Formulas with only one variable in Grzegorczyk logic

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Abstract. Grzegorczyk logic, **GRZ**, is the normal modal logic obtained by adding Grzegorczyk axiom $\Box(\Box(p \supset \Box p) \supset p) \supset p$ to the smallest normal modal logic **K**. The quotient set of the set of formulas modulo the provability of **GRZ** is Boolean with respect to the derivation of **GRZ** (cf. Chagrov and Zakharyaschev [CZ97]). Here we give an inductive construction of the representatives of the quotient set of the set of formulas with only one propositional variable p and with a finite number of occurrences of $\Box$.

1 Preliminaries

We use lower case Latin letters p, q, r for propositional variables. Formulas are defined inductively, as usual, from the propositional variables and $\bot$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\Box$ (necessitation). By $\mathbf{S}(p)$, we mean the set of formulas constructed from p by using $\wedge$, $\vee$ $\supset$ and $\Box$. By **GRZ**, we mean the smallest set of formulas containing all the tautologies and the axioms

$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$,

$Grz : \Box(\Box(p \supset p) \supset p) \supset p$ (Grzegorczyk axiom),

and closed under modus ponens, substitution and necessitation.

We introduce a sequent system for **GRZ** given in Avron [Avr84]. We use Greek letters, Γ and Δ, possibly with suffixes, for finite sets of formulas. The expression $\Box\Gamma$ denotes the set $\{\Box A \mid A \in \Gamma\}$. By a sequent, we mean the expression $\Gamma \to \Delta$. For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

By **GGRZ**, we mean the system defined by the following axioms and inference rules in the usual way.

Axioms of GGRZ:

$$A \to A$$

$$\bot \to$$

Inference rules of GGRZ:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to w)$$

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}(cut)$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Pi \to \Lambda}{A \supset B, \Gamma, \Pi \to \Delta, \Lambda}(\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to\supset)$$

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$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to) \qquad\qquad \frac{\Box(A \supset \Box A), \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\to \Box)$$

Definition 1.1. The set $\mathsf{SubFig}(\mathcal{P})$ of a proof figure in **GGRZ** is defined as follows:
(1) $\mathsf{SubFig}(\mathcal{P}) = \{\mathcal{P}\}$ if $\mathcal{P}$ consists of only one axiom,
(2) $\mathsf{SubFig}(\frac{\mathcal{P}_1}{S}) = \mathsf{SubFig}(\mathcal{P}_1) \cup \{\mathcal{P}\}$ if $\mathcal{P} = \frac{\mathcal{P}_1}{S}$,
(3) $\mathsf{SubFig}(\frac{\mathcal{P}_1 \quad \mathcal{P}_2}{S}) = \mathsf{SubFig}(\mathcal{P}_1) \cup \mathsf{SubFig}(\mathcal{P}_2) \cup \{\mathcal{P}\}$ if $\mathcal{P} = \frac{\mathcal{P}_1 \quad \mathcal{P}_2}{S}$.

Let $\mathcal{P}$ be a proof figure in **GGRZ**. We note that each element in $\mathsf{SubFig}(\mathcal{P})$ is a proof figure in **GGRZ**. A proof figure in $\mathsf{SubFig}(\mathcal{P})$ is called a subfigure of $\mathcal{P}$. A subfigure $\mathcal{Q}$ of $\mathcal{P}$ is called a proper subfigure of $\mathcal{P}$ if $\mathcal{P} \neq \mathcal{Q}$.

Lemma 1.2([Avr84]).
(1) $\Gamma \to \Delta \in$ **GGRZ** *if and only if* $\bigwedge_{A \in \Gamma} A \supset \bigvee_{B \in \Delta} B \in$ **GRZ**.
(2) *If* $\Gamma \to \Delta \in$ **GGRZ**, *then there exists a cut-free proof figure for* $\Gamma \to \Delta$ *in* **GGRZ** .

By the lemma above, we can identify **GGRZ** with **GRZ**. So, if there is no confusion, we use the sequent system **GGRZ** instead of **GRZ**.

Definition 1.3. The depth $d(A)$ of a formula $A \in \mathbf{S}(p)$ is defined inductively as follows:
(1) $d(p) = 0$,
(2) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
(3) $d(\Box B) = d(B) + 1$.

We put $\mathbf{S}^n(p) = \{A \in \mathbf{S}(p) \mid d(A) \leq n\}$. Immediately, we note that $\mathbf{S}(p) = \bigcup_{n=0}^{\infty} \mathbf{S}^n(p)$.

2 Main results

For formulas A and B, we use the expression $A \equiv B$ instead of $(A \supset B) \wedge (B \supset A) \in$ **GRZ**. We note that $\equiv$ is an equivalence relation on a set $\mathbf{S}$ of formulas. We write $[A] \leq [B]$ if there exist $A' \in [A]$ and $B' \in [B]$ such that $B' \supset A' \in$ **GRZ**. Our main purpose is to give a concrete representative of each equivalence class of $\mathbf{S}^n(p)$ in an inductive way and elucidate the structure $\langle \mathbf{S}^n(p), \leq \rangle$.

Definition 2.1. Formulas F_n $(n = 0, 1, 2, \cdots)$ are defined inductively as follows:
$F_0 = p$,
$F_1 = p \supset \Box p$,
$F_{k+2} = F_k \vee \Box F_{k+1}$.

We note that $F_n \in \mathbf{S}^n(p)$.

Definition 2.2. The sets $\mathbf{G}_n$ $(n = 0, 1, 2, \cdots)$ of formulas are defined inductively as follows:
$\mathbf{G}_0 = \{F_0\}$,
$\mathbf{G}_1 = \{F_0, F_1\}$,
$\mathbf{G}_{k+2} = (\mathbf{G}_{k+1} - \{F_k\}) \cup \{F_{k+2}, F_k \vee (\Box F_{k+1} \supset \Box p)\}$.

