^\beta 2,$$
$$\text{If } r_1 \Vdash_Q \text{“}\dot b\lceil\beta = \sigma_1\text{” and } r_2 \Vdash_Q \text{“}\dot b\lceil\beta = \sigma_2\text{”, then } \sigma_1 = \sigma_2.$$

Since $Q$ is $\sigma$-Baire, for each $\beta \in [\alpha_0, \omega_1)$, may fix $q^\beta \leq q_0$ and $\sigma^\beta$ such that $q^\beta \Vdash_Q$ "$\dot b\lceil\beta = \sigma^\beta$".

**4.3.1 Claim.** *If $\alpha_0 \leq \beta_1 < \beta_2 < \omega_1$, then $\sigma^{\beta_1} \subset \sigma^{\beta_2}$.*

*Proof.* Take $r \leq q^{\beta_1}$ in $Q$ and $\sigma \in {}^{\beta_2}2$ such that $r \Vdash_Q$ "$\dot b\lceil\beta_2 = \sigma$". Then $r, q^{\beta_2} \leq q_0$, $q^{\beta_2} \Vdash_Q$ "$\dot b\lceil\beta_2 = \sigma^{\beta_2}$". Therefore, $\sigma^{\beta_1} \subset \sigma = \sigma^{\beta_2}$ holds.

Let $b = \bigcup\{\sigma^\beta \mid \alpha_0 \leq \beta < \omega_1\}$. Then $b : \omega_1 \longrightarrow 2$ and

**4.3.2 Claim.** $q_0 \Vdash_Q$ "$\dot b = b$" *and so this is a contradiction.*

*Proof.* Fix any $\beta$ with $\alpha_0 \leq \beta < \omega_1$. Take any $d \leq q_0$ in $Q$. Take $d' \leq d$ and $\sigma \in {}^\beta 2$ such that $d' \Vdash_Q$ "$\dot b\lceil\beta = \sigma$". Since $d', q^\beta \leq q_0$, we have $\sigma = \sigma^\beta$ and so $d' \Vdash_Q$ "$\dot b\lceil\beta = b\lceil\beta$". Since $\beta$ and $d$ are arbitrary, we conclude $q_0 \Vdash_Q$ "$\dot b = b$".

□

**4.4 Lemma.** *Let $Q$ be $E$-complete and $q \Vdash_Q$ "$\dot{b} : \omega_1 \longrightarrow 2, \dot{b} \notin V$". Then for all sufficiently large regular cardinals $\theta$ and all countable elementary substructures $N$ of $H_\theta$ such that $q, Q, E, \dot{b} \in N$ and $N \cap \xi \in E$, we may construct a map $\langle f \mapsto \sigma_f \mid f \in {}^\omega 2\rangle$ from ${}^\omega 2$ into ${}^{(N\cap\omega_1)}2$ which is one-to-one and an associated $q_f \leq q$ such that $q_f$ is $(Q, N)$-generic and $q_f \Vdash_Q$ "$\dot{b}\lceil(N \cap \omega_1) = \sigma_f$".*

*Proof.* The following is routine.

Let us first denote $\delta = N \cap \omega_1$. Let $\langle D_n \mid n < \omega\rangle$ enumerate the open dense subsets $D \in N$ of $Q$. Fix a strictly increasing sequence $\langle \delta_n \mid n < \omega\rangle$ of ordinals such that $\delta_0 = 0$ and $\sup\{\delta_n \mid n < \omega\} = \delta$.
Then construct $\langle s \mapsto (q_s, \sigma_s) \mid s \in {}^{<\omega}2\rangle$ such that

- $q_\emptyset = q$, $\sigma_\emptyset = \emptyset$,
- $q_s \in Q \cap N$, $\sigma_s \in ({}^{|\sigma_s|}2) \cap N$ and $\delta_{|s|} \leq |\sigma_s|$,
- $q_s \leq q$ and $q_s \Vdash_Q$"$\dot{b}\lceil|\sigma_s| = \sigma_s$",
- $q_{s\frown\langle 0\rangle}, q_{s\frown\langle 1\rangle} \leq q_s$ and $q_{s\frown\langle 0\rangle}, q_{s\frown\langle 1\rangle} \in D_n \cap N$,
- $|\,\sigma_{s\frown\langle 0\rangle}\,| = |\,\sigma_{s\frown\langle 1\rangle}\,|$ but $\sigma_{s\frown\langle 0\rangle} \neq \sigma_{s\frown\langle 1\rangle}$.

For $f \in {}^\omega 2$, $\langle q_{f\lceil n} \mid n < \omega\rangle$ is a $(Q, N)$-generic sequence and $N \cap \xi \in E$. Hence we may take a lower bound $q_f \in Q$ of the $\{q_{f\lceil n} \mid n < \omega\}$. Let $\sigma_f = \bigcup\{\sigma_{f\lceil n} \mid n < \omega\}$. Then $q_f \Vdash_Q$"$\dot{b}\lceil\delta = \sigma_f$" holds. By construction, we see that $f \mapsto \sigma_f$ is one-to-one.

□

Here is our main lemma.

