A Forcing Axiom for The Second Uncountable Cardinal Must Fail

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25th of January 2003

Abstract

The Forcing Axiom for the p.o. sets which are σ-closed, ω_2-Baire and preserve the stationary subsets of ω_2 with ω_2-many dense subsets must fail. This is a straightforward simplification of a construction due to S. Shelah.

Introduction

We consider Forcing Axioms for the second uncountable cardinal. For positive answers, we may consult [B], [S1] or [W]. We summarize the failure in this context. Recall that a notion of forcing is ω_2-Baire, if it adds no new sequences of ordinals of length ω_1 to the ground model. The following is known.

Theorem. (p. 856 in [W]) (CH) The Forcing Axiom for the p.o. sets which are σ-closed and ω_1-centered with ω_2-many dense subsets must fail.

CH used to show the stronger form of the ω_2-c.c. The following is a very specific case among others in [S2].

Theorem. ([S2]) (CH is not assumed) The Forcing axiom for the p.o. sets which are σ-closed, ω_2-Baire and preserve the stationary subsets of ω_2 with ω_2-many dense subsets must fail.

Main set theoretic structures in the construction of [S2] are what are termed witnesses and strong witnesses. We observe that it suffices to consider $\square_{\omega_1}$ instead of those objects in the specific case of ω_2. More specifically,

(1) If $\square_{\omega_1}$ fails, then the Forcing Axiom for the notion of p.o. set to force a generic $\square_{\omega_1}$-sequence via the possible initial segments must fail. It is well-known that the p.o. set is σ-closed, ω_2-Baire and preserves the stationary subsets of ω_2.

(2) If $\square_{\omega_1}$ holds, then we may turn it into a stronger one (see lemma below) and consider a p.o. set which forces a counter example to this stronger property to fail. This notion of forcing turns out to be in the same category as above.

(3) Hence either (1) or (2), we have the failure of this type of Forcing Axiom.

It appears that the main new point here is that we use $\square_{\omega_1}$ instead of CH. This situation is somewhat analoguous to the constructions of ω_2-Souslin trees. Namely, one may construct asssuming CH together with $\diamondsuit_{\omega_2}(S^2_1)$, while the others may use $\square_{\omega_1}$ and $\diamondsuit_{\omega_2}(S^2_1)$. (see pp. 140-143 in [D])

We appreciate a series of talks by [F] on this subject in the Set Theory Seminar, Nagoya University, May through July, 2002.

§1. Turning the $\square_{\omega_1}$-sequences into stronger ones

In this section we formulate a stronger form of $\square_{\omega_1}$. This corresponds to a strong witness of [S2]. Given any club E of ω_2, a strong $\square_{\omega_1}$-sequence $\mathcal{C} = \langle C_\delta \mid \delta \in \text{limit} \cap \omega_2\rangle$ would capture E in a specific manner at quite many C_δ with $\delta \in S^2_1$. The manner C_δ captures E is that $C_\delta \setminus \text{acc}(C_\delta)$ hits E cofinally often below

δ. Since $\delta \in S^2_1$, it is necessary that $\mathrm{acc}(C_\delta)$ hits E club often below δ as long as $\delta \in \mathrm{acc}(E)$, though. Here $\mathrm{acc}(C_\delta)$ denotes the accumulation points of C_δ.

1.1 Definition. $\mathcal{C} = \langle C_\delta \mid \delta \in \mathrm{limit} \cap \omega_2 \rangle$ is a $\square_{\omega_1}$-*sequence*, if

- C_δ is a closed unbounded subset of δ,
- If $\mathrm{cf}(\delta) < \omega_1$, then $\mathrm{o.t.}(C_\delta) < \omega_1$,
- If $\alpha \in \mathrm{acc}(C_\delta)$, then $C_\alpha = C_\delta \cap \alpha$,

1.2 Definition. Let $\mathcal{C} = \langle C_\delta \mid \delta \in \mathrm{limit} \cap \omega_2 \rangle$ be a $\square_{\omega_1}$-sequence.
$\mathcal{C}$ is a *strong* $\square_{\omega_1}$-sequence, if for any club $E \subseteq \omega_2$, the following is stationary in ω_2

$$\{\delta \in S^2_1 \mid \ \sup\{\alpha \in C_\delta \mid \mathrm{suc}_{C_\delta}(\alpha) \in E\} = \delta\}$$

where, $\mathrm{suc}_{C_\delta}(\alpha)$ denotes the least member of C_δ strictly above $\alpha \in C_\delta$. Namely, the next element of α in C_δ.

