A note on the consistency of ϕ_{AC}

MIYAMOTO Tadatoshi

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Abstract

We show the consistency of the combinatorial principle ϕ_{AC} starting from the ground model where there exists a regular cardinal below which there are cofinally many measurable cardinals. It is known that these two hypotheses are equiconsistent.

§1. Introduction

Two combinatorial principles ϕ_{AC} and φ_{AC} are introduced in [W] and a similar enumeration principle θ_{AC} in [T]. We have formulated the principles CODE(S) for stationary and costationary subsets S of ω_1 and θ^*_{AC} as our counterparts to φ_{AC} and θ_{AC} respectively. For precise definitions, see next section. We know that ([M], [M1] and [M2])

(1) If there exists a stationary and costationary subset S of ω_1 such that CODE(S) holds, then $2^\omega = 2^{\omega_1} = \omega_2$ holds.

(2) If CODE(S) hold for all stationary and costationary subsets S of ω_1, then φ_{AC} holds.

    It is known that φ_{AC} implies $2^\omega = 2^{\omega_1} = \omega_2$ ([W]).

(3) If CODE(S) hold for all stationary and costationary subsets S of ω_1, then it holds the so called complete bounding (*CB*) of functions from ω_1 to ω_1.

    However, it turns out that (3) gets interpolated by φ_{AC}.

(4) ([A], [A-W]) φ_{AC} implies CB.

    Concerning the consistency strengths of these principles, we have

(5) ([D-D]) CB entails the existence of a regular cardinal below which there are cofinally many measurable cardinals.

We simply denote this large cardinal assumption by *LC*.

(6) LC is sufficient to construct a model of set theory where CODE(S) hold for all stationary and costationary subsets S of ω_1.

Turning to the remaining collection of principles, we have

(7) θ^*_{AC} implies both θ_{AC} and CB.

(8) LC is sufficient to construct a model of set theory where θ^*_{AC} holds.

Hence, most of these are equiconsistent. The only possible exception would be θ_{AC}.

However, the following have been known to D. Aspero and told accordingly to us during his visit to Nagoya and Kobe University late 2003.

(1) ([A], [A-W], [W]) Both ϕ_{AC} and φ_{AC} imply CB.

(2) ([A]) LC is sufficient to construct a model of set theory where a strong form of ϕ_{AC}, denoted by ϕ^*_{AC}, holds.

Therefore, we have the following.

(1.1) Theorem. *The following seven are all equiconsistent.*

(1) ϕ_{AC}.

(2) φ_{AC}.

(3) CODE(S) *for all stationary and costationary subsets* S *of* ω_1.

(4) CB.

(5) θ^*_{AC}.

(6) ϕ^*_{AC}.

(7) LC.

We present a consistency proof of ϕ^*_{AC} starting from the large cardinal assumption LC. This note is based on a short but essential conversation with D. Aspero during the Symposium on Mathematical Logic 03 held at Kobe University on 17th through 19th of December in 2003.

Before closing this section, let me mention that the following appears to be still open.

(1.2) Question. (1) ([W]) MM implies ϕ_{AC}. Does MM imply ϕ^*_{AC} ?

(2) Does ϕ^*_{AC} (or ϕ_{AC}) imply $2^\omega = 2^{\omega_1} = \omega_2$?

§2. A Quick Review of Definitions

We prepare a list of relevant definitions for the sake of clarity. We begin with remarks on notations used in this note.

(2.1) Definition. For a countable set X of ordinals, o.t.(X) denotes the order type of X. Hence o.t.$(X) < \omega_1$. For γ with $\omega_1 < \gamma < \omega_2$, a sequence $\langle X_\delta \mid \delta < \omega_1 \rangle \nearrow \gamma$ means the following;

- Each X_δ is a countable subset of γ,
- The X_δ's are continuously increasing,
- $\bigcup \{X_\delta \mid \delta < \omega_1\} = \gamma$.

We allow $X_\delta = X_{\delta+1}$ to occur. But notice that there is some δ with $\omega_1 \in X_\delta$ and that $\omega_1 = \bigcup \{X_\delta \cap \omega_1 \mid \delta < \omega_1\}$ holds.

The following is from [W] where a weaker system of Set Theory is intended. Here we work with an equivalent formulation in the usual system of Set Theory, i.e, ZFC. We are simply concerned with the values of 2^ω and 2^{ω_1} in the $\aleph$'s.

(2.2) Definition. ϕ_{AC} holds, if for any sequence $\langle S_n \mid n < \omega \rangle$ of stationary subsets of ω_1 and any partition $\langle T_n \mid n < \omega \rangle$ of ω_1, there exist η and a function $F : \omega_1 \longrightarrow \eta$ such that $\omega_1 < \eta < \omega_2$, F is strictly increasing and continuous and cofinal below η and for all $n < \omega$, $F''T_n \subseteq \widetilde{S_n}$ hold, where $\gamma \in \widetilde{S_n}$, if $\omega_1 < \gamma < \omega_2$ and there exists a sequence $\langle X_\delta \mid \delta < \omega_1 \rangle \nearrow \gamma$ such that for each $\delta < \omega_1$, we have

$$\mathrm{o.t.}(X_\delta) \in S_n.$$

We consider a somewhat stronger principle than ϕ_{AC} due to [A]. This principle deals with stationary subsets of ω_1 indexed by ω_1. I do not know whether these two can be separated.