**4.5 Lemma.** *Let $\xi$ be an ordinal with $\omega_1 \leq \xi$ and $E \subseteq [\xi]^\omega$ be stationary in $[\xi]^\omega$. Let $\langle S_\alpha \mid \alpha < \omega_1\rangle$ be a sequence such that each $S_\alpha$ is countable and $S_\alpha \subseteq {}^\alpha 2$. Let $Q$ be a notion of forcing which is $E$-complete. Then we have $\Vdash_Q$ "<u>if</u> $\dot{b} : \omega_1 \longrightarrow 2$, $\dot{C} \subseteq \omega_1$ is a club and for all $\alpha \in \dot{C}$, $\dot{b}\lceil\alpha \in S_\alpha$, <u>then</u> $\dot{b} \in V$".*

*Proof.* Suppose not. Then we have $q \in Q$ such that $q \Vdash_Q$"$\dot{b} : \omega_1 \longrightarrow 2$, $\dot{C} \subseteq \omega_1$ is a club and for all $\alpha \in \dot{C}$, $\dot{b}\lceil\alpha \in S_\alpha$, yet $\dot{b} \notin V$". Let $\theta$ be a sufficiently large regular cardinal and $N$ be a countable elementary substructure of $H_\theta$ with $N \cap \xi \in E$. Let $\delta = N \cap \omega_1$. We may assume relevant parameters are all in $N$. In particular, $\dot{C} \in N$. By 4.4 Lemma, we have a one-to-one map $\langle f \mapsto \sigma_f \mid f \in {}^\omega 2\rangle$ from ${}^\omega 2$ into ${}^\delta 2$ and the associated $q_f \leq q$ such that $q_f \Vdash_Q$"$\dot{b}\lceil\delta = \sigma_f$".
Since $q_f$ is $(Q, N)$-generic, we have $q_f \Vdash_Q$"$\delta \in \dot{C}$ and so $\dot{b}\lceil\delta \in S_\delta$". Therefore we have $\sigma_f \in S_\delta$. Since $S_\delta$ is countable, this is a contradiction.

□

Here is a related lemma.

**4.6 Lemma.** *Let $F$ be a stationary subset of $\omega_1$. Let $\langle S_\alpha \mid \alpha < \omega_1\rangle$ be a sequence such that each $S_\alpha$ is countable and $S_\alpha \subseteq {}^\alpha 2$. Let $Q$ be a notion of forcing which is $F$-complete. Then we have $\Vdash_Q$ "<u>if</u> $\dot{b} : \omega_1 \longrightarrow 2$, $\dot{C} \subseteq \omega_1$ is a club and for all $\alpha \in \underline{F \cap} \dot{C}$, $\dot{b}\lceil\alpha \in S_\alpha$, <u>then</u> $\dot{b} \in V$".*

*Proof.* Similar. Since $q_f \Vdash_Q$"$\delta \in F \cap \dot{C}$", we would be done.

□

In closing this section, we record two additional lemmas for later use.

Suppose we have a p.o. set $P$ and want to show it is $E$-complete. Then it would be sufficient to find a large regular cardinal $\theta$ and club-many $N$'s such that every $(P, N)$-generic sequence has a lower bound in $P$.

**4.7 Lemma.** *Let $P$ be a p.o. set and let $E \subseteq [\xi]^\omega$ be stationary with $\omega_1 \leq \xi$. Then the following are equivalent.*

*(1) $P$ is $E$-complete.*

*(2) The following set contains a club in $[\xi \cup \mathcal{P}(P) \cup P]^\omega$.*

$$\{X \in [\xi \cup \mathcal{P}(P) \cup P]^\omega \mid X \cap \xi \in E \Rightarrow \forall \langle p_n \mid n < \omega\rangle \ (P, X) - \text{generic sequence} \, \exists p \in P \ \forall n < \omega \ p \leq p_n\}$$

*Proof.* The point here is definabilities with the parameters $E$ and $P$.

□

The notions of forcing which are $E$-complete iterates under the countable support ([S], in a more general setting). We also pay attention to the following.

**4.8 Lemma.** *Let $P$ and $Q$ be p.o. sets and let $E \subseteq [\xi]^\omega$ be stationary with $\omega_1 \leq \xi$. Then the following are equivalent.*

*(1) Both $P$ and $Q$ are $E$-complete.*

*(2) The product $P \times Q$ is $E$-complete.*

*(3) The two stage iteration $P * \check{Q}$ is $E$-complete.*

*(4) $P$ is $E$-complete and $\Vdash_P$ "$\check{Q}$ is $E$-complete".*

*Proof.* It is mostly routine. We just note that if $\langle p_n \mid n < \omega\rangle$ is a $(P, N)$-generic sequence, then there exist a descending sequence $\langle q_k \mid k < \omega\rangle$ of conditions from $Q$ and a strictly increasing sequence $\langle n_k \mid k < \omega\rangle$ of natural numbers such that $\langle (p_{n_k}, q_k) \mid k < \omega\rangle$ is a $(P \times Q, N)$-generic sequence.

□

Even if $P * \dot{Q}$ is $E$-complete, we may not have $\Vdash_P$"$\dot{Q}$ is $E$-complete". (A good example to try to look at is; $P =$ Adding a Sousline tree $T$ by its initial segments and force with $\omega_2$-many copies of $T$ under countable support product. Then force with $T$, again. Namely, $\dot{Q} = T$. Then $P * \dot{Q}$ would be $\omega_1$-complete but $\dot{Q}$ would kill a stationary subset of $[\omega_2]^\omega$ and so not proper. In particular, $\dot{Q}$ is never $\omega_1$-complete in $V[P]$.) So it is important that the $Q$ above is a fixed one in the ground model $V$.

## §5. **Forcing a family of almost disjoint functions**

In this section, we introduce our notion of forcing which adds a family $\langle g_\beta \mid \beta < \kappa\rangle$ of almost disjoint functions from $\omega_1$ into $\omega$, where $\kappa$ is a strongly inaccessible cardinal. The main point here is to understand the quotients with respect to this notion of forcing. Namely, we consider how does $\langle g_\beta \mid \beta < \xi\rangle$ get lengthened to acquire $\langle g_\beta \mid \xi \leq \beta < \kappa\rangle$ for any $\xi$ with $\omega_1 \leq \xi < \kappa$. This section is hinted on and reformulates [B].