The following entails that we once have a $\square_{\omega_1}$-sequence $\mathcal{C}$, then we may assume that it is strong.

1.3 Lemma. Let $\mathcal{C} = \langle C_\delta \mid \delta \in \mathrm{limit} \cap \omega_2 \rangle$ be a $\square_{\omega_1}$-sequence. Then there exists a club $E^* \subseteq \omega_2$ such that for any club $E \subseteq \omega_2$, the following is stationary in ω_2.

$$\{\delta \in S^2_1 \mid \ \sup\{\alpha \in C_\delta \mid \alpha < \sup(E^* \cap \mathrm{suc}_{C_\delta}(\alpha)) \in E\} = \delta\}$$

Proof. By contradiction. Suppose not and construct $\langle E^*_n \mapsto E_n \mid n < \omega \rangle$ such that

(1) $E^*_0 = \omega_2$,

(2) $E^*_n \mapsto E_n$ are clubs in ω_2 such that the following is non-stationary.

$$A_n = \{\delta \in S^2_1 \mid \sup\{\alpha \in C_\delta \mid \alpha < \sup(E^*_n \cap \mathrm{suc}_{C_\delta}(\alpha)) \in E_n\} = \delta\}.$$

(3) $E^*_{n+1} \subset \mathrm{acc}(E_n \cap E^*_n)$ and $E^*_{n+1} \cap A_n = \emptyset$.

In particular,

$$E^*_{n+1} \subset E^*_n \text{ and } E^*_{n+1} \subset E_n.$$

Define clubs E^* and E^{**} in ω_2 as follows;

$$E^* = \bigcap \{E^*_n \mid n < \omega\},$$

and

$$E^{**} = \mathrm{acc}(E^* \cap S^2_1).$$

Take

$$\delta^* \in S^2_1 \cap E^{**}.$$

Since $\mathrm{o.t.}(C_{\delta^*}) = \omega_1$, notice that $(E^* \cap S^2_1 \cap \delta^*) \subset (\delta^* \setminus \mathrm{acc}(C_{\delta^*}))$ is cofinal in δ^*.
And for any $n < \omega$, we have

$$\delta^* \in E^*_{n+1} \text{ and so } \delta^* \notin A_n.$$

Define

$$\beta_n = \mathrm{Max}\{\mathrm{Min}(C_{\delta^*}),\ \sup\{\alpha \in C_{\delta^*} \mid \alpha < \sup(E_n^* \cap \mathrm{suc}_{C_{\delta^*}}(\alpha)) \in E_n\}\} < \delta^*$$

so that

$$\beta_n \in C_{\delta^*}.$$

Let

$$\beta^* = \sup\{\beta_n \mid n < \omega\} \in C_{\delta^*}.$$

Pick any γ^* such that

$$\mathrm{suc}_{C_{\delta^*}}(\beta^*) < \gamma^* \in \delta^* \cap E^* \cap S_1^2.$$

Take ζ^* so that

$$\mathrm{suc}_{C_{\delta^*}}(\beta^*) \leq \zeta^* = \mathrm{Max}(C_{\delta^*} \cap \gamma^*) \in C_{\delta^*}.$$

Since $\gamma^* \in S_1^2$, we have

$$\zeta^* < \gamma^*.$$

Let

$$\xi^* = \mathrm{suc}_{C_{\delta^*}}(\zeta^*).$$

Then by the definition of ζ^*, we have

$$\gamma^* \leq \xi^*.$$

Case 1. $\gamma^* < \xi^*$: Fix any $n < \omega$. Since $\gamma^* \in E^* \subset E_n^*$, we have

$$\zeta^* < \sup(E_n^* \cap \xi^*).$$

But $\beta_n < \zeta^*$, so

$$\sup(E_n^* \cap \xi^*) \notin E_n.$$

But $E_{n+1}^* \subset E_n$, so

$$\sup(E_n^* \cap \xi^*) \notin E_{n+1}^*.$$

Since $E_{n+1}^* \subset E_n^*$, we conclude

$$\gamma^* \leq \sup(\xi^* \cap E_{n+1}^*) < \sup(\xi^* \cap E_n^*).$$

Therefore, we have a strictly descending infinite sequence of ordinals. This is a contradiction.