We note that $\mathbf{G}_n$ has just $n + 1$ elements.

Theorem 2.3.
(1) $\mathbf{S}^n(p)/\equiv = \{[\bigwedge_{A \in \mathbf{S}} A] \mid \mathbf{S} \subseteq \mathbf{G}_n\}$.
(2) *For subsets* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *of* $\mathbf{G}_n$,

(2.1) $\mathbf{S}_1 \subseteq \mathbf{S}_2$ *if and only if* $[\bigwedge_{A\in\mathbf{S}_1} A] \leq [\bigwedge_{A\in\mathbf{S}_2} A]$,

(2.2) $\mathbf{S}_1 = \mathbf{S}_2$ *if and only if* $[\bigwedge_{A\in\mathbf{S}_1} A] = [\bigwedge_{A\in\mathbf{S}_2} A]$.

(3) $\mathbf{S}^n(p)/\equiv$ *has just* 2^{n+1} *elements.*

To prove the theorem above, we need some lemmas.

Lemma 2.4.
(1) $\Box A \supset A \in \mathbf{GRZ}$,
(2) $\Box\Box A \equiv \Box A$,
(3) $\Box(A \wedge B) \equiv (\Box A \wedge \Box B)$.

Proof. By **GGRZ**. $\dashv$

Lemma 2.5. *Let A and B be formulas in $\mathbf{S}(p)$. Then*
(1) $\Box p \supset A \in \mathbf{GRZ}$,
(2) $(A \supset B) \equiv ((A \supset \Box p) \vee B)$,
(3) $A \equiv ((A \supset \Box p) \supset \Box p)$.

Proof. For (1). We use an induction on A. If $A = p$, then (1) is clear from Lemma 2.1(1). Also by the following four figures, if $\Box p \supset B$ and $\Box p \supset C$ are provable in **GRZ**, then so are four formulas $\Box p \supset B \wedge C, \Box p \supset B \vee C, \Box p \supset B \supset C$ and $\Box p \supset \Box B$.

$$\dfrac{\Box p \to B \quad \Box p \to C}{\Box p \to B \wedge C}\,(\to \wedge) \qquad \dfrac{\Box p \to C}{\Box p \to B \vee C}\,(\to \vee_2) \qquad \dfrac{\dfrac{\Box p \to C}{B, \Box p \to C}\,(w \to)}{\Box p \to B \supset C}\,(\to\supset) \qquad \dfrac{\dfrac{\Box p \to B}{\Box(B \to \Box B), \Box p \to B}\,(w \to)}{\Box p \to \Box B}\,(\to \Box)$$

For (2). By the following figures and (1), we obtain (2).

$$\dfrac{\dfrac{\dfrac{A \to A \quad B \to B}{A, A \supset B \to B}\,(\supset\to)}{A \supset B \to A \supset \Box p, B}\,(\to\supset), (\to w)}{A \supset B \to (A \supset \Box p) \vee B}\,(\to \vee_2), (\to \vee_1) \qquad \dfrac{\dfrac{\dfrac{A \to A \quad \Box p \to B}{A, A \supset \Box p \to B}\,(\supset\to)}{A \supset \Box p \to A \supset B}\,(\to\supset) \quad \dfrac{B \to B}{B \to A \supset B}\,(\to\supset), (w \to)}{(A \supset \Box p) \vee B \to A \supset B}\,(\vee \to)$$

For (3). By the following figures and (1), we obtain (3).

$$\dfrac{\dfrac{A \to A \quad \Box p \to \Box p}{A, A \supset \Box p, \Box p}\,(\supset\to)}{A \to (A \supset \Box p) \supset \Box p}\,(\to\supset) \qquad \dfrac{\dfrac{A \to A}{\to A \supset \Box p, A}\,(\to\supset), (\to w) \quad \Box p \to A}{(A \supset \Box p) \supset \Box p \to A}\,(\supset\to)$$

$\dashv$

Lemma 2.6. *For $n > 0$,*

$$\mathbf{G}_n = \{F_n, F_{n-1}\} \cup \{F_k \vee (\Box F_{k+1} \supset \Box p) \mid 0 \leq k \leq n-2\}.$$

Proof. We use an induction on n. If $n = 1$, then the lemma is clear by the definition. Suppose that $n > 1$ and

$$\mathbf{G}_{n-1} = \{F_{n-1}, F_{n-2}\} \cup \{F_k \vee (\Box F_{k+1} \supset \Box p) \mid 0 \leq k \leq n-3\}.$$

Then

$$\mathbf{G}_n = (\mathbf{G}_{n-1} - \{F_{n-2}\}) \cup \{F_n, F_{n-2} \vee (\Box F_{n-1} \supset \Box p)\}$$
$$= \{F_{n-1}\} \cup \{F_k \vee (\Box F_{k+1} \supset \Box p) \mid 0 \leq k \leq n-3\} \cup \{F_n, F_{n-2} \vee (\Box F_{n-1} \supset \Box p)\}$$

$$= \{F_{n-1}, F_n\} \cup \{F_k \vee (\Box F_{k+1} \supset \Box p) \mid 0 \le k \le n-2\}.$$

⊣

Lemma 2.7. $\Box F_n \supset \Box F_{n+1} \in$ **GRZ**.