**5.1 Definition.** Let $\kappa$ be a strongly inaccessible cardinal and let $\xi$ be an ordinal with $\omega_1 \leq \xi \leq \kappa$. We define $R_\xi$ as follows:

$p \in R_\xi$, if $p = \langle g^p_\beta \mid \beta \in X^p\rangle$ such that

- $X^p \in [\xi]^{\leq\omega}$,
- There exists a unique ordinal $\alpha^p < \omega_1$ such that for all $\beta \in X^p$, it holds that $g^p_\beta$ is a function from $\alpha^p$ into $\omega$.

We set $R = R_\kappa$. It is clear that $R_\xi \subset R$ holds.

For $q, p \in R_\xi$, we define $q \leq_\xi p$, if

- $X^q \supseteq X^p$,
- For all $\beta \in X^p$, it holds that $g^q_\beta \lceil \alpha^p = g^p_\beta$,

- The maps $\langle \beta \mapsto g^q_\beta(\alpha) \mid \beta \in X^p \rangle$ are all one-to-one for all $\alpha$ with $\alpha^p \leq \alpha < \alpha^q$.

We write $R_\xi$ for $(R_\xi, \leq_\xi)$ and $\leq$ for $\leq_\xi$. There is no confusion. We may draw pictures to observe the following.

**5.2 Lemma.** *$R_\xi$ is a p.o. set.*

We make important remarks.

**5.3 Note.** (1) If $p \in R_\xi$ and $\alpha_0 < \alpha^p$, then $\langle g^p \lceil \alpha_0 \mid \beta \in X^p \rangle \in R_\xi$. But this condition may not be extended to $p$ in $R_\xi$.

(2) If $p \in R$ and $\omega_1 \leq \xi \leq \kappa$, then $\langle g^p_\beta \mid \beta \in X^p \cap \xi \rangle \in R_\xi$ and if we denote this condition by $p\lceil\xi$, then $p \leq p\lceil\xi$ holds in $R$.

(3) The $R_\xi$ are upward-absolute as long as extensions are by notions of forcing which are $\sigma$-Baire.

We begin our analysis of this $R_\xi$.

**5.4 Lemma.** *$R_\xi$ is $\sigma$-closed.*

*Proof.* Let $\langle p_n \mid n < \omega \rangle$ be descending in $R_\xi$. To define a lower bound $p$ of the $p_n$, let $X^p = \bigcup \{X^{p_n} \mid n < \omega\}$ and for each $\beta \in X^p$, let $g^p_\beta = \bigcup \{g^{p_n}_\beta \mid \beta \in X^{p_n}, n < \omega\}$.

Then we may check $p \in R_\xi$ with $\alpha^p = \sup\{\alpha^{p_n} \mid n < \omega\}$ and that for all $n < \omega$, we have $p \leq p_n$.

□

**5.5 Lemma.** *(1) (CH) $R_\xi$ has the $\omega_2$-c.c.*

*(2) $R_\xi$ has the $\kappa$-c.c.*

*Proof.* The point here is that if $p, q \in R_\xi$ are two conditions such that $\alpha^p = \alpha^q$ and $g^p_\beta = g^q_\beta$ for all $\beta \in X^p \cap X^q$, then $p \cup q$ is a common extension of $p, q$ in $R_\xi$. The rest is either by the $\Delta$-system Lemma under CH or by the assumption that $\kappa$ is strongly inacccessible.

□

**5.6 Corollary.** *(CH) $R_\xi$ preserves the cofinalities and so the cardinalities.*

We observe $R_\xi$ indeed adds what we intend.

**5.7 Lemma.** *Let $G_\xi$ be $R_\xi$-generic over the ground model $V$. For each $\beta < \xi$, let*

$$g_\beta = \bigcup \{g^p_\beta \mid \beta \in X^p, p \in G_\xi\}.$$

*Then the $g_\beta$'s are almost disjoint functions from $\omega_1$ into $\omega$. In particular, the map $\langle \beta \mapsto g_\beta \mid \beta < \xi \rangle$ is one-to-one. Hence, $2^{\omega_1} \geq |\xi|$ holds in $V[G_\xi]$.*

*Proof.* By genericities. In particular, we have $p \Vdash_{R_\xi}$ "the maps $\langle \beta \mapsto \dot{g}_\beta(\alpha) \mid \beta \in X^p \rangle$ are one-to-one for all $\alpha$ with $\alpha^p \leq \alpha < \omega_1$".

□

We observe $G_\xi$ gets recovered from $\langle g_\beta \mid \beta < \xi \rangle$.

**5.8 Lemma.** *Let $G_\xi$ be $R_\xi$-generic over the ground model $V$ and define $\langle g_\beta \mid \beta < \xi \rangle$ as above. Then for any $p \in R_\xi$, we have $p \in G_\xi$ iff the following (1)-(2) hold.*

*(1) For all $\beta \in X^p$, we have $g_\beta \lceil \alpha^p = g^p_\beta$,*

*(2) For all $\alpha$ with $\alpha^p \leq \alpha < \omega_1$, the maps $\langle \beta \mapsto g_\beta(\alpha) \mid \beta \in X^p \rangle$ are one-to-one.*

*In particular, we have $V[G_\xi] = V[\langle g_\beta \mid \beta < \xi \rangle]$.*

*Proof.* We explicitly observe one direction. Suppose (1) and (2) hold. We take $w \in G_\xi$ such that $w \Vdash_{R_\xi}$ "(1)-(2)". We may assume that $X^w \supseteq X^p$ and $\alpha^w \geq \alpha^p$. Then we may check that $w \leq p$ in $R_\xi$. Hence $p \in G_\xi$.

$\square$

**5.9 Note.** Even if we had $p, q \in G_\xi$ and $\alpha^p = \alpha^q$, it does not automatically imply $p \cup q \in G_\xi$.