Case 2. $\gamma^* = \xi^* = \mathrm{suc}_{C_{\delta^*}}(\zeta^*)$: Take any $n < \omega$. Since $\gamma^* \in E^* \subset E_{n+1}^* \subset \mathrm{acc}(E_n^*)$, we have

$$\zeta^* < \sup(E_n^* \cap \xi^*) = \xi^* = \gamma^* \in E_n.$$

Hence by the definition of β_n, we have

$$\zeta^* < \beta_n.$$

This is a contradiction.

□

We may always turn a given $\square_{\omega_1}$-sequence into a stronger one.

1.4 Lemma. Let $\mathcal{C}$ and E^* be as in the previous lemma. We set

$$C'_\delta = C_\delta \cup \{\beta \mid \alpha \in C_\delta, \alpha < \beta = \sup(E^* \cap \mathrm{suc}_{C_\delta}(\alpha))\}.$$

Then $\mathcal{C}' = \langle C'_\delta \mid \delta \in \mathrm{limit} \cap \omega_2\rangle$ is a strong $\square_{\omega_1}$-sequence.

Proof. We neeed to check the following.

(1) C'_δ is a club in δ,

(2) If $\mathrm{cf}(\delta) < \omega_1$, then $\mathrm{o.t.}(C'_\delta) < \omega_1$,

(3) If $\alpha \in \mathrm{acc}(C'_\delta)$, then $C'_\alpha = C'_\delta \cap \alpha$,

And

(4) For any club $E \subseteq \omega_2$, if $\alpha \in C_\delta$ and $\alpha < \sup(E^* \cap \mathrm{suc}_{C_\delta}(\alpha)) \in E$, then $\alpha \in C'_\delta$ and $\mathrm{suc}_{C'_\delta}(\alpha) = \sup(E^* \cap \mathrm{suc}_{C_\delta}(\alpha)) \in E$.

They are mostly routine to check and left to the reader.

$\square$

§ 2. Forcing a counter-example to make sure a given $\square_{\omega_1}$-sequence non-strong

While any $\square_{\omega_1}$-sequence $\mathcal{C}$ would produce to a strong one $\mathcal{C}'$, we may force a club of ω_2 to make sure that $\mathcal{C}$ is not strong in the generic extensions.

2.1 Definition. Let $\langle C_\alpha \mid \alpha \in \mathrm{limit} \cap \omega_2\rangle$ be a $\square_{\omega_1}$-sequence. We define $p \in P$, if

(1) p is a closed bounded subset of ω_2,

(2) For any $\delta \in \mathrm{acc}(p) \cap S^2_1$ there exists $\bar\delta < \delta$ such that $(p \cap \delta) \setminus \bar\delta \subset \mathrm{acc}(C_\delta) \cup (\delta \setminus C_\delta)$.

For $p_1, p_2 \in P$, we set $p_2 \le p_1$, if p_2 end-extends p_1.

2.2 Lemma. (1) P is σ-closed.

(2) P is ω_2-Baire.

Proof. For (1): Let $\langle p_n \mid n < \omega\rangle$ be a descending sequence in P. For each $n < \omega$, let $\alpha_n = \sup p_n$. We may assume the α_n's are strictly increasing. Let $\alpha = \sup\{\alpha_n \mid n < \omega\}$ and let $q = \bigcup\{p_n \mid n < \omega\} \cup \{\alpha\}$. Since $\alpha \in S^2_0$, we have $q \in P$ and so q is a lower bound of the p_n's.