(2.3) Definition. ([A]) ϕ^*_{AC} holds, if for any sequence $\langle S_i \mid i < \omega_1 \rangle$ of stationary subsets of ω_1, there exist η and a function $F : \omega_1 \longrightarrow \eta$ such that $\omega_1 < \eta < \omega_2$, F is strictly increasing and continuous and cofinal below η and for all $i < \omega_1$, $F(i) \in \widetilde{S}_i$ hold.

Now we go back to [W] where a simpler principle φ_{AC} than ϕ_{AC} is also activated.

(2.4) Definition. ([W]) φ_{AC} holds, if for any two stationary and costationary subsets S and T of ω_1, there exist γ with $\omega_1 < \gamma < \omega_2$ and a club C of ω_1 such that

$$T \cap C = \{\delta \in C \mid \mathrm{o.t.}(X_\delta) \in S\}.$$

We formulate a somewhat stronger principle than φ_{AC}. This new principle exactly calculates every subset of ω_1 with no appearances of club subsets C of ω_1. I do not know whether these two can be separated.

(2.5) Definition. ([M]) For any stationary and costationary subset S of ω_1, CODE(S) holds, if for any $B \subseteq \omega_1$, there exists γ with $\omega_1 < \gamma < \omega_2$ and a sequence $\langle X_\delta \mid \delta < \omega_1 \rangle \nearrow \gamma$ such that

$$B = \{\delta < \omega_1 \mid \mathrm{o.t.}(X_\delta) \in S\}.$$

The following also appears in [W] and considered by many including this author. This principle has found its exact consistency strength ([D-D], [M1]) and its relation to CH ([L-S]). We remark that this simple formulation and its naming are one of the local dialects.

(2.6) Definition. ([A], [D-D], [L-S], [M1], [W]) The *complete bounding* (CB) means that for any $f : \omega_1 \longrightarrow \omega_1$, there exist γ and a sequence $\langle X_\delta \mid \delta < \omega_1 \rangle \nearrow \gamma$ such that for all $\delta < \omega_1$, we have

$$f(\delta) < \mathrm{o.t.}(X_\delta).$$

We prepare for our last principle.

(2.7) Definition. A *one-to-one list* $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ in ${}^{\omega}2$ means that for all $i < \omega_1$, $r_i : \omega \longrightarrow 2$ and for all $i, j < \omega_1$, if $i \neq j$, then $r_i \neq r_j$. In this case, we denote $\Delta(r_i, r_j) = \text{Min}\,\{n < \omega \mid r_i(n) \neq r_j(n)\}$. For any ordinals $\alpha < \beta$, if o.t.$(X \cap \alpha) <$ o.t.$(X \cap \beta) < \omega_1$, we denote $\Delta^{\mathbf{r}}_X(\alpha, \beta) = \Delta(r_{\text{o.t.}(X\cap\alpha)}, r_{\text{o.t.}(X\cap\beta)})$. We usually simply write $\Delta_X(\alpha, \beta)$ instead of $\Delta^{\mathbf{r}}_X(\alpha, \beta)$. For any ordinals α, β and γ, if o.t.$(X \cap \alpha) <$ o.t.$(X \cap \beta) <$ o.t.$(X \cap \gamma) < \omega_1$, then we denote

$$\text{Max}\ \Delta_X(\alpha, \beta, \gamma) = \text{Max}\ \{\Delta_X(\alpha, \beta), \Delta_X(\alpha, \gamma), \Delta_X(\beta, \gamma)\}.$$

An enumeration principle θ_{AC} along the same line as φ_{AC} and ϕ_{AC} of [W] has been considered by [T]. The Bounded Martin's Maximum (BMM) implies θ_{AC}. And θ_{AC} in turn implies $2^{\omega} = 2^{\omega_1} = \omega_2$. But no large cardinal lower bound to this principle is known yet. The following θ^*_{AC} due to us is somewhat stronger than θ_{AC}. The Bounded Semi-Proper Forcing Axiom (BSPFA) with a measurable cardinal implies θ^*_{AC}.

(2.8) Definition. ([T], [M2]) θ^*_{AC} holds, if for any $\mathbf{r}$ one-to-one list in ${}^{\omega}2$ and any $B \subseteq \omega_1$, there exist β and γ with $\omega_1 < \beta < \gamma < \omega_2$ and $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$ such that

$$B = \{i < \omega_1 \mid \Delta_{X_i}(\omega_1, \beta) = \text{Max}\ \Delta_{X_i}(\omega_1, \beta, \gamma)\}.$$

We restate the large cardinal assumption LC which has found many but similar equiconsistent principles. Under many of these principles, $2^{\omega_1} = \omega_2$ holds and sometimes even $2^{\omega} = 2^{\omega_1} = \omega_2$ can be concluded via the Weak Diamond. A known exception is CB. We may construct a model of Set Theory where CB holds together with CH via a highly-semiproper revised countable support iterated forcing ([L-S]).

(2.9) Definition. *LC* stands for the following large cardinal assumption.

- There exists a regular cardinal ρ such that $\{\kappa < \rho \mid \kappa \text{ is a measurable cardinal}\}$ is a cofinal subset of ρ.

