A sequent system for a sublogic of the smallest interpretability logic

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Abstract. An interpretability logic is an extension of provability logic **GL** with a binary modal operator $\rhd$. The smallest interpretability logic **IL** is obtained by adding axioms concerning $\rhd$ to **GL** (cf. Visser [Vis97] and Japaridze and de Jongh [JJ98]). The logic **IK4** is a sublogic of **IL** and is obtained by adding the same axioms to normal modal logic **K4** as the addtional axioms of **IL**. [Sas02] gave a cut-free sequent system for **IK4** (see also [Sas01]). Here we give another cut-free sequent system for **IK4**. Both of the system in [Sas02] and the system here satisfy kinds of subformula property, however, our new system has nicer one. In the system in [Sas02], a formula $B \rhd D$ possibly occurs in a cut-free proof figure for $A \rhd B \to C \rhd D$, while in the new system doesn't.

1 Preliminaries

The language of the logic **IK4** contains two modal operators $\Box$ and $\rhd$. However, we can show the equivalence between $\Box A$ and $(A \supset \bot) \rhd \bot$. Hence, we do not have to treat $\Box$ as a primary operator. Systems for interpretability logics with two primary modal operators are much more complicated than the ones with one primary modal operator. So, we treat $\Box A$ as an abbreviation of $(A \supset \bot) \rhd \bot$.

Definition 1.1. The set **WFF** of formulas are defined inductively as follows.
(1) a propositional variable belongs to **WFF**,
(2) $\bot \in$ **WFF**,
(3) $A, B \in$ **WFF** implies $(A \wedge B), (A \vee B), (A \supset B), (A \rhd B) \in$ **WFF**.

An element of **WFF** is said to be a formula, especially a formula of the form $A \rhd B$ is said to be a $\rhd$-formula. The expressions $\neg A$, $\Box A$ and $\Diamond A$ are abbreviations for $A \supset \bot, \neg A \rhd \bot$ and $\neg(A \rhd \bot)$, respectively. By **IK4**, we mean the smallest set of formulas containing all the tautologies and axioms
$(K) : \Box(p \supset q) \supset (\Box p \supset \Box q)$,
$(4) : \Box p \supset \Box\Box p$,
$(J1) : \Box(p \supset q) \supset (p \rhd q)$,
$(J2) : (p \rhd q) \wedge (q \rhd r) \supset (p \rhd r)$,
$(J3) : (p \rhd r) \wedge (q \rhd r) \supset ((p \vee q) \rhd r)$,
$(J5) : (\Diamond p) \rhd p$,
and closed under modus ponens, substitution and necessitation. The axiom $(J4)$ is necessary of the logic in the language with $\Box$ as a primary one, while it does not necessary of the logic in our language. So, we do not need $(J4)$.

To introduce a sequent system, we need some preparations. We use Greek letters, possibly with suffixes, for finite sets of formulas, especially we use Σ, possibly with suffixes, for finite sets of $\rhd$-formulas. For each prefix $\odot \in \{\Box, \Diamond, \neg\}$, the expression $\odot\Gamma$ denotes the set $\{\odot A \mid A \in \Gamma\}$. Similarly, $\Gamma \rhd \bot$ denotes $\{A \rhd \bot \mid A \in \Gamma\}$. By a sequent, we mean the expression

$$\Gamma \to \Delta.$$

For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

We put

$$\mathbf{an}(\{A\} \to \Delta) = A \text{ (antecedent).}$$

$$\mathbf{Su}(\Gamma \to \Delta) = \Delta \text{ (succedent).}$$

Definition 1.2. Let

$$S_1 \quad \cdots \quad S_n$$

be a sequence of n sequents. *The depth* $d_{S_1 \ \dots \ S_n}(o_i)$ of the occurrence o_i of S_i denotes $n+1-i$.

We use **SEQ**, possibly with suffixes, for sequences of sequents. Let **SEQ** be a sequence of sequents. For an occurrence o of a sequent S in **SEQ**, we also use the expressions $\mathbf{an}(o)$ and $\mathbf{Su}(o)$ for $\mathbf{an}(S)$ and $\mathbf{Su}(S)$, respectively.

Definition 1.3. A sequent is said to be *on* a sequent $\Sigma \to A \rhd B$ if it is of the form

$$C \to \Delta,$$

where $C \in \{A\} \cup \{E \mid D \rhd E \in \Sigma\}$ and $B \in \Delta \subseteq \{D \mid D \rhd E \in \Sigma\} \cup \{B\}$.

Remark 1.4.