For (2): Let $p \Vdash_P$ "$\dot f : \omega_1 \longrightarrow V$". We want to find $q \le p$ such that $q \mid\mid \dot f$. To this end let θ be a sufficiently large regular cardinal and $\langle N_i \mid i \le \omega_1\rangle$ be a sequence such that

- N_i is an elementary substructure of H_θ, $\mid N_i \mid = \omega_1$, $N_i \cap \omega_2 < \omega_2$ and for any i which is non-limit, we demand $N_i \cap \omega_2 \in S^2_1$,
- $\langle N_k \mid k \le i\rangle \in N_{i+1}$ and the N_i's are continuously increasing,
- In particular, the $N_i \cap \omega_2$'s are strictly increasing and forms a club in ω_2,
- N_0 contains every relevant parameters.

For each $i < \omega_1$, let

$$\delta_i = N_i \cap \omega_2 \text{ and } \delta = \sup\{\delta_i \mid i < \omega_1\} = N_{\omega_1} \cap \omega_2 \in S^2_1.$$

And let

$$W = \{i < \omega_1 \mid N_i \cap \omega_2 \in \mathrm{acc}(C_\delta)\} \subset S^2_0.$$

We have

$$\text{If } i \in W, \text{ then } W \cap (i+1) \in N_{i+1}.$$

This is because $\delta_i \in \mathrm{acc}(C_\delta)$ and so $C_{\delta_i} = C_\delta \cap \delta_i$. Hence

$$W \cap i = \{k < i \mid \delta_k \in \mathrm{acc}(C_\delta)\} = \{k < i \mid \delta_k \in \mathrm{acc}(C_{\delta_i})\}.$$

Since $\langle N_k \mid k \leq i\rangle, C_{\delta_i}$ are in N_{i+1} and since $i \in N_{i+1}$, we conclude $W \cap (i+1) \in N_{i+1}$.

We build $\langle p_i \mid i \in W\rangle$ by recursion on i so that

- For $i_0 = \mathrm{Min}(W)$, we set $p_{i_0} = F(p, \sup(C_\delta \cap \delta_{i_0+1}) + 1, 0)$,
- For $i \in W$ such that $j = \mathrm{Max}(W \cap i) < i$, we set $p_i = F(p_j, \sup(C_\delta \cap \delta_{i+1}) + 1, \mathrm{o.t.}(W \cap i))$,
- For $i \in \mathrm{acc}(W)$, we set $p_i = \bigcup\{p_k \mid k \in W \cap i\} \cup \{\sup(\bigcup\{p_k \mid k \in W \cap i\})\}$.

Here, $F : P \times \omega_2 \times \omega_1 \longrightarrow P$ such that for (p, ξ, i) with $\sup p < \xi$, if we let $q = F(p, \xi, i)$, then

$$q \leq p \cup \{\xi\} \text{ and } q \mid\mid \dot{f}(i).$$

We may assume that $F \in N_0$.

Claim. We have 4 items in accordance with the recursive construction.

($\underline{i = i_0}$): For $i_0 = \mathrm{Min}(W)$, we have

- $p_{i_0} \in P$,
- $\delta_{i_0} < \sup p_{i_0}$,
- $p_{i_0} \in N_{i_0+1}$,
- $p_{i_0} \mid\mid \dot{f}(0)$,
- $p_{i_0} \leq p$,
- $(p_{i_0} \setminus p) \cap C_\delta = \emptyset$,

($\underline{i \text{ is the successor of } j \text{ in } W}$): For $i \in W$ with $j = \mathrm{Max}(W \cap i) < i$, we have

- $p_i \in P$,
- $\delta_i < \sup p_i$,
- $p_i \in N_{i+1}$,
- $p_i \mid\mid \dot{f}(\mathrm{o.t.}(W \cap i))$,
- $p_i \leq p_j$,
- $(p_i \setminus p_j) \cap C_\delta = \emptyset$,

- $\langle p_k \mid k \in W \cap (i+1)\rangle \in N_{i+1}$,

  ($\underline{i \in \mathrm{acc}(W)}$): For $i \in \mathrm{acc}(W)$, we have

- $p_i \in P$,
- For all $k \in W \cap i$, $p_i \leq p_k$,
- $\sup p_i = \delta_i \in \mathrm{acc}(C_\delta)$,
- $\langle p_k \mid k \in W \cap (i+1)\rangle \in N_{i+1}$,

  (conclusion): Let $q = \bigcup\{p_i \mid i \in W\} \cup \{\delta\}$, then $q \in P$, $q \leq p$ and $q \mid\mid \dot{f}$.