§3. Easy Implications

We begin to list easier relations among the principles introduced so far. For those claims mentioned in the introduction but whose proofs are not found in this section, we may consult [M], [M1] and [M2].

(3.1) Proposition. *ϕ^*_{AC} implies ϕ_{AC}.*

Proof. Let $\langle S_n \mid n < \omega \rangle$ and $\langle T_n \mid n < \omega \rangle$ be as in ϕ_{AC}. Define a map $\langle i \mapsto n_i \mid i < \omega_1 \rangle$ such that $i \in T_{n_i}$ and apply ϕ^*_{AC} to $\langle S_{n_i} \mid i < \omega_1 \rangle$. Get γ with $\omega_1 < \gamma < \omega_2$ and a strictly increasing continuous function $F : \omega_1 \longrightarrow \gamma$ with a cofinal image in γ such that for all $i < \omega_1$, we have $F(i) \in \widetilde{S_{n_i}}$. Hence $F''T_n \subseteq \widetilde{S_n}$ holds.

$\square$

We have seen that ϕ^*_{AC} implies ϕ_{AC}. It is known that ϕ_{AC} implies $2^{\omega_1} = \omega_2$ ([W]). Therefore the following is not new but we include it for the sake of completeness.

(3.2) Proposition. *ϕ^*_{AC} implies $2^{\omega_1} = \omega_2$.*

Proof. We provide a one-to-one map g from $\mathcal{P}(\omega_1)$ into ω_2. To this end, we first partition ω_1 into $\langle S_i \mid i < \omega_1 \rangle$ so that each S_i is stationary in ω_1. Hence $\omega_1 = \dot{\bigcup}\{S_i \mid i < \omega_1\}$, where $\dot{\bigcup}$ denotes the disjoint union of the S_i's. We also fix a stationary and costationary subset T of ω_1 so that we have

$$\widetilde{T} \cap \widetilde{(\omega_1 \setminus T)} = \emptyset.$$

Now given $A \subseteq \omega_1$, we define $S(A) = \dot{\bigcup}\{S_i \mid i \in A\}$. Notice that for any $i \in \omega_1$, we have

$$S_i \subseteq S(A) \text{ iff } S(A) \cap S_i \neq \emptyset \text{ iff } i \in A.$$

We next define a sequence $\langle S^A_\alpha \mid \alpha < \omega_1 \rangle$ of stationary subsets of ω_1 so that for any $\alpha < \omega_1$,

$$S^A_\alpha = \begin{cases} T, & \text{if } \alpha \in S(A), \\ \omega_1 \setminus T, & \text{otherwise.} \end{cases}$$

Apply ϕ^*_{AC} to this sequence of S^A_α's. Then we have η^A with $\omega_1 < \eta^A < \omega_2$ and a strictly increasing continuous function $F^A : \omega_1 \longrightarrow \eta^A$ with the cofinal range in η^A such that for all $\alpha < \omega_1$, we have $F^A(\alpha) \in \widetilde{S^A_\alpha}$. Hence

$$(*)_A \quad \begin{cases} F^A(\alpha) \in \widetilde{T}, & \text{if } \alpha \in S(A), \\ F^A(\alpha) \in (\widetilde{\omega_1 \setminus T}), & \text{otherwise.} \end{cases}$$

We define $g(A) = \eta^A$. This completes the definition of g. Note that we do not associate F^A to A. We just associate η^A for which there exists such F^A. To see this g works, let $A, B \subseteq \omega_1$ and suppose $\eta^A = \eta^B$. Take any F^A, F^B and set

$$C = \{\alpha < \omega_1 \mid F^A(\alpha) = F^B(\alpha)\}.$$

Since F^A and F^B are continuous and strictly increasing cofinally in $\eta^A = \eta^B$, we know C is a club. To conculde $A = B$, we may observe, say, $A \subseteq B$ as follows;

Let $i \in A$. Then $S_i \subseteq S(A)$ holds. Since S_i is sationary, there exists $\alpha \in S_i \cap C$ and so $\alpha \in S(A) \cap C$. By $(*)_A$, we have $F^B(\alpha) = F^A(\alpha) \in \widetilde{T}$. In turn by $(*)_B$, we have $\alpha \in S_i \cap S(B)$ and so $S_i \subseteq S(B)$. Hence $i \in B$.

□

(3.3) Proposition. ([A]) ϕ_{AC} *implies CB.*

Proof. Let $f : \omega_1 \longrightarrow \omega_1$ and $C(f) = \{i < \omega_1 \mid f''i \subseteq i\}$. Then $C(f)$ is a club in ω_1. Apply ϕ_{AC} to $\langle C(f) \mid n < \omega\rangle$ and $\langle \omega_1, \emptyset, \cdots, \emptyset, \cdots\cdots\rangle$. We have η with $\omega_1 < \eta < \omega_2$ and a strictly increasing continuous function $F : \omega_1 \longrightarrow \eta$ such that $F''\omega_1 \subset \widetilde{C(f)}$.

Pick (any) one $\gamma \in F''\omega_1$. Then there exists a sequence $\langle X_\delta \mid \delta < \omega_1\rangle \nearrow \gamma$ such that for all $\delta < \omega_1$, we have o.t.$(X_\delta) \in C(f)$. Let

$$D = \{\delta < \omega_1 \mid \omega_1 \in X_\delta,\ \omega_1 \cap X_\delta = \delta\}.$$

Then D is a club in ω_1. It suffices to show that for all $\delta \in D$, $f(\delta) < \text{o.t.}(X_\delta)$ hold. But $\delta < \text{o.t.}(X_\delta) \in C(f)$, so this is immediate.