Proof. The case $n = 0$ is shown by the following figure on the left-hand side, and other cases, by the figure on the right-hand side (see also Figure 1):

$$\dfrac{\dfrac{\Box p \to \Box p}{\Box p \to p \supset \Box p}\,(\supset\to), (w\to)}{\Box p \to \Box(p \supset \Box p)}\,(\to\Box), (w\to) \qquad \dfrac{\dfrac{\Box F_n \to \Box F_n}{\Box F_n \to F_{n-1} \vee \Box F_n}\,(\to\vee_2)}{\Box F_n \to \Box(F_{n-1} \vee \Box F_n)}\,(\to\Box), (w\to)$$

⊣

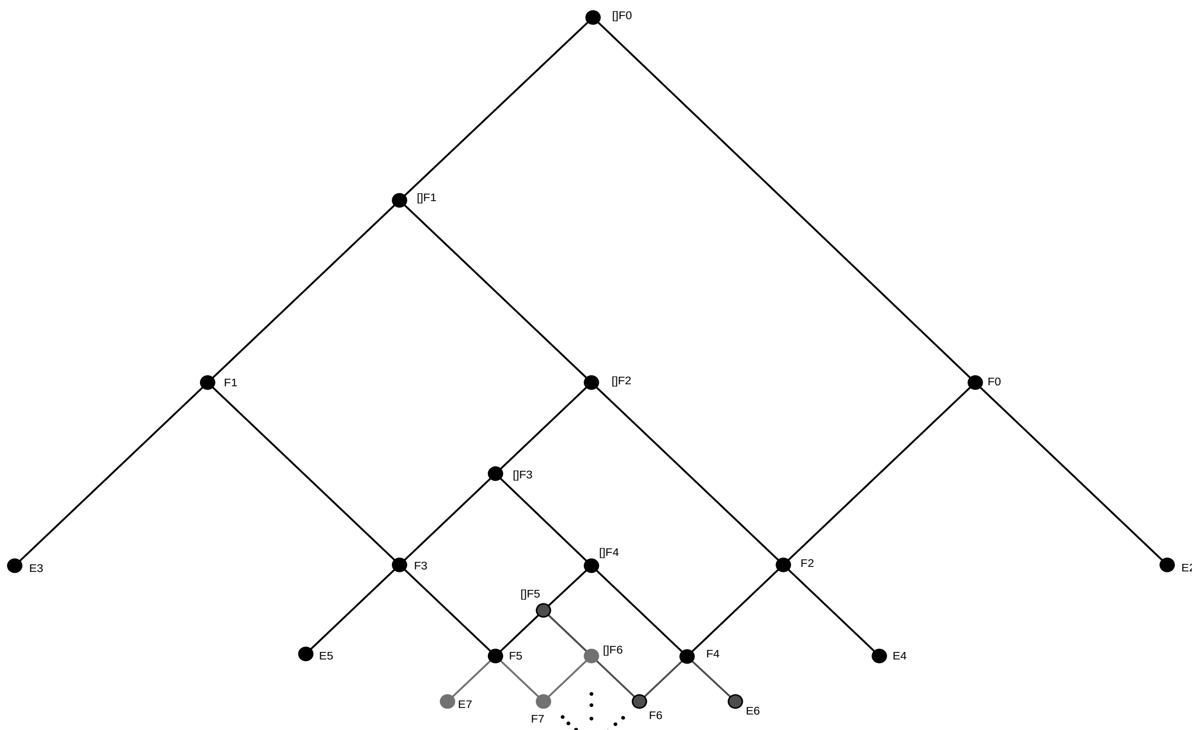

Figure 1: Hasse diagram of $\langle \bigcup_{k=1}^{n} \mathbf{G}_k, \le \rangle$, where $E_{k+2} = F_k \vee (\Box F_{k+1} \supset \Box p)$

For $m \ge -1$, we put

$$\mathbf{F}_{2m} = \{F_{2i} \mid 0 \le i \le m\},$$

$$\mathbf{F}_{2m+1} = \{F_{2i+1} \mid 0 \le i \le m\}.$$

For $n \ge 0$, we put

$$\mathbf{Ant}_n = \{F_i \mid n < i\} \cup \{\Box F_i \mid n < i\} \cup \{F_i \supset \Box F_i \mid n \le i\} \cup \{\Box(F_i \supset \Box F_i) \mid n \le i\},$$

$$\mathbf{Seq}_n = \{\Gamma \to \Delta \mid \Gamma \text{ is a finite subset of } \mathbf{Ant}_n \cup \mathbf{F}_{n-1}, \Delta \text{ is a finite subset of } \mathbf{F}_n \cup \Box\mathbf{F}_{n-1} \cup \{\Box p\}\},$$

$$\mathbf{Seq} = \bigcup_{n=0}^{\infty} \mathbf{Seq}_n.$$

Lemma 2.8. *Let $\mathcal{P}$ be a cut-free proof figure in* **GGRZ**. *Then none of the sequents in* **Seq** *is the end sequent of $\mathcal{P}$.*

Proof. We use an induction on $\mathcal{P}$.

If $\mathcal{P}$ consists of only one axiom, then the end sequent of $\mathcal{P}$ is an axiom. However, we note that

$$(\mathbf{Ant}_n \cup \mathbf{F}_{n-1}) \cap (\mathbf{F}_n \cup \Box\mathbf{F}_{n-1} \cup \{\Box p\}) = \emptyset \text{ and } \bot \notin \mathbf{Ant}_n \cup \mathbf{F}_{n-1}.$$

So, none of the sequents in **Seq** is an axiom, and hence the end sequent of $\mathcal{P}$.

Suppose that $\mathcal{P}$ has proper subfigures $\mathcal{P}_1, \cdots, \mathcal{P}_k$ $(k = 1, 2)$ such that

$$\mathcal{P} = \frac{\mathcal{P}_1 \quad \cdots \quad \mathcal{P}_k}{S},$$

where S is the end sequent of $\mathcal{P}$ and none of the sequents in **Seq** is the end sequents of any proper subfigures of $\mathcal{P}$. Let I be the inference rule introducing the end sequent of $\mathcal{P}$. We also suppose that the end sequent of $\mathcal{P}$ belongs to $\mathbf{Seq}_n$ for some $n \geq 0$. We divide the cases.