(1) *There are finitely many sequents on $\Sigma \to A \rhd B$. If the number of formulas in Σ is n, then the number of sequents on $\Sigma \to A \rhd B$ is less than or equal to $(n+1) \times 2^n$.*

(2) *If S is a sequent on $\Sigma \to A \rhd B$, then it is also on $\Sigma \cup \Sigma' \to A \rhd B$.*

Definition 1.5. We say that a sequence **SEQ** of sequents is *on* a sequent $\Sigma \to A \rhd B$ if each sequent in **SEQ** is on $\Sigma \to A \rhd B$ and $\mathbf{an}(S) = A$ for some sequent S in **SEQ**.

Definition 1.6. Let **SEQ** be a sequence on $\Sigma \to A \rhd B$ and let Σ' be a subset of Σ. Let **OCC** be the set of occurrences of sequents in **SEQ**.

(1) A mapping

$$f : \{(o, C) \mid o \in \mathbf{OCC}, C \in \mathbf{Su}(o)_B\} \to \mathbf{OCC}$$

is called *a mapping on* $\mathbf{SEQ}/\Sigma'$ if $C \rhd \mathbf{an}(f(o, C)) \in \Sigma'$ and $d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f(o, C))$ for any occurrence o and any formula $C \in \mathbf{Su}(o)_B$. By $Im(f)$, we mean the range of f.

(2) A mapping

$$f : \{(o, C) \mid o \in \mathbf{OCC}, C \in \mathbf{Su}(o)_B\} \to \mathbf{OCC}$$

is called *a pseudo-onto mapping on* $\mathbf{SEQ}/\Sigma' \to A \rhd B$ if f is on $\mathbf{SEQ}/\Sigma'$ and

$$\{o \mid o \in \mathbf{OCC}, \mathbf{an}(o) \neq A\} \subseteq Im(f).$$

Definition 1.7. We say that a sequence **SEQ** is *essential* on a sequent $\Sigma \to A \rhd B$ if the following three conditions hold:

(1) **SEQ** is on $\Sigma \to A \rhd B$,

(2) there exists an pseudo-onto mapping on $\mathbf{SEQ}/\Sigma \to A \rhd B$,

(3) $\mathbf{an}(o_1) \neq \mathbf{an}(o_2)$ for any different occurrences o_1 and o_2 in **SEQ**.

Corollary 1.8. *Let n be the number of elements in Σ.*

(1) *An essential sequence on $\Sigma \to A \rhd B$ has at most $n+1$ sequents.*

(2) *the number of essential sequences on $\Sigma \to A \rhd B$ is less than $(n+1)! \times 2^{n \times (n+1)}$.*

Proof. (1) is from Definition 1.7(3), and (2) from Remark 1.4(1). $\dashv$

Lemma 1.9. *Let*

$$S_1 \quad \cdots \quad S_n$$

be an essential sequence on $\Sigma \to A \rhd B$. Then
(1) $\mathbf{Su}(S_n) = \{B\}$,
(2) $\mathbf{an}(S_1) = A$,
(3) $\mathbf{an}(S_i) = A$ *if and only if* $i = 1$.

Proof. Let o_i be the occurrence of S_i. Then

$$n \geq d_{S_1 \ \cdots \ S_n}(o_i) \geq 1. \cdots (*1)$$

Since the sequence is essential, there exists a pseudo-onto mapping f on

$$(S_1 \quad \cdots \quad S_n)/\Sigma \to A \rhd B.$$

For (1): Suppose that $\mathbf{Su}(S_n) \neq \{B\}$. Since S_n is on $\Sigma \to A \rhd B$, we have $\{B\} \subsetneq \mathbf{Su}(S_n)$, and so, $D \in \mathbf{Su}(S_n)_B$ for some D. Using Definition 1.6, $1 = d_{S_1 \ \cdots \ S_n}(o_n) > d_{S_1 \ \cdots \ S_n}(f(o_n, D))$. This is in contradiction with $(*1)$.

For (2): Suppose that $\mathbf{an}(S_1) \neq A$. Since f is pseudo-onto, there exit an occurrence o' and a formula $D \in \mathbf{Su}(o')_B$ such that $o_1 = f(o', D)$. Also we have $d_{S_1 \ \cdots \ S_n}(o') > d_{S_1 \ \cdots \ S_n}(f(o', D)) = d_{S_1 \ \cdots \ S_n}(o_1) = n$. This is in contradiction with $(*1)$.