$\square$

(3.4) Proposition. *If CODE(S) hold for all stationary and costationary subsets S of ω_1, then φ_{AC} holds.*

Proof. Let S and T be two stationary and costationary subsets of ω_1. Apply CODE(S). We have γ with $\omega_1 < \gamma < \omega_2$ and $\langle X_\delta \mid \delta < \omega_1 \rangle \nearrow \gamma$ such that $T = \{\delta < \omega_1 \mid \text{o.t.}(X_\delta) \in S\}$. Hence φ_{AC} holds.

$\square$

(3.5) Proposition. ([A], [A-W]) φ_{AC} *implies CB.*

Proof. Let $f : \omega_1 \longrightarrow \omega_1$ and $C(f) = \{i < \omega_1 \mid f''i \subseteq i\}$. Then $C(f)$ is a club in ω_1. Partition $C(f)$ into two stationary sets S and T. So $C(f) = S \cup T$ and $S \cap T = \emptyset$. Apply φ_{AC} to (S, T) and $(S, \omega_1 \setminus T)$. So there exist γ_1, γ_2, C_1, C_2, $\langle X^1_\delta \mid \delta < \omega_1 \rangle \nearrow \gamma_1$ and $\langle X^2_\delta \mid \delta < \omega_1 \rangle \nearrow \gamma_2$ such that

$$T \cap C_1 = \{\delta \in C_1 \mid \text{o.t.}(X^1_\delta) \in S\},$$

$$(\omega_1 \setminus T) \cap C_2 = \{\delta \in C_2 \mid \text{o.t.}(X^2_\delta) \in S\}.$$

Since $\gamma_1 \neq \gamma_2$, we may assume $\omega_1 < \gamma_1 < \gamma_2 < \omega_2$. Let

$$D = C_1 \cap C_2 \cap \{\delta < \omega_1 \mid \omega_1 \in X^1_\delta,\ \gamma_1 \in X^2_\delta,\ X^1_\delta = X^2_\delta \cap \gamma_1,\ X^1_\delta \cap \omega_1 = \delta\}.$$

Then D is a club in ω_1. It suffices to show that for all $\delta \in D$, we have

$$f(\delta) < \text{o.t.}(X^2_\delta).$$

Case 1. $\delta \in T$: $\delta < \text{o.t.}(X^1_\delta) \in S \subset C(f)$. So we have $f(\delta) < \text{o.t.}(X^1_\delta) < \text{o.t.}(X^2_\delta)$.

Case 2. $\delta \notin T$: $\delta < \text{o.t.}(X^1_\delta) < \text{o.t.}(X^2_\delta) \in S \subset C(f)$. So we have $f(\delta) < \text{o.t.}(X^2_\delta)$.

$\square$

§4. Measurable Cardinals and Semiproperness

We review combinatorial arguments which involve measurable cardinals and elementary substructures in this section.

(4.1) Lemma. (End-Extension) *Let κ be a measurable cardinal with a normal measure $\mathcal{U}$. Let θ be a regular cardinal with $\theta \geq (2^\kappa)^+$ so that $\mathcal{U} \in H_\theta$. For any countable elementary substructure N of H_θ with $\mathcal{U} \in N$ and any $s \in \bigcap(\mathcal{U} \cap N)$, we may define $N(s) = \{f(s) \mid f \in N\}$. Then we have*

(1) $\sup(N \cap \kappa) < s$,

(2) $\{s\} \cup N \subset N(s)$, $N(s)$ is a countable elementary substructure of H_θ,

(3) For all $a \in N \cap H_\kappa$, we have $a \cap N = a \cap N(s)$,

(4) s is the least ordinal in $N(s) \setminus N$.

Proof. For (1): (Set Theory Seminar at Nagoya University) Let $\xi \in N \cap \kappa$. Then $(\xi, \kappa) = \{x < \kappa \mid \xi < x\} \in \mathcal{U} \cap N$ and so $\xi < s$ holds. Hence $\sup(N \cap \kappa) \leq s$. Since $\{\eta < \kappa \mid \eta \text{ is regular}\} \in \mathcal{U} \cap N$, we know s is regular and so $\sup(N \cap \kappa) < s$ holds.

For (2): Let $f = \{(i,i) \mid i < \kappa\}$. Then $f \in N$ and $s = f(s) \in N(s)$. Let $y \in N$ and set $g = \{(i,y) \mid i < \kappa\}$. Then $g \in N$ and $y = g(s) \in N(s)$. Hence $N \subset N(s)$ holds.