Proof. By induction on $i \in W$.

($i = i_0$): Since $p \in N_0 \subset N_{i_0}$, we have $\sup p < \delta_{i_0} \in C_\delta$ and so

$$\delta_{i_0} \leq \sup(C_\delta \cap \delta_{i_0+1}) < \sup(C_\delta \cap \delta_{i_0+1}) + 1 < \delta_{i_0+1} \in S^2_1.$$

Hence $p_{i_0} \in P$, $\delta_{i_0} < \sup p_{i_0}$, $p_{i_0} \mid\mid \dot{f}(0)$, $p_{i_0} \leq p$, $(p_{i_0} \setminus p) \cap C_\delta = \emptyset$ and $p_{i_0} \in N_{i_0+1}$.

(i is the successor of j in W): We have $\langle p_k \mid k \in W \cap (j+1)\rangle \in N_{j+1}$. Since $p_j \in N_{j+1} \subset N_i$, we have $\sup p_j < \delta_i \in C_\delta$. And so

$$\delta_i \leq \sup(C_\delta \cap \delta_{i+1}) < \sup(C_\delta \cap \delta_{i+1}) + 1 < \delta_{i+1} \in S^2_1.$$

Hence $p_i \in P$, $\delta_i < \sup p_i$, $p_i \mid\mid \dot{f}(\mathrm{o.t.}(W \cap i))$, $p_i \leq p_j$, $(p_i \setminus p_j) \cap C_\delta = \emptyset$ and $p_i \in N_{i+1}$. And so $\langle p_k \mid k \in W \cap (i+1)\rangle \in N_{i+1}$.

($i \in \mathrm{acc}(W)$): We have constructed $\langle p_k \mid k \in W \cap i\rangle$. We observe that $\langle p_k \mid k \in W \cap i\rangle$ is definable from parameters which are all in N_{i+1}. And so we would have

$$\langle p_k \mid k \in W \cap i\rangle \in N_{i+1}.$$

Some details follow. Notice first $\delta_i \in \mathrm{acc}(C_\delta)$ and so

$$C_{\delta_i} = C_\delta \cap \delta_i.$$

Since both $\langle N_k \mid k \leq i\rangle$ and $W \cap (i+1)$ are in N_{i+1}, we have

$$\langle N_{k+1} \mid k \in W \cap i\rangle \in N_{i+1}.$$

Since C_{δ_i} is in N_{i+1}, we have

$$\langle \sup(C_\delta \cap \delta_{k+1}) \mid k \in W \cap i\rangle = \langle \sup(C_{\delta_i} \cap N_{k+1}) \mid k \in W \cap i\rangle \in N_{i+1}.$$

Note that $\langle p_k \mid k \in W \cap i\rangle$ is definable in H_θ from $W \cap i$, F, p and $\langle \sup(C_\delta \cap \delta_{k+1}) \mid k \in W \cap i\rangle$ as follows;

- For $k_0 = \mathrm{Min}(W \cap i)$, $p_{k_0} = F(p, \sup(C_\delta \cap \delta_{k_0+1}) + 1, 0)$,
- For $k \in W \cap i$ with $j = \mathrm{Max}((W \cap i) \cap k) < k$, $p_k = F(p_j, \sup(C_\delta \cap \delta_{k+1}) + 1, \mathrm{o.t.}((W \cap i) \cap k))$,
- For $k \in \mathrm{acc}(W \cap i)$, $p_k = \bigcup\{p_{\bar{k}} \mid \bar{k} \in (W \cap i) \cap k\} \cup \{\sup(\bigcup\{p_{\bar{k}} \mid \bar{k} \in (W \cap i) \cap k\})\}$.