For (3): From (2), we have "if" part. We show "only if" part. Suppose that $\mathbf{an}(S_i) = A$. By (2) and Definition 1.7(3), we have $o_i = o_1$. Hence $i = 1$. ⊣

Example 1.10. Here we show some examples of essential sequences.
(1) A sequence of a sequent

$$p \to q$$

is only one essential sequence on

$$\to p \rhd q.$$

(2) Two sequences of sequents

$$p \to q, p_1 \qquad q_1 \to q$$

and

$$p \to q$$

are essential on

$$p_1 \rhd q_1 \to p \rhd q.$$

(3) Two sequences

$$p \to q, p_1 \qquad q_1 \to q, p_2 \qquad q_2 \to q$$

and

$$p \to q, p_1, p_2 \qquad q_1 \to q \qquad q_2 \to q$$

are essential on

$$p_1 \rhd q_1, p_2 \rhd q_2 \to p \rhd q.$$

(4) A sequence

$$p \rhd \bot \to p, q \qquad q \rhd \bot \to p, q \qquad \bot \to p$$

is essential on

$$q \rhd (q \rhd \bot), q \rhd \bot \to (p \rhd \bot) \rhd p.$$

(5) A sequence

$$p \to q, \neg r \qquad \bot \to q$$

is essential on

$$\Box r \to p \rhd q.$$

(6) A sequence

$$\neg\Box p \rightarrow \bot, \neg p \qquad \bot \rightarrow \bot$$

is essential on

$$\Box p \rightarrow \Box\Box p.$$

Definition 1.11. Let $S, S_1, \cdots S_n$ be sequents.

(1) By S^*, we mean the sequent

$$\Gamma, \Delta \rhd \bot \rightarrow \Delta$$

if S is $\Gamma \rightarrow \Delta$.

(2) By $\mathbf{SEQ}^*$, we mean the sequence

$$S_1^* \qquad \cdots \qquad S_n^*$$

if **SEQ** is

$$S_1 \qquad \cdots \qquad S_n.$$

Our system **GIK4** is defined from the following axioms and inference rules in the usual way.

Axioms of GIK4

$$A \rightarrow A$$

$$\bot \rightarrow$$

Inference rules of GIK4

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}(T \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}(\rightarrow T)$$

$$\frac{\Gamma \rightarrow \Delta, A \qquad A, \Pi \rightarrow \Lambda}{\Gamma, \Pi_A \rightarrow \Delta_A, \Lambda}(\mathrm{cut})$$

$$\frac{A_i, \Gamma \rightarrow \Delta}{A_1 \wedge A_2, \Gamma \rightarrow \Delta}(\wedge \rightarrow_i) \qquad \frac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}(\rightarrow \wedge)$$

$$\frac{A, \Gamma \rightarrow \Delta \qquad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}(\vee \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, A_i}{\Gamma \rightarrow \Delta, A_1 \vee A_2}(\rightarrow \vee_i)$$

$$\frac{\Gamma \rightarrow \Delta, A \qquad B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta}(\supset\rightarrow) \qquad \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B}(\rightarrow\supset)$$

$$\frac{\mathbf{SEQ}^*}{\Sigma \rightarrow A \rhd B}(\rhd_{IK4})$$

where **SEQ** is essential on $\Sigma \rightarrow A \rhd B$.

2 Equivalence between IK4 and GIK4

The main theorem in this section is

Theorem 2.1. $A \in \mathbf{IK4}$ if and only if $\rightarrow A \in \mathbf{GIK4}$.

To prove the theorem above, we provide some preparations.

By **GK4**, we mean the system obtained from **GIK4** by replacing $(\rhd_{IK4})$ by

$$\frac{\Gamma, \Box\Gamma \rightarrow A}{\Box\Gamma \rightarrow \Box A}(\Box_{K4}).$$

By $\mathbf{GK4} + J$, we mean the system obtained from $\mathbf{GK4}$ by adding following four axioms:
$(GJ1) : \Box(A \supset B) \to A \rhd B$,
$(GJ2) : A \rhd B, B \rhd C \to A \rhd C$,
$(GJ3) : A \rhd C, B \rhd C \to (A \vee B) \rhd C$,
$(GJ5) :\to \Diamond A \rhd A$.

It is known that $\mathbf{GK4}$ enjoys cut-elimination theorem and that $\to A \in \mathbf{GK4}$ if and only if $A \in \mathbf{K4}$. So, we have

Lemma 2.2. $A \in \mathbf{IK4}$ *if and only if* $\to A \in \mathbf{GK4} + J$.

Lemma 2.3. $A \in \mathbf{GK4} + J$ implies $\to A \in \mathbf{GIK4}$.

Proof. It is sufficient to show that four axioms $(J1), (J2), (J3)$ and $(J5)$ are provable in $\mathbf{GIK4}$ and $(\Box_{K4})$ holds in $\mathbf{GIK4}$.