To show that $N(s)$ is an elementary substructure of H_θ, we resort to the Tarski's criterion. Namely, suppose $f_1, \cdots, f_n \in N$ and

$$H_\theta \models \text{“}\exists\, y\ \varphi\big(y, f_1(s), \cdots, f_n(s)\big)\text{”}.$$

It suffices to find $f \in N$ with

$$H_\theta \models \text{“}\varphi\big(f(s), f_1(s), \cdots, f_n(s)\big)\text{”}.$$

Since ${}^\kappa H_\theta \subset H_\theta$, we have $f : \kappa \longrightarrow H_\theta$ such that $f \in H_\theta$ and

$$H_\theta \models \text{“}\forall x < \kappa\ \forall y\ \Big(\varphi\big(y, f_1(x), \cdots, f_n(x)\big) \Longrightarrow \varphi\big(f(x), f_1(x), \cdots, f_n(x)\big)\Big)\text{”}.$$

Since $\kappa, f_1, \cdots, f_n \in N$, we may assume $f \in N$. Hence we have

$$H_\theta \models \text{``}\varphi\big(f(s), f_1(s), \cdots, f_n(s)\big)\text{''}.$$

For (3): We first mention the following preliminary

Claim. For all $\eta \in N \cap \kappa$, we have $\eta \cap N = \eta \cap N(s)$.

We observe this claim suffices as follows; Let $a \in N \cap H_\kappa$ and $e : |a| \longrightarrow a$ be a bijection. We may assume $e \in N, |a| \in N \cap \kappa$. Let $x \in a \cap N(s)$. Then there exists $i \in |a| \cap N(s) = |a| \cap N$ with $e(i) = x$. Hence $x \in N$ and so $a \cap N = a \cap N(s)$ holds.

Proof of Claim. Let $\eta \in N \cap \kappa$ and $i \in \eta \cap N(s)$. There is $f \in N$ with $i = f(s)$. Since $i < \eta < s$, we may assume that $f \in N$ is regressive. Hence there exists $v \in N \cap \kappa$ such that $f^{-1}\text{``}\{v\} \in \mathcal{U} \cap N$. Therefore $s \in f^{-1}\text{``}\{v\}$ and so $i = f(s) = v \in N$. Hence $\eta \cap N = \eta \cap N(s)$ holds.

For (4): A similar proof as above works. Let $i \in N(s) \cap s$. We want to show $i \in N$. There is $f \in N$ with $f(s) = i$. Since $f(s) = i < s$, we may assume $f \in N$ is regressive. Hence there exists $v \in N \cap \kappa$ with $f^{-1}\text{``}\{v\} \in \mathcal{U} \cap N$. Since $s \in f^{-1}\text{``}\{v\}$, we have $i = f(s) = v \in N$.

□

By applying Lemma (End-Extension) repeatedly, we may form a chain of elementary substructures. Therefore we may shoot into a given stationary set in the following sense.

(4.2) Lemma. (End-Extending into Stationary Sets) *Let κ be a measurable cardinal with a normal measure $\mathcal{U}$. Let θ be a regular cardinal with $\theta \geq (2^\kappa)^+$ so that $\mathcal{U} \in H_\theta$. For any countable elementary substructure N of H_θ with $\mathcal{U} \in N$, any stationary subset S of ω_1 and any $t < \omega_1$, there exists M such that*

(1) M is a countable elementary substructure of H_θ with $N \subset M$,

(2) For all $a \in N \cap H_\kappa$, we have $a \cap M = a \cap N$,

(3) $t < \text{o.t.}(M \cap \kappa) \in S$.

Proof. Let $t < \omega_1$ and S be a stationary subset of ω_1. Let N be a countable elementary substructure of H_θ with $\mathcal{U} \in N$. We make use of Lemma (End-Extension) repeatedly to construct $\langle N_i \mid i < \omega_1 \rangle$ and $\langle s_i \mid i < \omega_1 \rangle$ such that

- $N_0 = N$ and $s_0 \in \bigcap (\mathcal{U} \cap N)$,
- $N \subset N_i$, N_i is a countable elementary substructure of H_θ and $s_i \in \bigcap (\mathcal{U} \cap N_i)$,
- $\sup(N_i \cap \kappa) < s_i < \kappa$,
- $\{s_i\} \cup N_i \subset N_{i+1} = N_i(s_i) = \{f(s_i) \mid f \in N_i\}$ is a countable elementary substructure of H_θ,
- For all $a \in N_i \cap H_\kappa$, we have $a \cap N_i = a \cap N_{i+1}$,
- For limit i, $N_i = \bigcup\{N_j \mid j < i\}$.

Then, it is easy to conclude

- $\langle \text{o.t.}(N_i \cap \kappa) \mid i < \omega_1 \rangle$ is a club in ω_1,
- For all $i < \omega_1$ and all $a \in N \cap H_\kappa$, we have $a \cap N = a \cap N_i$.

Now take $i < \omega_1$ so that $t < \text{o.t.}(N_i \cap \kappa) \in S$ and set $M = N_i$. This M works.

$\square$

Now we are ready for the main technical Lemma in this note. This observation is due to [A].

(4.3) Lemma. *Let $\langle S_i \mid i < \omega_1 \rangle$ be a sequence of stationary subsets of ω_1 and $\langle \kappa_i \mid i \leq \omega_1 \rangle$ be a continuously strictly increasing sequence of cardinals such that for all non-limit i, κ_i are measurable cardinals. Let us define*

$$\begin{aligned} S^* &= S^*(\langle S_i \mid i < \omega_1 \rangle,\ \langle \kappa_i \mid i \leq \omega_1 \rangle) \\ &= \{X \in [\kappa_{\omega_1}]^\omega \mid \forall\, i \leq X \cap \omega_1 \ \text{o.t.}(X \cap \kappa_i) \in S_i\}. \end{aligned}$$

Then S^* is semiproper. Namely, let θ be any regular cardinal with $\theta \geq (\kappa_{\omega_1})^+$ so that $\langle \kappa_i | \ i \leq \omega_1 \rangle \in H_\theta$. Then for any countable elementary substructure N of H_θ with $\langle \kappa_i | \ i \leq \omega_1 \rangle \in N$, there exists a countable elementary substructure M of H_θ such that $N \subset M$, $M \cap \omega_1 = N \cap \omega_1$ and $M \cap \kappa_{\omega_1} \in S^*$.