The case that I is either $(w \rightarrow)$ or $(\rightarrow w)$. We note the upper sequent of I, the end sequent of $\mathcal{P}_1$, also belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\Box \rightarrow)$. The principal formula of I is $\Box F_i$ $(n < i)$ or $\Box(F_i \supset \Box F_i)$ $(n \leq i)$ in $\mathbf{Ant}_n$. We note that the auxiliary formula, which is either F_i $(n < i)$ or $F_i \supset \Box F_i$ $(n \leq i)$, belongs to $\mathbf{Ant}_n$. Hence the upper sequent of I, the end sequent of $\mathcal{P}_1$, also belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\supset\rightarrow)$ and the principal formula of I is $F_n \supset \Box F_n$. Then the auxiliary formula F_n occurring in the succeedent of the left upper sequent of I belongs to $\mathbf{F}_n$. So, the left upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\supset\rightarrow)$ and the principal formula of I is $F_i \supset \Box F_i$ $(n < i)$. Then the auxiliary formula $\Box F_i$ occurring in the antecedent of the right upper sequent of I belongs to $\mathbf{Ant}_n$. So, the right upper sequent of I, the end sequent of $\mathcal{P}_2$, belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\supset\rightarrow)$ and the principal formula of I is F_1. We note that $F_1 \in \mathbf{F}_{n-1} \cup \{F_i \mid n < i\}$. If $F_1 \in \mathbf{F}_{n-1}$, then $n = 2, 4, 6, \cdots$. If $F_1 \in \{F_i \mid n < i\}$, then $n = 0$. So, we have $n = 0, 2, 4, \cdots$. Hence the auxiliary formula p $(= F_0)$ occurring in the succeedent of the left upper sequent of I belongs to $\mathbf{F}_n$. So, the left upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\vee \rightarrow)$ and the principal formula of I is $F_i = F_{i-2} \vee \Box F_{i-1} \in \mathbf{F}_{n-1}$. Then the auxiliary formula F_{i-2} occurring in the antecedent of the left upper sequent of I belongs to $\mathbf{F}_{n-1}$. So, the left upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\vee \rightarrow)$ and the principal formula of I is $F_{n+1} = F_{n-1} \vee \Box F_n \in \{F_j \mid n < j\}$. Then the auxiliary formula F_{n-1} occurring in the antecedent of the left upper sequent of I belongs to $\mathbf{F}_{n-1}$. So, the left upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\vee \rightarrow)$ and the principal formula of I is $F_i = F_{i-2} \vee \Box F_{i-1} \in \{F_j \mid n+1 < j\}$. Since $n + 1 < i$, we have $n < i - 1$. So the auxiliary formula $\Box F_{i-1}$ occurring in the antecedent of the right upper sequent of I belongs to $\mathbf{Ant}_n$. So, the right upper sequent of I, the end sequent of $\mathcal{P}_2$, belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\rightarrow \vee_1)$ and the principal formula of I is $F_i = F_{i-2} \vee \Box F_{i-1} \in \mathbf{F}_n$. Then the auxiliary formula F_{i-2} occurring in the succeedent of the upper sequent of I belongs to $\mathbf{F}_n$. So, the upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_n$ $(\subseteq \mathbf{Seq})$. This is in contradiction with the induction hypothesis.

The case that I is $(\rightarrow \vee_2)$ and the principal formula of I is $F_i = F_{i-2} \vee \Box F_{i-1} \in \mathbf{F}_n$. Then the auxiliary formula $\Box F_{i-1}$ occurring in the succeedent of the upper sequent of I belongs to $\Box\mathbf{F}_{n-1}$. So, the upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_n$ ($\subseteq$ **Seq**). This is in contradiction with the induction hypothesis.

The case that I is ($\rightarrow\supset$) and the principal formula of I is $F_1 = p \supset \Box p \in \mathbf{F}_n$. We note that n is an odd number. So, the auxiliary formula p ($= F_0$) occurring in the antecedent of the upper sequent of I belongs to $\mathbf{F}_{n-1}$. Also, the auxiliary formula $\Box p$ ($= \Box F_0$) occurring in the succeedent of the upper sequent of I belongs to $\Box\mathbf{F}_{n-1}$. So, the upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_n$ ($\subseteq$ **Seq**).

The case that I is ($\rightarrow \Box$) and the principal formula of I is $\Box F_i \in \Box\mathbf{F}_{n-1} \cup \{\Box p\}$. We note that $i \leq n-1$ and I is of the form of the following figure:

$$\frac{\Box(F_i \supset \Box F_i), \Gamma, \Pi \rightarrow F_i}{\Gamma, \Pi \rightarrow \Box F_i}\ I$$

where Γ and Π are finite subsets of $\{\Box F_j \mid n < j\}$ and $\{\Box(F_j \supset \Box F_j) \mid n \leq j\}$, respectively. The upper sequent of I, the end sequent of $\mathcal{P}_1$, belongs to $\mathbf{Seq}_i$ ($\subseteq$ **Seq**). This is in contradiction with the induction hypothesis. $\dashv$

Corollary 2.9. *None of the formulas in* $\mathbf{G}_n$ *is provable in* **GRZ**.