We now pay attention to a stationary subset of $[\xi]^\omega$ forced. This stationarity reflects the genericity of the $g_\beta$' s to some extent. We make heavy use of this property to analyze the quotients $R/G_\xi$ in the next section. ([B])

**5.10 Lemma.** *Let $G_\xi$ and $g_\beta$ be as above. Let $E_\xi \subseteq [\xi]^\omega$ be defined as follows:*
*$X \in E_\xi$, if the following (1)-(3) are satisfied.*

*(1) $X \in [\xi]^\omega$ and $X \cap \omega_1 < \omega_1$,*

*(2) If $X \cap \omega_1 \leq \alpha < \omega_1$, then $|\,\omega \setminus \{g_\beta(\alpha) \mid \beta \in X\}\,| = \omega$,*

*(3) If $X \cap \omega_1 \leq \alpha < \omega_1$, then the map $\langle \beta \mapsto g_\beta(\alpha) \mid \beta \in X \rangle$ is one-to-one.*

*Then this $E_\xi$ is a stationary subset of $[\xi]^\omega$ in $V[G_\xi]$.*

*Proof.* Suppose $p \Vdash_{R_\xi}$ "$\dot{\varphi} : {}^{<\omega}\xi \longrightarrow \xi$". Let $\theta$ be a sufficiently large regular cardinal and $N$ be a countable elementary substructure of $H_\theta$ with $p, R_\xi, \dot{\varphi} \in N$. Let $q \leq p$ be $(R_\xi, N)$-generic such that $\alpha^q = N \cap \omega_1$ and $X^q = N \cap \xi$. Since $\omega_1 \leq \xi$ and $X^q$ is countable, we may fix $q^+ \in R_\xi$ such that $\alpha^{q^+} = \alpha^q$ but $X^q \subset X^{q^+}$ and $|\,X^{q^+} \setminus X^q\,| = \omega$. And so $q^+ \leq q$ in $R_\xi$. Then

- $q^+ \Vdash_{R_\xi}$ "$N \cap \xi = N[\dot{G}_\xi] \cap \xi$ is $\dot{\varphi}$-closed",
- $q^+ \Vdash_{R_\xi}$ "$\omega \setminus \{\dot{g}_\beta(\alpha) \mid \beta \in N \cap \xi\} \supseteq \{\dot{g}_\beta(\alpha) \mid \beta \in X^{q^+} \setminus X^q\}$ for all $\alpha$ with $\alpha^q \leq \alpha < \omega_1$"
- $q^+ \Vdash_{R_\xi}$ "$\langle \beta \mapsto \dot{g}_\beta(\alpha) \mid \beta \in N \cap \xi \rangle$ are one-to-one for all $\alpha$ with $\alpha^q \leq \alpha < \omega_1$".

Hence $q^+ \Vdash_{R_\xi}$ "$N \cap \xi \in \dot{E}_\xi$". Therefore, $E_\xi$ is a stationary subset of $[\xi]^\omega$ in $V[G_\xi]$.

$\square$

## §6. Viewing $R$ as two stage iterations $R_\xi * \dot{R}^\xi$

We factor $R$ and examine the quotients. This analysis closely follows [B].

**6.1 Lemma.** *The identity map $p \mapsto p$ from $R_\xi$ into $R$ is a complete embedding. Namely,*

*(1) If $q \leq p$ in $R_\xi$, then so in $R$.*

*(2) If $p$ and $q$ are incompatible in $R_\xi$, then so are in $R$.*

*(3) For any $p \in R$, there exists $a \in R_\xi$ such that for any $b \leq a$ in $R_\xi$, it holds that $b$ and $q$ are compatible in $R$.*

*Proof.* For (1): We actually have $q \leq p$ in $R_\xi$ iff $q \leq p$ in $R$.

For (2): Suppose $p, q \in R_\xi$ and there exists $r \in R$ such that $r \leq p, q$ in $R$. Then $r \lceil \xi = \langle g^r_\beta \mid \beta \in X^r \cap \xi \rangle \leq p, q$ in $R_\xi$.

For (3): Let $c : X^p \setminus \xi \longrightarrow Y$ be a bijection such that $Y \subset \xi$ is disjoint with $X^p \cap \xi$. This is possible, as $\omega_1 \leq \xi$ and $X^p \cap \xi$ is countable. Let $a = p\lceil \xi \cup \{(c(\beta), g^p_\beta) \mid \beta \in X^p \setminus \xi\}$. Then it is not hard to show that this $a$ works.

$\square$

The following are routine (see pp. 243-244 in [Ku]).

**6.2 Definition.** Let $G_\xi$ be $R_\xi$-generic over $V$. Define a suborder $R^\xi$ of $R$ as follows; $p \in R^\xi$, if the following two are satisfied.

- $p \in R$,
- For all $w \in G_\xi$, it holds that $w$ and $p$ are compatible in $R$.

So the quotient $R/G_\xi$ is simply denoted by $R^\xi$. Let $\dot{R}^\xi$ denote the canonical $R_\xi$-name of $R^\xi$.

**6.3 Lemma.** *$R$ and $R_\xi * \dot{R}^\xi$ are forcing equivalent. More precisely, we have*

*(1) If $G$ is $R$-generic over $V$, then $G \cap R_\xi$ is $R_\xi$-generic over $V$ and the $G$ itself is $\dot{R}^\xi_{(G\cap R_\xi)}$-generic over $V[G \cap R_\xi]$. In particular, we have $V[G] = V[G \cap R_\xi][G]$.*

*(2) If $G_\xi$ is $R_\xi$-generic over $V$ and $G$ is $\dot{R}^\xi_{G_\xi}$-generic over $V[G_\xi]$, then $G$ is $R$-generic over $V$ and $G \cap R_\xi = G_\xi$ holds. In particular, we have $V[G_\xi][G] = V[G]$.*

□

We identify $R^\xi$ in $V[G_\xi]$.