We have $\langle p_k \mid k \in W \cap i\rangle \in N_{i+1}$.

Let $p_i = \bigcup\{p_k \mid k \in (W \cap i)\} \cup \{\sup(\bigcup\{p_k \mid k \in (W \cap i)\})\}$. We know $p_i \in N_{i+1}$ and $\langle p_k \mid k \in W \cap (i+1)\rangle \in N_{i+1}$. Since $\delta_k < \sup p_k < \delta_{k+1}$ for all $k \in (W \cap i) \setminus \mathrm{acc}(W)$, we know p_i as defined is a bounded closed set. Since $\sup p_i = \delta_i \in \mathrm{acc}(C_\delta) \cap S^2_0$, we have $p_i \in P$.

For (conclusion): Since $\delta_i < \sup p_i < \delta_{i+1}$ for $i \in W \setminus \mathrm{acc}(W)$, we see that q as defined is a closed subset of ω_2 with $\sup q = \delta$. To see $q \in P$, we may check that

$$q \setminus p \subset \mathrm{acc}(C_\delta) \cup (\delta \setminus C_\delta).$$

But this holds by construction. Since q sits below every p_i, we have $q \leq p$ and $q \mid\mid \dot{f}$.

$\square$

2.3 Lemma. P preserves every stationary subset of ω_2.

Proof. Since P is σ-closed, P is proper. Hence P preserves every stationary subset of S^2_0. Therefore we need to take care of stationary subsets T with $T \subseteq S^2_1$.

Let $T \subseteq S^2_1$ be stationary and $p \Vdash_P$ "$\dot{f} : \omega_2 \longrightarrow \omega_2$". We want to find $q \leq p$ and $\delta \in T$ such that $q \Vdash_P$ "$\forall \alpha < \delta\ \dot{f}(\alpha) < \delta$". To this end let θ be a sufficiently large regular cardinal and $\langle N_i \mid i < \omega_2\rangle$ be a continuously increasing sequence of elementary substructures of H_θ such that

- $|N_i| = \omega_1$ and $\delta_i = N_i \cap \omega_2 < \omega_2$.
- If $i = 0$ or i is a sucessor, then $\delta_i \in S^2_1$.
- $\langle N_i \mid i \leq j\rangle \in N_{j+1}$.

Take $i^* \in S^2_1$ such that

- $\delta^* = \delta_{i^*} = N_{i^*} \cap \omega_2 \in T \subseteq S^2_1$.

Let

$$W^* = \{i < i^* \mid \delta_i \in \mathrm{acc}(C_{\delta^*})\} \subset S^2_0.$$

Claim 1. (1) The order type of W^* is exactly ω_1.

(2) If $i \in W^*$, then $W^* \cap (i+1) \in N_{i+1}$.

Proof. For (1): Since $\delta^* \in S^2_1$, we know that $\langle \delta_i | i < i^*\rangle$ is a club in δ^* and so is $\mathrm{acc}(C_{\delta^*})$. Hence o.t.$(W^*) = \omega_1$.

For (2): This is where we need $\square_{\omega_1}$. Since $i \in W^*$, we have

$$C_{\delta_i} = C_{\delta^*} \cap \delta_i.$$

Then for any $k < i$, we have $k \in W^*$ iff $\delta_k \in \mathrm{acc}(C_{\delta^*})$ iff $\delta_k \in \mathrm{acc}(C_{\delta_i})$ iff $\langle N_k \mid k \leq i\rangle(k) \cap \omega_2 \in \mathrm{acc}(C_{\delta_i})$. But $\langle N_k \mid k \leq i\rangle, \omega_2, C_{\delta_i} = C_{N_i \cap \omega_2}$ are all in N_{i+1}. Hence $W^* \cap (i+1) = (W^* \cap i) \cup \{i\} \in N_{i+1}$.