Proof. We note that

$$\rightarrow F_k \in \mathbf{Seq} \text{ and } \Box F_{k+1} \rightarrow F_k, \Box p \in \mathbf{Seq}.$$

Using Lemma 2.8, none of the above sequents is provable in **GGRZ**. So, none of the formulas

$$F_k \text{ and } F_k \vee (\Box F_{k+1} \supset \Box p)$$

is provable in **GRZ**. By Lemma 2.6, each member of $\mathbf{G}_n$ is of the form of the above two. So, we obtain the corollary. $\dashv$

Lemma 2.10. *For any different formulas* $A, B \in \mathbf{G}_n$*,* $A \vee B \in$ **GRZ**.

Proof. We use an induction on n. There is no two different formulas in $\mathbf{G}_0$. Also $F_0 \vee F_1$ is a tautology. So, the lemma holds if $n = 0, 1$. Suppose that $n > 1$ and the lemma holds for any $k < n$. Since

$$\mathbf{G}_n = (\mathbf{G}_{n-1} - \{F_{n-2}\}) \cup \{F_n, F_{n-2} \vee (\Box F_{n-1} \supset \Box p)\},$$

either one of the following holds, for different formulas A and B,

(1) both belong to $\mathbf{G}_{n-1}$,

(2) one belongs to $\mathbf{G}_{n-1} - \{F_{n-2}\}$ and the other belongs to $\{F_n, F_{n-2} \vee (\Box F_{n-1} \supset \Box p)\}$,

(3) one is F_n and the other is $F_{n-2} \vee (\Box F_{n-1} \supset \Box p)$.

If (1) holds, then by the induction hypothesis, we obtain the lemma. If (3) holds, then we also obtain the lemma since $F_n \vee (F_{n-2} \vee (\Box F_{n-1} \supset \Box p))$ is a tautology. Suppose that (2) holds. Without loss of the generality, we assume that $A \in \mathbf{G}_{n-1} - \{F_{n-2}\}$ and $B \in \{F_n, F_{n-2} \vee (\Box F_{n-1} \supset \Box p)\}$. By Lemma 2.6, we have $F_{n-2} \in \mathbf{G}_{n-1}$. Using the induction hypothesis, we have $A \vee F_{n-2} \in$ **GRZ**. On the other hand, we note that $F_{n-2} \supset B$ is a tautology. Hence we have $A \vee B \in$ **GRZ**. $\dashv$

Lemma 2.11. *For* $n > 0$, $\displaystyle\bigwedge_{C \in \mathbf{G}_n} C \equiv \Box p$.

Proof. We use an induction on n. We can easily see

$$(p \wedge (p \supset \Box p)) \equiv \Box p.$$

So, we have $(F_0 \wedge F_1) \equiv \Box p$.

Suppose that $n > 1$ and

$$\bigwedge_{C \in \mathbf{G}_{n-1}} C \equiv \Box p.$$

Since $F_{n-2} \in \mathbf{G}_{n-1}$,

$$(\bigwedge_{C \in \mathbf{G}_{n-1} - \{F_{n-2}\}} C) \wedge F_{n-2} \equiv \Box p$$

On the other hand, by Lemma 2.5(1), we can see

$$F_{n-2} \equiv ((F_{n-2} \vee \Box F_{n-1}) \wedge (F_{n-2} \vee (\Box F_{n-1} \supset \Box p))),$$

and hence

$$F_{n-2} \equiv (F_n \wedge (F_{n-2} \vee (\Box F_{n-1} \supset \Box p))).$$

Hence we obtain the lemma. $\dashv$

Lemma 2.12. *For any subset* $\mathbf{S}$ *of* $\mathbf{G}_n$*,*

$$(\bigwedge_{C \in \mathbf{S}} C) \supset \Box p \equiv \bigwedge_{D \in \mathbf{G}_n - \mathbf{S}} D.$$

Proof. By Lemma 2.11, we have

$$\bigwedge_{D \in \mathbf{G}_n - \mathbf{S}} D \supset ((\bigwedge_{C \in \mathbf{S}} C) \supset \Box p) \in \mathbf{GRZ}.$$

By Lemma 2.10, for different formulas $A, B \in \mathbf{G}_n$,

$$A \vee B \in \mathbf{GRZ}.$$

Using Lemma 2.5(3),

$$((A \supset \Box p) \supset \Box p) \vee B \in \mathbf{GRZ}.$$

Using Lemma 2.5(2),

$$(A \supset \Box p) \supset B \in \mathbf{GRZ}.$$

Hence for any $D \notin \mathbf{S}$,

$$\bigwedge_{C \in \mathbf{S}} ((C \supset \Box p) \supset D) \in \mathbf{GRZ},$$

and so,

$$((\bigwedge_{C \in \mathbf{S}} C) \supset \Box p) \supset D \in \mathbf{GRZ}.$$

Hence

$$((\bigwedge_{C \in \mathbf{S}} C) \supset \Box p) \supset (\bigwedge_{D \in \mathbf{G}_n - \mathbf{S}} D) \in \mathbf{GRZ}.$$

$\dashv$

Lemma 2.13. *For* $0 \leq m \leq n$*,*

$$F_{2m} \equiv (\bigwedge_{k=m}^{n-1} (F_{2k} \vee (\Box F_{2k+1} \supset \Box p))) \wedge F_{2n},$$

$$F_{2m+1} \equiv (\bigwedge_{k=m}^{n-1} (F_{2k+1} \vee (\Box F_{2k+2} \supset \Box p))) \wedge F_{2n+1},$$

Proof. We use an induction on n. If $n = m$, then the lemma is clear. Suppose that $n > m$ and

$$F_{2m} \equiv (\bigwedge_{k=m}^{n-2} (F_{2k} \vee (\Box F_{2k+1} \supset \Box p))) \wedge F_{2n-2},$$

$$F_{2m+1} \equiv (\bigwedge_{k=m}^{n-2} (F_{2k+1} \vee (\Box F_{2k+2} \supset \Box p))) \wedge F_{2n-1}.$$