**6.4 Lemma.** *Let $p \in R$. Then $p \in R^\xi$ iff the following (1) and (2) hold.*

*(1) $p \lceil \xi \in G_\xi$. Namely,*
*(1.1) $\forall \beta \in X^p \cap \xi \; g_\beta \lceil \alpha^p = g^p_\beta$,*
*(1.2) $\forall \alpha \in [\alpha^p, \omega_1) \; \langle \beta \mapsto g_\beta(\alpha) \mid \beta \in X^p \cap \xi \rangle$ is one-to-one.*

*(2) $\forall \alpha \in [\alpha^p, \omega_1) \; |\, \omega \setminus \{g_\beta(\alpha) \mid \beta \in X^p \cap \xi\} \,| \geq |\, X^p \setminus \xi \,|$.*

*Proof.* Suppose $p \in R^\xi$. Take a $R^\xi$-generic filter $G$ over $V[G_\xi]$ with $p \in G$. Then it is rather easy to see that (1) and (2) hold in $V[G]$. Then in turn, by absoluteness, (1) and (2) hold in $V[G_\xi]$.

Conversely, suppose (1) and (2). We want to show $p \in R^\xi$. To this end, let $a \in G_\xi$. We may assume that $X^a \supset X^p \cap \xi$ and $\alpha^a > \alpha^p$. Then it is straightforward to construct $r \in R$ such that $\alpha^r = \alpha^a$, $r\lceil \xi = a$ and $r \leq a, p$ in $R$.

□

We study the quotients $R^\xi$ in $V[G_\xi]$. However, rather than directly making use of the genericity of $G_\xi$, we may formulate the current universe of set theory as follows;

(1) We have a family of almost disjoint functions $\langle g_\beta \mid \beta < \xi \rangle$ with $\omega_1 \leq \xi$.
(2) $E_\xi$ is stationary in $[\xi]^\omega$, where $E_\xi$ is defined from $\langle g_\beta \mid \beta < \xi \rangle$ as in 5.10 Lemma.
(3) $R^\xi$ is explicitly defined from $\langle g_\beta \mid \beta < \xi \rangle$ as in the equivalent manner of 6.4 Lemma.

Notice that (1) does not imply (2) in general. Now (1)-(3) suffice to investigate the properties of $R^\xi$, possibly except the chain conditions.

**6.5 Lemma.** *The p.o. set $R^\xi$ enjoys the following density. For any $p \in R^\xi$, any $\alpha_0 < \omega_1$ and any $Y \in [\kappa]^\omega$, there exists $r \in R^\xi$ such that $r \leq p$ in $R^\xi$ (namely, as in $R$), $\alpha^r > \alpha_0$ and $X^r \supset Y$.*

*Proof.* Given $p \in R^\xi$, $\alpha_0 < \omega_1$ and $Y \in [\kappa]^\omega$. Take $X \in E_\xi$ such that $(Y \cup X^p) \cap \xi \subset X$ and $\alpha_0 < X \cap \omega_1 < \omega_1$. Construct $r \in R$ such that $\alpha^r = X \cap \omega_1$, $X^r \cap \xi = X$, $X^r \supset (X^p \cup Y)$, $r\lceil \xi = \langle \beta \mapsto g_\beta \lceil (X \cap \omega_1) \mid \beta \in X^r \cap \xi \rangle$ and $\langle \beta \mapsto g^r_\beta \mid \beta \in X^p \setminus \xi \rangle$ can be constructed so that $r \lceil X^p \leq p$ in $R$. Since $X \in E_\xi$, the maps $\langle \beta \mapsto g_\beta(\alpha) \mid \beta \in X \rangle$ are one-to-one and $|\, \omega \setminus \{g_\beta(\alpha) \mid \beta \in X\} \,| = \omega \geq |\, X^r \setminus \xi \,|$ for all $\alpha$ with $X \cap \omega_1 \leq \alpha < \omega_1$. Therefore, we conclude $r \in R^\xi$.

□

**6.6 Lemma.** *The p.o. set $R^\xi$ is $E_\xi$-complete. Namely, for all sufficiently large regular cardinals $\theta$ and all countable elementary substructures $N$ of $H_\theta$ such that $E_\xi, R^\xi \in N$ and $N \cap \xi \in E_\xi$, if $\langle p_n \mid n < \omega \rangle$ is a*

*$(R^\xi, N)$-generic sequence, then there exists $p \in R^\xi$ such that for all $n < \omega$, we have $p \leq p_n$. In addition to this, we may construct $p$ with $\alpha^p = N \cap \omega_1$ and $X^p = N \cap \kappa$.*

*Proof.* Let $\theta$ be a regular cardinal with $\theta \geq \kappa^+$ so that $R^\xi \in H_\theta$. Let $N$ and $p_n$ be as in the hypothesis. To define $p \in R^\xi$, let $X^p = \bigcup\{X^{p_n} \mid n < \omega\}$. For $\beta \in X^p$, let $g^p_\beta = \bigcup\{g^{p_n}_\beta \mid \beta \in X^{p_n}, n < \omega\}$. Then by the density of $R^\xi$, we indeed have $p \in R$ with $X^p = N \cap \kappa$ and $\alpha^p = N \cap \omega_1$. Since $N \cap \xi \in E_\xi$, the maps $\langle \beta \mapsto g_\beta(\alpha) \mid \beta \in X^p \cap \xi \rangle$ are one-to-one and $|\,\omega \setminus \{g_\beta(\alpha) \mid \beta \in X^p \cap \xi\}\,| = \omega \geq |\,X^p \setminus \xi\,|$ for all $\alpha$ with $X^p \cap \omega_1 \leq \alpha < \omega_1$. Therefore, we conclude $p \in R^\xi$ and $p \leq p_n$ for all $n < \omega$.