Proof. We formulate a couple of intermediary technical Claims.

Claim 1. Let $P(t, \alpha, N, \beta)$ denote the following;

- $t < \omega_1$,
- $\alpha < \beta < \omega_1$,
- N is a countable elementary substructure of H_θ with $\langle \kappa_i \mid i \leq \omega_1 \rangle \in N$,
- $\alpha, \beta \in N$ and so $\kappa_\alpha, \kappa_\beta \in N$,

Let $Q(t, \alpha, N, \beta, M)$ denote the following;

- $N \subseteq M$, M is a countable elementary substructure of H_θ,
- $N \cap H_{\kappa_\alpha} = M \cap H_{\kappa_\alpha}$,
- For all i with $\alpha < i \leq \beta$, we have $\text{o.t.}(M \cap \kappa_i) \in S_i$,
- $t < \text{o.t.}(M \cap \kappa_\beta)$.

Then for all $\beta < \omega_1$, we have

$$\forall\ t\ \forall \alpha \forall N\ \big(P(t, \alpha, N, \beta) \Longrightarrow \exists\ M\ Q(t, \alpha, N, \beta, M)\big).$$

Proof. By induction on $\beta < \omega_1$.

Case 1. $\beta + 1$: Suppose $P(t, \alpha, N, \beta + 1)$.

Subcase 1.1. $\alpha < \beta$:

Then $P(t, \alpha, N, \beta)$ holds. By induction, there exists M' such that $Q(t, \alpha, N, \beta, M')$ holds. By Lemma (End-Extending into Stationary Sets) with $(M', \kappa_{\beta+1}, S_{\beta+1})$, we have M such that

(1) M is a countable elementary substructure of H_θ with $M' \subset M$,
(2) For all $a \in M' \cap H_{\kappa_{\beta+1}}$, we have $a \cap M' = a \cap M$,
(3) $t < \text{o.t.}(M \cap \kappa_{\beta+1}) \in S_{\beta+1}$.

In particular, we have $M' \cap H_{\kappa_\beta} = M \cap H_{\kappa_\beta}$. It is routine to check $Q(t, \alpha, N, \beta + 1, M)$. Namely,

- $N \subseteq M$, M is a countable elementary substructure of H_θ,
- $N \cap H_{\kappa_\alpha} = M \cap H_{\kappa_\alpha}$,
- For all i with $\alpha < i \leq \beta + 1$, we have $\text{o.t.}(M \cap \kappa_i) \in S_i$,

- $t < \text{o.t.}(M \cap \kappa_{\beta+1})$.

Subcase 1.2. $\alpha = \beta$:

For $(N, \kappa_{\beta+1}, S_{\beta+1})$, we apply Lemma (End-Extending into Stationary Sets) so that we have M as wished.

Case 2. β is a limit ordinal: Suppose $P(t, \alpha, N, \beta)$. Take a regular cardinal χ so that $H_\theta \in H_\chi$. Take a countable elementary substructure M^* of H_χ such that $t, N, H_\theta \in M^*$ and

$$M^* \cap \omega_1 \in S_\beta.$$

This is possible, since S_β is stationary. Let $\langle \beta_n \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals such that $\beta_0 = \alpha$ and $\sup\{\beta_n \mid n < \omega\} = \beta$. Note that $\beta_n \in N \cap \omega_1$ holds. Let $\langle t_n \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals such that $t_0 = t$ and $\sup\{t_n \mid n < \omega\} = M^* \cap \omega_1$. Note that $t_n \in M^* \cap \omega_1$ holds.

Now construct $\langle N_n \mid n < \omega \rangle$ such that

- $N_0 = N$,
- $N_n \in M^*$ is a countable elementary substructure of H_θ such that $N \subseteq N_n$, $N \cap H_{\kappa_\alpha} = N_n \cap H_{\kappa_\alpha}$ and $P(t_n, \beta_n, N_n, \beta_{n+1})$ holds,
- $Q(t_n, \beta_n, N_n, \beta_{n+1}, N_{n+1})$ holds and $N_{n+1} \in M^*$.

Let $M = \bigcup\{N_n \mid n < \omega\}$. Notice that

$$\text{o.t.}(M \cap \kappa_\beta) = \sup\{\text{o.t.}(N_n \cap \kappa_{\beta_n}) \mid n < \omega\}$$

$$= \sup\{t_n \mid n < \omega\} = M^* \cap \omega_1 \in S_\beta.$$

Now it is routine to check that $Q(t, \alpha, N, \beta, M)$ holds.

□

Claim 2. Let $P(t, N, \beta)$ denote the following;

- $t < \omega_1$,
- N is a countable elementary substructure of H_θ such that $\langle \kappa_i \mid i \leq \omega_1 \rangle \in N$,

- $\beta < \omega_1 \cap N$.