By Lemma 2.5(1), we note the following two:

$$F_{2n-2} \equiv (F_{2n-2} \vee \Box F_{2n-1}) \wedge (F_{2n-2} \vee (\Box F_{2n-1} \supset \Box p)),$$

$$F_{2n-1} \equiv (F_{2n-1} \vee \Box F_{2n}) \wedge (F_{2n-1} \vee (\Box F_{2n} \supset \Box p)),$$

and so,

$$F_{2n-2} \equiv F_{2n} \wedge (F_{2n-2} \vee (\Box F_{2n-1} \supset \Box p)),$$

$$F_{2n-1} \equiv F_{2n+1} \wedge (F_{2n-1} \vee (\Box F_{2n} \supset \Box p)).$$

Using the induction hypothesis, we obtain the lemma. $\dashv$

Lemma 2.14.
(1) $F_n \wedge F_{n+1} \equiv \Box F_n$,
(2) $\Box(F_n \vee (\Box F_{n+1} \supset \Box p)) \equiv \Box F_n$.

Proof.
For (1). By Lemma 2.4(1),

$$\Box F_n \supset F_n \in \mathbf{GRZ}.$$

Also by Lemma 2.7 and Lemma 2.4(1), two formulas $\Box F_n \supset \Box F_{n+1}$ and $\Box F_{n+1} \supset F_{n+1}$ are provable in **GRZ**, and hence

$$\Box F_n \supset F_{n+1} \in \mathbf{GRZ}.$$

Hence

$$\Box F_n \supset F_n \wedge F_{n+1} \in \mathbf{GRZ}.$$

To prove $F_n \wedge F_{n+1} \supset \Box F_n \in \mathbf{GRZ}$, we show

$$F_n, F_{n+1} \to \Box F_n \in \mathbf{GGRZ},$$

by an induction on n. Clearly,

$$F_0, F_1 \to \Box F_1 \in \mathbf{GGRZ} \ (p, p \supset \Box p \to \Box p \in \mathbf{GGRZ}).$$

Suppose that $n > 0$ and

$$F_{n-1}, F_n \to \Box F_{n-1} \in \mathbf{GGRZ}.$$

Then by Lemma 2.7 and the following figure, we obtain (2).

$$\dfrac{\dfrac{F_n, F_{n-1} \to \Box F_{n-1} \quad \Box F_{n-1} \to \Box F_n}{F_n, F_{n-1} \to \Box F_n}\,(cut) \qquad \dfrac{\Box F_n \to \Box F_n}{F_n, \Box F_n \to \Box F_n}\,(w \to)}{F_n, F_{n+1} \to \Box F_n}\,(\vee \to)$$

For (2). By the figure

$$\dfrac{\dfrac{\dfrac{F_n \to F_n}{\Box F_n \to F_n}\,(\Box \to)}{\Box F_n \to F_n \vee (\Box F_{n+1} \supset \Box p)}\,(\to \vee_1)}{\Box F_n \to \Box(F_n \vee (\Box F_{n+1} \supset \Box p))}\,(\to \Box), (w \to),$$

we obtain

$$\Box F_n \supset \Box(F_n \vee (\Box F_{n+1} \supset \Box p)) \in \mathbf{GRZ}.$$

For the other direction, by Lemma 2.5(1) and the figure

$$
\dfrac{\dfrac{\dfrac{F_n \to F_n \qquad \dfrac{\dfrac{\dfrac{F_n \supset \Box F_n \to F_{n+1}}{\Box(F_n \supset \Box F_n) \to F_{n+1}}\,(\Box \to)}{\Box(F_n \supset \Box F_n) \to \Box F_{n+1}}\,(\to \Box),(w \to) \qquad \Box p \to F_n}{\Box(F_n \supset \Box F_n), \Box F_{n+1} \supset \Box p \to F_n}\,(\supset\to)}{\Box(F_n \supset \Box F_n), F_n \vee (\Box F_{n+1} \supset \Box p) \to F_n}\,(\vee \to),(w \to)}{\Box(F_n \supset \Box F_n), \Box(F_n \vee (\Box F_{n+1} \supset \Box p)) \to F_n}\,(\Box \to)}{\Box(F_n \vee (\Box F_{n+1} \supset \Box p)) \to \Box F_n}\,(\to \Box)
$$
,

it is sufficient to show

$$F_n \supset \Box F_n \to F_{n+1} \in \mathbf{GGRZ}.$$

If $n = 0$, then it is an axiom. If $n > 0$, then we consider the following figure

$$
\dfrac{\to F_{n+1}, F_n \qquad \dfrac{\Box F_n \to \Box F_{n+1} \quad \Box F_{n+1} \to F_{n+1}}{\Box F_n \to F_{n+1}}\,(cut)}{F_n \supset \Box F_n \to F_{n+1}}\,(\supset\to)
$$
.

By Lemma 2.6, $F_{n+1}, F_n \in \mathbf{G}_{n+1}$, and using Lemma 2.10, $\to F_{n+1}, F_n \in \mathbf{GGRZ}$. Also by Lemma 2.4 and Lemma 2.7, every sequent at the leaves of the above figure is provable. Hence we obtain $F_n \supset \Box F_n \to F_{n+1} \in \mathbf{GGRZ}$. $\dashv$

Lemma 2.15.
(1) *If* $A \in \Box\mathbf{G}_n$, *then* $A \equiv \Box F_i$ *for some* $i \in \{0, 1, \cdots, n\}$,
(2) *If* $0 \le m \le n$, *then there exists a subset* $\mathbf{S}$ *of* $\mathbf{G}_n$ *such that*

$$F_m \equiv \bigwedge_{B \in \mathbf{S}} B.$$

(3) *If* $A \in \Box\mathbf{G}_n$, *then there exists a subset* $\mathbf{S}$ *of* $\mathbf{G}_{n+1}$ *such that*

$$A \equiv \bigwedge_{B \in \mathbf{S}} B.$$

Proof. For (1). By Lemma 2.6,

$$A \in \{\Box F_n, \Box F_{n-1}\} \cup \{\Box(F_k \vee (\Box F_{k+1} \supset \Box p)) \mid 0 \le k \le n-2\}.$$

If $A \in \{\Box F_n, \Box F_{n-1}\}$, then (1) is clear. So, we assume that $A = \Box(F_k \vee (\Box F_{k+1} \supset \Box p))$ for some $k \in \{0, 1, \cdots, n-2\}$. Using Lemma 2.14(2), $A \equiv \Box F_k$, and hence we obtain (1).