□

Suppose we have two conditions $p_1$ and $p_2$ in $R^\xi$ such that $\alpha^{p_1} = \alpha^{p_2}$ and for all $\beta \in X^{p_1} \cap X^{p_2}$, we have $g^{p_1}_\beta = g^{p_2}_\beta$. Let $p = p_1 \cup p_2$, then $p \in R$ and is a common extension of $p_1$ and $p_2$. But we can not expect $p \in R^\xi$ in general, as it may not hold that $|\,\omega \setminus \{g_\beta(\alpha) \mid \beta \in X^p \cap \xi\}\,| \geq |\,X^p \setminus \xi\,|$. The maps $\langle \beta \mapsto g_\beta(\alpha) \mid \beta \in X^p \cap \xi \rangle$ may not be one-to-one, either. However the following is handy, when we establish the chain condition of $R^\xi$ under the $\Delta$-system Lemma.

**6.7 Corollary.** *For any $p \in R^\xi$, there exists $q \leq p$ in $R^\xi$ such that for all $\alpha$ with $\alpha^q \leq \alpha < \omega_1$, we have $|\,\omega \setminus \{g_\beta(\alpha) \mid \beta \in X^q \cap \xi\}\,| = \omega$.*

*Proof.* For $p$, take $q \leq p$ which is $(R^\xi, N)$-generic as in 6.6 Lemma.

□

We show that $R^\xi$ has the $\kappa$-c.c., assuming that $\kappa$ is strongly inaccessible. This would be sufficient for our purposes. However, since $R$ has the $\omega_2$-c.c. (under CH) and $R$, $R_\xi * \dot{R}^\xi$ are forcing equivalent, we may improve it to the $\omega_2$-c.c. in the extension $V[G_\xi]$.

**6.8 Lemma.** *Let us recall $\kappa$ is assumed strongly inaccessible. Then $R^\xi$ has the $\kappa$-c.c. More precisely, given $\langle p_i \mid i < \kappa \rangle$, there exists $\langle q_i \mid i \in I \rangle$ such that $I \in [\kappa]^\kappa$, for all $i \in I$, $q_i \leq p_i$ in $R^\xi$ and the $q_i$'s are pairwise compatible in $R^\xi$.*

*Proof.* If two conditions $q_1, q_2$ in $R^\xi$ satisfy the following, then $q_1 \cup q_2 \in R^\xi$ and is a common extension in $R^\xi$.

- $\alpha^{q_1} = \alpha^{q_2}$,
- $X^{q_1} \cap \xi = X^{q_2} \cap \xi$,
- $g^{q_1}_\beta = g^{q_2}_\beta$ for all $\beta \in (X^{q_1} \cap X^{q_2}) \setminus \xi$,
- For all $\alpha$ with $\alpha^{q_1} \leq \alpha < \omega_1$, we have $|\,\omega \setminus \{g_\beta(\alpha) \mid \beta \in X^{q_1} \cap \xi\}\,| = \omega$. So the same must hold for $q_2$, too.

Notice that we have $\omega \geq |\,(X^{q_1} \setminus \xi) \cup (X^{q_2} \setminus \xi)\,|$.

Since $\kappa$ is strongly inaccessible, given $\langle p_i \mid i < \kappa \rangle$, we may get the $q_i$ and $I$ as claimed.

□

We prepare for intermediate stages.

**6.9 Corollary.** *Let $R^\xi$ and $\kappa$ be as above. Then in $V[R_\xi][\mathrm{Lv}(\xi, \omega_1)]$, it holds that*

*(1)* $R^\xi \times \mathrm{Lv}([\xi, \kappa), \omega_1)$ *is $E_\xi$-complete.*

*(2)* $R^\xi \times \mathrm{Lv}([\xi, \kappa), \omega_1)$ *has the $\kappa$-c.c.*

*Proof.* For (1): Since $R^\xi$ remains $E_\xi$-complete and $\mathrm{Lv}([\xi, \kappa), \omega_1)$ is $\sigma$-closed, we may show their product is $E_\xi$-complete.

For (2): Since $R^\xi$ still has the stronger type of the $\kappa$-c.c. and so does $\mathrm{Lv}([\xi, \kappa), \omega_1)$, their product certainly has the $\kappa$-c.c.

□

We extract a combinatorial property of $\langle g_\beta \mid \beta < \xi\rangle$ in $V[G_\xi]$ which would guarantee an $\omega_2$-c.c. of $R^\xi$.

**6.10 Definition.** Let $X_1, X_2 \in E_\xi$. We say $X_1$ and $X_2$ are *amalgable*, if there exists $(X, h)$ such that

- $(X_1 \cup X_2) \subseteq X \in E_\xi$ and $X_1 \cap \omega_1 = X_2 \cap \omega_1 \leq X \cap \omega_1$,
- $h$ is a bijection from $X_1$ onto $X_2$ such that $h$ is the identity on $X_1 \cap X_2$,
- For all $\beta \in X_1 \setminus (X_1 \cap X_2)$, $g_\beta$ and $g_{h(\beta)}$ agree on the interval $[X_1 \cap \omega_1, X_\cap \omega_1)$.

For any $X \in E_\xi$, $X$ and $X$ are amalgable. It is also easy to see that if $X_1$ and $X_2$ are amalgable, then so are $X_2$ and $X_1$.

In addition to the stationarity, $E_\xi$ has the following property of amalgable pairs in $V[G_\xi]$.