Let $R(t, N, \beta, M)$ denote the following;

- M is a countable elementary substructure of H_θ such that $N \subseteq M$ and $N \cap \omega_1 = M \cap \omega_1$,
- For all $i \leq \beta$, we have o.t.$(M \cap \kappa_i) \in S_i$,
- $t <$ o.t.$(M \cap \kappa_\beta)$.

Then for all $\beta < \omega_1$, we have

$$\forall t \; \forall N \; \big(P(t, N, \beta) \Longrightarrow \exists \; M \; R(t, N, \beta, M)\big).$$

Proof. By induction on $\beta < \omega_1$.

Case 1. $\beta = 0$: Suppose $P(t, N, 0)$. Apply Lemma (End-Extending into Stationary Sets) so that there exists M such that $R(t, N, 0, M)$ holds.

Case 2. $\beta + 1$: Suppose $P(t, N, \beta + 1)$. Then $P(t, N, \beta)$ holds. By induction, there exists M' such that $R(t, N, \beta, M')$ holds.

With $(M', \kappa_{\beta+1}, S_{\beta+1})$, apply Lemma (End-Extending into Stationary Sets) so that there exists M such that $R(t, N, \beta + 1, M)$.

Case 3. β is a limit ordinal: Suppose $P(t, N, \beta)$. Take a regular cardinal χ with $H_\theta \in H_\chi$. Take a countable elementary substructure M^* of H_χ such that $t, N, H_\theta \in M^*$ and $M^* \cap \omega_1 \in S_\beta$. This is possible, since S_β is a stationary subset of ω_1. Fix a strictly increasing sequence $\langle \beta_n \mid n < \omega \rangle$ of ordinals such that $\beta_0 = 0, \beta_n \in N$ and $\sup\{\beta_n \mid n < \omega\} = \beta$. Fix also a strictly increasing sequence $\langle t_n \mid n < \omega \rangle$ of ordinals such that $t_0 = t, t_n \in M^*$ and $\sup\{t_n \mid n < \omega\} = M^* \cap \omega_1$. By induction (or Lemma (End-Extending into Stationary Sets)) and Claim 1, we may construct $\langle N_n \mid n < \omega \rangle$ such that

- $R(t_0, N, \beta_0, N_0)$ with $N_0 \in M^*$,
- $P(t_n, \beta_n, N_n, \beta_{n+1})$ with $N_n \in M^*$,
- $Q(t_n, \beta_n, N_n, \beta_{n+1}, N_{n+1})$ with $N_{n+1} \in M^*$.

Let $M = \bigcup\{N_n \mid n < \omega\}$. Then we have

$$\text{o.t.}(M \cap \kappa_\beta) = \sup\{\text{o.t.}(N_n \cap \kappa_{\beta_n}) \mid n < \omega\}$$

$$= \sup\{t_n \mid n < \omega\} = M^* \cap \omega_1 \in S_\beta.$$

It is routine to check $R(t, N, \beta, M)$.

$\square$

Proof of Lemma. Let θ be a regular cardinal with $\kappa_{\omega_1} \in H_\theta$. Let N be any countable elementary substructure of H_θ with $\langle \kappa_i \mid i \leq \omega_1 \rangle \in N$. We want to find a countable elementary substructure M of H_θ such that $N \subseteq M$, $N \cap \omega_1 = M \cap \omega_1$ and $M \cap \kappa_{\omega_1} \in S^* = S^*(\langle S_i \mid i < \omega_1 \rangle, \langle \kappa_i \mid i \leq \omega_1 \rangle)$. To this end, let us take a regular cardinal χ with $H_\theta \in H_\chi$. Let M^* be a countable elementary substructure of H_χ such that $H_\theta, N, \langle S_i \mid i < \omega_1 \rangle \in M^*$ and $M^* \cap \omega_1 \in S_{N \cap \omega_1}$. This is possible, since $S_{N \cap \omega_1}$ is a stationary subset of ω_1.

Fix a strictly increasing sequence $\langle t_n \mid n < \omega \rangle$ of ordinals such that $t_n \in M^* \cap \omega_1$ and $\sup\{t_n \mid n < \omega\} = M^* \cap \omega_1$. Fix also a strictly increasing sequence $\langle i_n \mid n < \omega \rangle$ of ordinals such that $i_n \in N \cap \omega_1$ and $\sup\{i_n \mid n < \omega\} = N \cap \omega_1$. By applying Claims 1 and 2, we may construct $\langle N_n \mid n < \omega \rangle$ such that

- $N \subset N_0$, $N_0 \in M^*$ is a countable elementary substructure of H_θ and $N \cap \omega_1 = N_0 \cap \omega_1$,
- For all i with $i \leq i_0$, we have o.t.$(N_0 \cap \kappa_i) \in S_i$,
- $t_0 <$ o.t.$(N_0 \cap \kappa_{i_0})$,
- $N \subset N_n$, $N_n \in M^*$ is a countable elementary substructure of H_θ and $N \cap \omega_1 = N_n \cap \omega_1$,
- For all i with $i \leq i_n$, we have o.t.$(N_n \cap \kappa_i) \in S_i$,
- $t_n <$ o.t.$(N_0 \cap \kappa_{i_n})$,
- $N_n \subset N_{n+1} \in M^*$, $N_n \cap H_{\kappa_{i_n}} = N_{n+1} \cap H_{\kappa_{i_n}}$,
- For all i with $i_n < i \leq i_{n+1}$, we have o.t.$(N_{n+1} \cap \kappa_i) \in S_i$,
- $t_{n+1} <$ o.t.$(N_{n+1} \cap \kappa_{i_{n+1}})$.