For (2). If $m = 2m'$ $n = 2n'$ for some n' and m', then we have $m' \le n'$, and by Lemma 2.13,

$$F_m \ (= F_{2m'}) \equiv (\bigwedge_{k=m'}^{n'-1} (F_{2k} \vee (\Box F_{2k+1} \supset \Box p))) \wedge F_{2n'}.$$

We note that every conjuncts of the formula on the right-hand side of the above equation belongs to $\mathbf{G}_{2n'}$ $(= \mathbf{G}_n)$.

Also such conjuncts belong to $\mathbf{G}_{2n'+1}$. So, in a similar way we can show (2) if $m = 2m'$ $n = 2n' + 1$ for some n' and m'.

If $m = 2m' + 1$ $n = 2n'$ for some n' and m', then we have $m' \le n' - 1$, and by Lemma 2.13,

$$F_m \ (= F_{2m'+1}) \equiv (\bigwedge_{k=m'}^{n'-2} (F_{2k+1} \vee (\Box F_{2k+2} \supset \Box p))) \wedge F_{2n'-1}.$$

We note that every conjuncts of the formula on the right-hand side of the above equation belongs to $\mathbf{G}_{2n'}$ $(= \mathbf{G}_n)$.

If $m = 2m' + 1$ $n = 2n' + 1$ for some n' and m', then we have $m' \leq n'$, and by Lemma 2.13,

$$F_m \ (= F_{2m'+1}) \equiv (\bigwedge_{k=m'}^{n'-1} (F_{2k+1} \vee (\Box F_{2k+2} \supset \Box p))) \wedge F_{2n'+1}.$$

We note that every conjuncts of the formula on the right-hand side of the above equation belongs to $\mathbf{G}_{2n'+1}$ $(= \mathbf{G}_n)$.

For (3). By (1), $A \equiv \Box F_i$ for some $i \leq n$. Using Lemma 2.14(1),

$$A \equiv F_i \wedge F_{i+1}.$$

Using (2),

$$A \equiv \bigwedge_{B \in \mathbf{S}_1} B \wedge \bigwedge_{C \in \mathbf{S}_2} C,$$

for some subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_{n+1}$. Hence

$$A \equiv \bigwedge_{B \in \mathbf{S}_1 \cup \mathbf{S}_2} B.$$

$\dashv$

Lemma 2.16. *For any $A \in \mathbf{S}^n(p)$, there exists a subset $\mathbf{S}$ of $\mathbf{G}_n$ such that*

$$A \equiv \bigwedge_{A' \in \mathbf{S}} A'.$$

Proof. We use an induction on A.

Basis($A = p$). We note that there exists a number m (≥ 0) such that $n \in \{2m, 2m+1\}$. We put

$$\mathbf{P}_m = \{F_{2k} \vee (\Box F_{2k+1} \supset \Box p) \mid 0 \leq k \leq m-1\} \cup \{F_{2m}\}.$$

Then by Lemma 2.13,

$$p \equiv \bigwedge_{A' \in \mathbf{P}_m} A'.$$

Also we can easily see

$$\mathbf{P}_m \subseteq \mathbf{G}_{2m} \text{ and } \mathbf{P}_m \subseteq \mathbf{G}_{2m+1},$$

and hence

$$\mathbf{P}_m \subseteq \mathbf{G}_n.$$

Induction step($A \neq p$). Suppose that the lemma holds for any proper subformula of A. We divide the cases.

The case that $A = B \wedge C$. By the induction hypothesis, there exist subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_n$ such that

$$B \equiv \bigwedge_{B' \in \mathbf{S}_1} B' \text{ and } C \equiv \bigwedge_{C' \in \mathbf{S}_2} C'.$$

Hence

$$A \equiv (\bigwedge_{B' \in \mathbf{S}_1} B' \wedge \bigwedge_{C' \in \mathbf{S}_2} C'),$$

and so,

$$A \equiv \bigwedge_{A' \in \mathbf{S}_1 \cup \mathbf{S}_2} A'.$$

We also note that

$$\mathbf{S}_1 \cup \mathbf{S}_2 \subseteq \mathbf{G}_n.$$

The case that $A = B \vee C$. By the induction hypothesis, there exist subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_n$ such that

$$B \equiv \bigwedge_{B' \in \mathbf{S}_1} B' \text{ and } C \equiv \bigwedge_{C' \in \mathbf{S}_2} C'.$$

Hence

$$A \equiv (\bigwedge_{B' \in \mathbf{S}_1} B') \vee (\bigwedge_{C' \in \mathbf{S}_2} C'),$$

and so,

$$A \equiv \bigwedge_{B' \in \mathbf{S}_1, C' \in \mathbf{S}_2} (B' \vee C'),$$

$$A \equiv (\bigwedge_{B' \in \mathbf{S}_1, C' \in \mathbf{S}_2, B' = C'} (B' \vee C') \wedge \bigwedge_{B' \in \mathbf{S}_1, C' \in \mathbf{S}_2, B' \neq C'} (B' \vee C')).$$