$\square$

We have seen that P adds no new sequences of ordinals of length less than ω_2. So we may construct $\langle p_i \mid i \in W^*\rangle$ so that

- For $i_0 = \mathrm{Min}(W^*)$, let $p_{i_0} = K(p, \sup(C_{\delta^*} \cap \delta_{i_0+1}) + 1, \delta_0)$,
- For i with $i > j = \mathrm{Max}(W^* \cap i)$, let $p_i = K(p_j, \sup(C_{\delta^*} \cap \delta_{i+1}) + 1, \delta_i)$,
- For $i \in \mathrm{acc}(W^*)$, let $p_i = \bigcup\{p_k \mid k \in W^* \cap i\} \cup \{\delta_i\}$,

where,

$$K : P \times \omega_2 \times \omega_2 \longrightarrow P$$

such that for (a, ξ, η) with $\sup(a) < \xi < \omega_2$, if we write $q = K(a, \xi, \eta)$, then $q \leq a \cup \{\xi\}$ and q decides $\dot{f}\lceil\eta$.

We may assume that $K \in N_0$.

Claim 2. For $i_0 = \mathrm{Min}(W^*)$, we have

- $p_{i_0} \in P$,
- $\delta_{i_0} < \sup(p_{i_0})$,
- $p_{i_0} \in N_{i_0+1}$,
- $p_{i_0} \Vdash_P$ "$\dot{f}\lceil\delta_{i_0} = f\lceil\delta_{i_0}$" for some (abusive notation) $f\lceil\delta_{i_0} \in N_{i_0+1}$,
- $p_{i_0} \leq p$,
- $(p_{i_0} \setminus p) \cap C_{\delta^*} = \emptyset$.

Proof. Note $K, p \in N_0 \subset N_{i_0+1}$. Also note that $\sup(C_{\delta^*}) \cap \delta_{i_0+1} < \delta_{i_0+1}$ and so $\sup(C_{\delta^*}) \cap \delta_{i_0+1} \in N_{i_0+1}$. Hence $p_{i_0} \in N_{i_0+1}$. The rest is more or less explicit in the definition of p_{i_0+1}.

□

Claim 3. For $i > j = \mathrm{Max}(W^* \cap i)$, we inductively suppose

- $\langle p_k \mid k \in W^* \cap (j+1) \rangle \in N_{j+1} \subseteq N_i \subset N_{i+1}$.
- In particular, $p_j \in N_{j+1} \subseteq N_i$ holds.

Then we have

- $p_i \in P$,
- $\delta_i < \sup(p_i)$,
- $p_i \in N_{i+1}$,
- $p_i \Vdash_P$ "$\dot{f}\lceil\delta_i = f\lceil\delta_i$" for some (abusive notation) $f\lceil\delta_i \in N_{i+1}$,
- $p_i \leq p_j$,
- $(p_i \setminus p_j) \cap C_{\delta^*} = \emptyset$,
- $\langle p_k \mid k \in W^* \cap (i+1) \rangle \in N_{i+1}$.

Proof. Since $i \in W^*$, we have $W^* \cap i \in N_{i+1}$. Hence $\mathrm{o.t.}(W^* \cap i) \in N_{i+1}$. Since $\delta_i < \sup(C_{\delta^*} \cap \delta_{i+1}) + 1 \in N_{i+1}$ as well, we have $p_i = K(p_j, \sup(C_{\delta^*} \cap \delta_{i+1}) + 1, \mathrm{o.t.}(W^* \cap i)) \in N_{i+1}$. Hence $\langle p_k \mid k \in W^* \cap (i+1) \rangle = \langle p_k \mid k \in W^* \cap (j+1) \rangle \cup \{(i, p_i)\} \in N_{i+1}$.

□

Claim 4. For $i \in \mathrm{acc}(W^*)$, we have

- $p_i \in P$,
- For all $k \in W^* \cap i$, $p_i \leq p_k$,
- $\sup(p_i) = \delta_i \in \mathrm{acc}(C_{\delta^*})$,
- $\langle p_k \mid k \in W^* \cap (i+1) \rangle \in N_{i+1}$.