Let $M = \bigcup\{N_n \mid n < \omega\}$. Then we have

$$\text{o.t.}(M \cap \kappa_{N \cap \omega_1}) = \sup\{\text{o.t.}(N_n \cap \kappa_{i_n}) \mid n < \omega\}$$

$$= \sup\{t_n \mid n < \omega\} = M^* \cap \omega_1 \in S_{N \cap \omega_1}.$$

It is routine to check that this M works.

$\square$

(4.4) Corollary. *Let $Q = Q(\langle S_i \mid i < \omega_1\rangle,\ \langle \kappa_i \mid i \leq \omega_1\rangle)$ be a partially ordered set for shooting a club through $S^* = S^*(\langle S_i \mid i < \omega_1\rangle,\ \langle \kappa_i \mid i \leq \omega_1\rangle)$, then Q is semiproper and in V^Q, there exists $\langle \dot{X}_\delta \mid \delta < \omega_1\rangle \nearrow \kappa_{\omega_1}$ such that for each $\delta < \omega_1$, we have $\dot{X}_\delta \in S^*$.*

Proof. We describe our relevant partially ordered set. $p = \langle X^p_i \mid i \leq \alpha^p\rangle \in Q$, if the $X^p_i \in S^*$ are continuously increasing with $\alpha^p < \omega_1$. For $p, q \in Q$, we set $q \leq p$, if $q \supseteq p$. Since S^* is semiproper iff Q is semiproper as a partially ordered set, we conclude Q is semiproper.

$\square$

(4.5) Theorem. ([A]) *Let ρ be a regular cardinal such that $\{\kappa < \rho \mid \kappa$ is a measurable cardinal $\}$ is cofinal in ρ. Then there is a semiproper preorder P_ρ such that P_ρ has the ρ-c.c. and in V^{P_ρ}, we have*

- *For any sequence $\langle S_i \mid i < \omega_1\rangle$ of stationary subsets of ω_1, there exist a strictly increasing continuous sequence $\langle \kappa_i \mid i \leq \omega_1\rangle$ of ordinals and a sequence $\langle X_\delta \mid \delta < \omega_1\rangle \nearrow \kappa_{\omega_1}$ such that for all $\delta < \omega_1$ and all $i \leq X_\delta \cap \omega_1$, we have* $\mathrm{o.t.}(X_\delta \cap \kappa_i) \in S_i$. *In particular, $\kappa_i \in \widetilde{S}_i$ and so ϕ^*_{AC} holds.*

Proof. We simply iterate relevant semiproper partially ordered set $Q = Q(\langle S_i \mid i < \omega_1\rangle, \langle \kappa_i \mid i \leq \omega_1\rangle)$. This is possible, since plenty of measurable cardinals remain in any intermediate stage. We accomplish the whole construction by book-keeping relevant names by the chain condition. The resulted semiproper simple ([M3]) iteration P_ρ is what we claimed.

$\square$

(4.6) Corollary. ([A], [D-D]) *The following are equiconsistent.*

(1) ϕ_{AC} holds,

(2) The large cardinal assumption LC holds. Namely, there exists a regular cardinal ρ such that $\{\kappa < \rho \mid \kappa$ is a measurable cardinal $\}$ is cofinal below ρ.

References

[A] D. Aspero, communication, at Nagoya University, November 2003 and Symposium on Mathematical Logic 03, Kobe University, December 17-19, 2003.

[A-W] D. Aspero, P. Welch, Bounded Martin's Maximum, Weak Erdős cardinals, and φ_{AC}, *Journal of Symbolic Logic*, vol. 67 (2002) pp. 1141-1152.

[D-D] O. Deiser, D. Donder, Canonical functions, non-regular ultrafilters and Ulam's problem on ω_1, *Journal of Symobolic Logic*, vol. 68 (2003) pp. 713-739.

[L-S] P. Larson, S. Shelah, Bounding by canonical functions, with CH, To Appear.

[M] T. Miyamoto, An Example of Iterated Forcing which adds Reals at Limit Stages, A typed note, May 3rd 2000.

[M1] ——, Forcing NS_{ω_1} completely bounded via semiproper iterations, *RIMS Kokyuroku*, Kyoto University 1143 (2000) pp. 8-18.

[M2] ——, BSPFA combined with one measurable cardinal, *RIMS Kokyuroku*, Kyoto University 1304 (2003) pp. 114-121.

[M3] ——, A Limit Stage Construction for Iterating Semiproper Preorders, *Proceedings of the 7th and 8th Asian Logic Conferences*, World Scientific (2003) pp. 303-327.

[T] S. Todorčević, Generic Absoluteness and the continuum, *Mathematical Research Letters* 9 (2002) pp. 465-472.

[W] H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Non-stationary Ideals*, Series in Logic and its applications, De Gruyter 1, Berlin, New York, 1999.

Mathematics
Nanzan University
27 Seirei-cho, Seto-shi
489-0863 Japan
miyamoto@nanzan-u.ac.jp