**6.11 Lemma.** *Suppose CH holds in $V$, then in $V[G_\xi]$, for any sequence $\langle X_i \mid i < \omega_2\rangle$ such that for all $i < \omega_2, X_i \in E_\xi$, there exist two distinct indices $i$ and $i'$ such that $X_i$ and $X_{i'}$ are amalgable.*

*Proof.* Suppose $p \Vdash_{R_\xi}$ "$\langle \dot{X}_i \mid i < \omega_2\rangle$ is a sequence such that for all $i < \omega_2$, $\dot{X}_i \in \dot{E}_\xi$". For each $i < \omega_2$, we take $p_i \leq p$ in $R_\xi$ to decide the value of $\dot{X}_i$ to be $X_i$. We may assume $X_i \subseteq X^{p_i}$ and $X_i \cap \omega_1 \leq \alpha^{p_i} = X^{p_i} \cap \omega_1$. Apply the $\Delta$-system Lemma to $\langle X^{p_i} \mid i < \omega_2\rangle$. We concentrate on two distinct indices $i$ and $i'$. We may assume the following.

- $\alpha^{p_i} = \alpha^{p_{i'}}$ and $X_i \cap \omega_1 = X_{i'} \cap \omega_1$,
- There exists a bijection $h : X^{p_i} \longrightarrow X^{p_{i'}}$ such that $h$ on $X^{p_i} \cap X^{p_{i'}}$ is the identity and $h''X_i = X_{i'}$,
- For all $\beta \in X^{p_i} \setminus (X^{p_i} \cap X^{p_{i'}})$, $g^{p_i}_\beta$ and $g^{p_{i'}}_{h(\beta)}$ agree on $[X_i \cap \omega_1, \alpha^{p_i})$,
- $p_i$ and $p_{i'}$ agree on $X^{p_i} \cap X^{p_{i'}}$.

Let $X = X^{p_i} \cup X^{p_{i'}}$. Then we may take $r \leq (p_i \cup p_{i'})$ in $R_\xi$ such that $\alpha^r = \alpha^{p_i}$ and $r \Vdash_{R_\xi}$ "$X \in \dot{E}_\xi$". Now we may observe $r \Vdash_{R_\xi}$ "$\dot{X}_i$ and $\dot{X}_{i'}$ are amalgable due to $(X, h\lceil X_i)$".

□

**6.12 Lemma.** *(CH) If $E_\xi$ satisfies the additional property on amalgable pairs as in 6.11 Lemma, then $R^\xi$ satisfies the $\omega_2$-c.c.*

*Proof.* Let $\langle p_i \mid i < \omega_2\rangle$ be a sequence of conditions of $R^\xi$. Since $E_\xi$ is stationary, we may take $q_i \leq p_i$ in $R^\xi$ such that $X^{q_i} \cap \xi \in E_\xi$. Apply the $\Delta$-system Lemma to $\langle X^{q_i} \mid i < \omega_2\rangle$. We may assume that $\alpha^{q_i}$ are constant and so are $q_i$ on the kernel of the $\Delta$-system. Then by assumption, there exist two distinct indices $i$ and $i'$ such that $X^{q_i} \cap \xi$ and $X^{q_{i'}} \cap \xi$ are amalgable with $(X, h)$. Then we may construct $r \in R^\xi$, by placing appropriate values to $g^r_\beta\lceil[\alpha^{q_i}, X \cap \omega_1)$ for $\beta \in (X^{q_i} \cup X^{q_{i'}}) \setminus \xi$, so that $r \leq q_i, q_{i'}$, $\alpha^r = X \cap \omega_1$, $X^r \cap \xi = X \in E_\xi$ and $X^r \setminus \xi = (X^{q_i} \cup X^{q_{i'}}) \setminus \xi$. In particular, $p_i$ and $p_{i'}$ are compatible in $R^\xi$.

□

I do not know whether $R^\xi$ satisfies the stronger $\omega_2$-c.c. Namely, given $\langle p_i \mid i < \omega_2\rangle$, there exists $I \in [\omega_2]^{\omega_2}$ such that for any two $i$ and $i'$ in $I$, $p_i$ and $p_{i'}$ are compatible in $R^\xi$.

## §7. Main Theorems

Now we are ready to provide details to our main theorem of section 3. We restate it as follows;

**7.1 Theorem.** *Let $\kappa$ be a strongly inaccessible cardinal. Then we have $\neg$club-wKH but TH in the generic extensions $V[\mathrm{Lv}(\kappa, \omega_1) \times R]$, where $R$ is defined in 5.1 Definition.*

*Proof.* By contradiction. Suppose $\langle C_\beta \mid \beta < \kappa\rangle$, $\langle b_\beta \mid \beta < \kappa\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ are given so that club-wKH holds in $V[\mathrm{Lv}(\kappa, \omega_1) \times R] = V[R \times \mathrm{Lv}(\kappa, \omega_1)]$. Then by the $\kappa$-c.c. of the product $R \times \mathrm{Lv}(\kappa, \omega_1)$, we have an ordinal $\xi$ such that $\omega_1 \leq \xi < \kappa$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ is in $V[R_\xi \times \mathrm{Lv}(\xi, \omega_1)]$.

In the intermediate stage $V[R_\xi \times \mathrm{Lv}(\xi, \omega_1)] = V[\mathrm{Lv}(\xi, \omega_1)][R_\xi]$, the tail $R^\xi \times \mathrm{Lv}([\xi, \kappa), \omega_1)$ is $E_\xi$-complete and has the $\kappa$-c.c. by 6.9 Corollary. Let $V_1 = V[R_\xi \times \mathrm{Lv}(\xi, \omega_1)]$ and $Q = R^\xi \times \mathrm{Lv}([\xi, \kappa), \omega_1)$ for a simpler notation.