Proof. For $k \in W^* \cap i$, we inductively have $p_k \in N_{k+1}$ and $\delta_k < \sup(p_k)$. Hence $\delta_k < \sup(p_k) < \delta_{k+1}$. Since $i \in \mathrm{acc}(W^*)$, we conclude

$$\sup\{\sup(p_k) \mid k \in W^* \cap i\} = \sup\{\delta_k \mid k \in W^* \cap i\} = \delta_i.$$

Since $\mathrm{cf}(\delta_i) = \omega$, we have $p_i \in P$.
$\langle p_k \mid k \in W^* \cap i\rangle$ is definable as follows.

- For $k_0 = \mathrm{Min}(W^* \cap i)$, $p_{k_0} = K(p, \sup(C_{\delta_i} \cap \delta_{k_0+1}) + 1, \delta_{k_0})$,
- For $k > j = \mathrm{Max}((W^* \cap i) \cap k)$, $p_k = K(p_j, \sup(C_{\delta_i} \cap \delta_{k+1}) + 1, \delta_k)$,
- For $k \in \mathrm{acc}(W^* \cap i)$, $p_k = \bigcup\{p_{\bar{k}} \mid \bar{k} \in (W^* \cap i) \cap k\} \cup \{\delta_k\}$.

This is in terms of K, $W^* \cap i$, $\langle N_k \mid k \in W^* \cap i\rangle$, $\mathcal{C} = \langle C_\delta \mid \delta \text{ is limit and } \delta < \omega_2\rangle$ and C_{δ_i} which are all in N_{i+1}. Hence $\langle N_k \mid k \in W^* \cap i\rangle \in N_{i+1}$. For this definabilty, we use the $\square_{\omega_1}$-ness of $\mathcal{C}$.

□

Now let $q = \bigcup\{p_k \mid k \in W^*\} \cup \{\delta^*\}$. Then this q is closed, as

$$\delta_k < \sup(p_k) < \delta_{k+1}.$$

And $q \in P$, as $q \setminus p \subset \mathrm{acc}(C_{\delta^*}) \cup (\delta^* \setminus C_{\delta^*})$. Since $p_k \Vdash_P$ "$\dot{f}\lceil\delta_k = f\lceil\delta_k$" with $f\lceil\delta_k \in N_{k+1}$, we conclude $q \Vdash_P$ "$\forall\alpha < \delta^*\ \dot{f}(\alpha) < \delta^*$".

□

2.4 Lemma. P adds a club $E \subset \omega_2$ such that $\langle C_\alpha \mid \alpha \in \text{limit} \cap \omega_2\rangle$ is non-strong due to E. Namely,

$$\forall\delta \in \mathrm{acc}(E) \cap S^2_1\ \{\alpha \in C_\delta \mid \mathrm{suc}_{C_\delta}(\alpha) \in E\} \text{ is bounded below } \delta.$$

Proof. We design P so that this holds. Let $E = \bigcup G$, where G is a P-generic filter over V. Then we have

$$\forall\delta \in \mathrm{acc}(E) \cap S^2_1\ \exists\bar{\delta} < \delta \text{ such that } E \cap (\delta \setminus \bar{\delta}) \subset (\delta \setminus C_\delta) \cup \mathrm{acc}(C_\delta).$$

□

Accordingly we have

2.5 Theorem. The forcing Axiom for the following class of p.o. sets $\mathcal{P}$ with ω_2-many dense subsets fails, where

$\mathcal{P}$ contains

- The notion of forcing to force $\square_{\omega_1}$ via the initial segments,
- The notions of forcing to kill the strongness of all $\square_{\omega_1}$-sequences, if any.

□

2.6 Note. (CH) We may directly force a generic strong $\square_{\omega_1}$-sequence via countable conditions.

Question 1. Give a single p.o. set which is σ-closed, ω_2-Baire, preserves the stationary subsets of ω_2 so that the Forcing Axiom with ω_2-many dense subsets fails. Does $< \omega_2$-support product of the above p.o. sets work ?

Question 2. Is it easy to generalize the argument in this note to higher cardinals ? Do we really need witnesses and strong witnesses of [S2] ?

Question 3. Does a non-reflecting stationary set $S \subset S^2_0 = \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ of any sort suffice to replace $\square_{\omega_1}$-sequence in the present context ? Can you view witnesses and strong witnesses of [S2] along this line ?

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