Using Lemma 2.10,

$$A \equiv \bigwedge_{B' \in \mathbf{S}_1, C' \in \mathbf{S}_2, B' = C'} (B' \vee C'),$$

and so,

$$A \equiv \bigwedge_{A' \in \mathbf{S}_1 \cap \mathbf{S}_2} A'.$$

We also note that

$$\mathbf{S}_1 \cap \mathbf{S}_2 \subseteq \mathbf{G}_n.$$

The case that $A = B \supset C$. By the induction hypothesis, there exist subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_n$ such that

$$B \equiv \bigwedge_{B' \in \mathbf{S}_1} B' \text{ and } C \equiv \bigwedge_{C' \in \mathbf{S}_2} C'.$$

Hence

$$A \equiv (\bigwedge_{B' \in \mathbf{S}_1} B') \supset (\bigwedge_{C' \in \mathbf{S}_2} C').$$

Using Lemma 2.5(2),

$$A \equiv (\bigwedge_{B' \in \mathbf{S}_1} B' \supset \Box p) \vee (\bigwedge_{C' \in \mathbf{S}_2} C').$$

Using Lemma 2.12,

$$A \equiv (\bigwedge_{B' \in \mathbf{G}_n - \mathbf{S}_1} B') \vee (\bigwedge_{C' \in \mathbf{S}_2} C'),$$

and similarly to the above case,

$$A \equiv (\bigwedge_{A' \in (\mathbf{G}_n - \mathbf{S}_1) \cup \mathbf{S}_2} A').$$

We also note that

$$(\mathbf{G}_n - \mathbf{S}_1) \cap \mathbf{S}_2 \subseteq \mathbf{G}_n.$$

The case that $A = \Box B$. Since $A \in \mathbf{S}^n(p)$, we have $B \in \mathbf{S}^{n-1}(p)$. By the induction hypothesis, there exists a subset $\mathbf{S}'$ of $\mathbf{G}_{n-1}$ such that

$$B \equiv \bigwedge_{B' \in \mathbf{S}'} B'.$$

Hence

$$A \equiv \Box \bigwedge_{B' \in \mathbf{S}'} B',$$

and using Lemma 2.4(3),

$$A \equiv \bigwedge_{B' \in \mathbf{S}'} \Box B'.$$

On the other hand, by Lemma 2.15, for any $C \in \Box \mathbf{G}_{n-1}$, there exists a subset $\mathbf{T}$(C) of $\mathbf{G}_n$ such that

$$C \equiv \bigwedge_{D \in \mathbf{T}(C)} D.$$

Since $\Box B' \in \Box \mathbf{S}'$ for $B' \in \mathbf{S}'$,

$$A \equiv \bigwedge_{B' \in \mathbf{S}'} \bigwedge_{D \in \mathbf{T}(\Box B')} D.$$

Hence

$$A \equiv \bigwedge_{D \in \bigcup_{B' \in \mathbf{S}'} \mathbf{T}(\Box B')} D.$$

We also note that

$$\bigcup_{B' \in \mathbf{S}'} \mathbf{T}(\Box B') \subseteq \mathbf{G}_n.$$

$\dashv$

Lemma 2.17. *For subsets $\mathbf{S}_1$ and $\mathbf{S}_2$ of $\mathbf{G}_n$,*

$$\mathbf{S}_1 \not\subseteq \mathbf{S}_2 \textit{ implies } [\bigwedge_{A \in \mathbf{S}_1} A] \not\leq [\bigwedge_{A \in \mathbf{S}_2} A].$$

Proof. Suppose that

$$\mathbf{S}_1 \not\subseteq \mathbf{S}_2 \text{ and } [\bigwedge_{A \in \mathbf{S}_1} A] \leq [\bigwedge_{A \in \mathbf{S}_2} A].$$

Then there exists a formula $B \in \mathbf{S}_1 - \mathbf{S}_2$ and

$$\bigwedge_{A \in \mathbf{S}_2} A \rightarrow \bigwedge_{A \in \mathbf{S}_1} A \in \mathbf{GGRZ}.$$

Since $B \in \mathbf{S}_1$, we have $\bigwedge_{A \in \mathbf{S}_1} A \rightarrow B \in \mathbf{GGRZ}$, and so,

$$\bigwedge_{A \in \mathbf{S}_2} A \rightarrow B \in \mathbf{GGRZ}.$$

Using Lemma 2.12,

$$(\bigwedge_{A \in \mathbf{G}_n - \mathbf{S}_2} A) \supset \Box p \rightarrow B \in \mathbf{GGRZ}.$$

Since $B \in \mathbf{G}_n - \mathbf{S}_2$, we have $B \supset \Box p \rightarrow \bigwedge_{A \in \mathbf{G}_n - \mathbf{S}_2} A \supset \Box p \in \mathbf{GGRZ}$, and so,

$$B \supset \Box p \rightarrow B \in \mathbf{GGRZ}.$$

Considering the figure

$$\dfrac{\dfrac{\dfrac{B \rightarrow B}{B \rightarrow B, \Box p}\ (\rightarrow w)}{\rightarrow B, B \supset \Box p}\ (\rightarrow\supset) \qquad \Box B \supset \Box p \rightarrow B}{\rightarrow B}\ (cut),$$

we obtain $B \in \mathbf{GRZ}$. This is in contradiction with Corollary 2.9. $\dashv$

Proof of Theorem 2.3. By Lemma 2.16, we obtain (1), The "if" part of (2.1) is clear and the "only of" part is from Lemma 2.17. From (2.1), we have (2.2). (3) is shown by (1) and (2). $\dashv$

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