Fix any $\beta < \kappa$ and let $b = b_\beta$ and $C = C_\beta$. Then $b : \omega_1 \longrightarrow 2$, $C$ is a club in $\omega_1$ and for all $\alpha \in C$, we assume $b\lceil\alpha \in S_\alpha$ in $V_1[Q]$. Since $\langle S_\alpha \mid \alpha < \omega_1 \rangle \in V_1$, we know that $b \in V_1$ by 4.5 Lemma. Since $Q$ has the $\kappa$-c.c., we have $\kappa = |\{b_\beta \mid \beta < \kappa\}| \leq |({}^{\omega_1} 2)^{V_1}| < \kappa$ in $V_1[Q]$. This is a contradiction.

□

**7.2 Theorem.** *Let $\kappa$ be a strongly inaccessible cardinal. Then in $V[\mathrm{Lv}(\kappa, \omega_1)]$, we have $(*)$-wKH but for all stationary subsets $F$ of $\omega_1$, we have the failure of club-wKH($F$).*

*Proof.* We have seen in [M] that $(*)$-wKH holds in $V[\mathrm{Lv}(\kappa, \omega_1)]$. We show that club-wKH($F$) must fail for all stationary $F \subseteq \omega_1$ in $V[\mathrm{Lv}(\kappa, \omega_1)]$. Our proof is identical to that of 7.1 Theorem. However, we write it down for the sake of clarity.

We proceed by contradiction. Suppose $F$, $\langle S_\alpha \mid \alpha < \omega_1 \rangle$, $\langle b_\beta \mid \beta < \kappa \rangle$ and $\langle C_\beta \mid \beta < \kappa \rangle$ are given so that club-wKH($F$) holds in $V[\mathrm{Lv}(\kappa, \omega_1)]$. Then by the $\kappa$-c.c. of $\mathrm{Lv}(\kappa, \omega_1)$, we have an ordinal $\xi$ such that $\omega_1 \leq \xi < \kappa$ and $F, \langle S_\alpha \mid \alpha < \omega_1 \rangle$ are in $V[\mathrm{Lv}(\xi, \omega_1)]$.

In the intermediate stage $V[\mathrm{Lv}(\xi, \omega_1)]$, the tail $\mathrm{Lv}([\xi, \kappa), \omega_1)$ is $\sigma$-closed and has the $\kappa$-c.c. Let $V_1 = V[\mathrm{Lv}(\xi, \omega_1)]$ and $Q = \mathrm{Lv}([\xi, \kappa), \omega_1)$ for a simpler notation. Note that $F$ is stationary, $\kappa$ remains strongly inaccessible and $Q$ is forcing equivalent to the whole $\mathrm{Lv}(\kappa, \omega_1)$ in $V_1$.

Fix any $\beta < \kappa$ and let $b = b_\beta$ and $C = C_\beta$. Then $b : \omega_1 \longrightarrow 2$, $C$ is a club in $\omega_1$ and for all $\alpha \in F \cap C$, we assume $b\lceil\alpha \in S_\alpha$ in $V_1[Q]$. Since $F$, $\langle S_\alpha \mid \alpha < \omega_1 \rangle \in V_1$, we know that $b \in V_1$ by 4.6 Lemma. Since $Q$ has the $\kappa$-c.c., we have $\kappa = |\{b_\beta \mid \beta < \kappa\}| \leq |({}^{\omega_1} 2)^{V_1}| < \kappa$ in $V_1[Q]$. This is a contradiction.

□

**7.3 Corollary.** *The following theories are all equiconsistent.*

*(1) Con(ZFC + there exists a strongly inaccessible cardinal).*

*(2) Con(ZFC + $\neg$club-wKH + TH).*

*(3) Con$\big($ZFC + for all stationary subsets $F$ of $\omega_1$, $\neg$club-wKH($F$) + $(*)$-wKH$\big)$.*

*(4) Con(ZFC + $\neg$wKH).*

*(5) Con(ZFC + $\neg$KH).*

*Proof.* (1) implies (2), (3), (4) and (5). Conversely, (2) or (3) or (4) imply (5). It is known ([Ku]) that (5) implies (1).

□

The following makes use of a measurable cardinal to separate the club-wKH($F$) and $(*)$-wKH.

**7.4 Theorem.** *Let $\kappa$ be a measurable cardinal. Then we have $(*)$-wKH and SCC in the generic extensions $V[\mathrm{Lv}(\kappa, \omega_1)]$.*

*Proof.* We know $(*)$-wKH holds in $V[\mathrm{Lv}(\kappa, \omega_1)]$ by [M]. We also know that SCC holds in $V[\mathrm{Lv}(\kappa, \omega_1)]$ by [S].

□

**7.5 Corollary.** *Con(ZFC + there exists a measurable cardinal) implies Con$\big($ $(*)$-wKH + for all stationary subsets $F$ of $\omega_1$, $\neg$TH($F$)$\big)$ and so Con($(*)$-wKH + $\neg$TH) holds.*

□

**A summary of implications**

$$\begin{array}{ccccccccccccc}
\mathrm{KH} & \rightarrow & \text{club-wKH} & \Rightarrow & \text{club-wKH}(F) & \rightarrow & (*)\text{-wKH} & \Rightarrow & \tilde{\tilde{\Diamond}} & \Rightarrow & \text{stat-wKH} & \Rightarrow & \mathrm{wKH} \\
 & & \downarrow & & \Downarrow & & & & & & & & \\
 & & \mathrm{TH} & & \mathrm{TH}(F) & & & & & & & & \\
 & & \Downarrow & & \Downarrow & & & & & & & & \\
 & & \neg\mathrm{CC} & & \neg\mathrm{SCC} & & & & & & & &
\end{array}$$

The symbol $\Rightarrow$ indicates a logical implication in ZFC.
The symbol $\rightarrow$ means $\Rightarrow$ and the converse consistently fails.

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