For $(J1)$: We note that the sequence

$$A \to B, \neg(A \supset B) \qquad\qquad \bot \to B$$

is essential on $\Box(A \supset B) \to A \rhd B$. On the other hand, it is easily seen that the following sequents are provable in $\mathbf{GIK4}$:

$$A, B \rhd \bot, \Box(A \supset B) \to B, \neg(A \supset B),$$

$$\bot, B \rhd \bot \to B.$$

So, using $(\rhd_{IK4})$, $(J1)$ is provable in $\mathbf{GIK4}$.

The other axioms and rules can be shown by the following $(\rhd_{IK4})$ in a similar way.

$$\frac{A, C \rhd \bot, A \rhd \bot \to C, A \quad B, C \rhd \bot, B \rhd \bot \to C, B \quad C, C \rhd \bot \to C}{A \rhd B, B \rhd C \to A \rhd C}$$

$$\frac{A \vee B, C \rhd \bot, A \rhd \bot, B \rhd \bot \to C, A, B \quad C, C \rhd \bot \to C}{A \rhd C, B \rhd C \to A \vee B \rhd C}$$

$$\frac{\Diamond A, A \rhd \bot \to A}{\to \Diamond A \rhd A}$$

$$\frac{\neg A, \Box\Gamma, \bot \rhd \bot \to \neg\Gamma, \bot \quad \bot, \bot \rhd \bot \to \bot}{\Box\Gamma \to \Box A}$$

$\dashv$

Lemma 2.4. *The following rules hold in* $\mathbf{GK4}$*+*J*.*
(1) $\Gamma \to \Box(A \supset B) \in \mathbf{GK4} + J$ *implies* $\Gamma \to A \rhd B \in \mathbf{GK4} + J$,
(2) *if* $\Gamma \to A \rhd B$ *and* $\Gamma \to B \rhd C$ *are provable in* $\mathbf{GK4} + J$*, then so is* $\Gamma \to A \rhd C$,
(3) *if* $\Gamma \to A \rhd C$ *and* $\Gamma \to B \rhd C$ *are provable in* $\mathbf{GK4} + J$*, then so is* $\Gamma \to A \vee B \rhd C$,
(4) $\to (B \vee \Diamond B) \rhd B \in \mathbf{GK4} + J$,
(5) $\Gamma \to A \rhd (B \vee \Diamond B) \in \mathbf{GK4} + J$ *implies* $\Gamma \to A \rhd B \in \mathbf{GK4} + J$,
(6) $\Gamma \to A \rhd B \in \mathbf{GK4} + J$ *implies* $\Gamma \to (A \vee \Diamond A) \rhd B \in \mathbf{GK4} + J$.

Proof. We obtain (1), (2) and (3), from $(GJ1)$, $(GJ2)$ and $(GJ3)$, respectively. (5) and (6) are from (2) and (4).

So, we only show (4). By $(GJ1)$, it is easily seen that $\to B \rhd B \in \mathbf{GK4} + J$. Using $(GJ5)$ and (3), we obtain (4). $\dashv$

Lemma 2.5. *Let* $\mathbf{SEQ}$ *be an essential sequence on* $\Sigma \to A \rhd B$ *such that each sequent occurring in* $\mathbf{SEQ}^*$ *is provable in* $\mathbf{GK4} + J$ *and let* o *be an occurrence of a sequent in* $\mathbf{SEQ}$*. Then*

$$\Sigma \to \mathbf{an}(o) \rhd B \in \mathbf{GK4} + J.$$

Proof. The sequent occurring at o is of the form

$$\mathbf{an}(o) \to \mathbf{Su}(o),$$

where $B \in \mathbf{Su}(o)$, and so,

$$\mathbf{an}(o), \mathbf{Su}(o) \rhd \bot \to \mathbf{Su}(o)$$

occurs in $\mathbf{SEQ}^*$ and is provable in $\mathbf{GIK4} + J$. Remember $\Diamond D = \neg(D \rhd \bot)$ (cf. Notation 6.1.2), and we have

$$\mathbf{an}(o) \to \bigvee_{D \in \mathbf{Su}(o)} (D \vee \Diamond D) \in \mathbf{GK4} + J.$$

Using $(\to\supset)$, $(\Box_{K4})$

$$\to \Box(\mathbf{an}(o) \supset \bigvee_{D \in \mathbf{Su}(o)} (D \vee \Diamond D)) \in \mathbf{GK4} + J.$$

Using Lemma 2.4(1),

$$\to \mathbf{an}(o) \rhd (\bigvee_{D \in \mathbf{Su}(o)} (D \vee \Diamond D)) \in \mathbf{GK4} + J. \cdots (*1)$$

Now, we use an induction on $d_{\mathbf{SEQ}}(o)$.

If $d_{\mathbf{SEQ}}(o) = 1$, then o occurs at the end of $\mathbf{SEQ}$. By Lemma 1.9(1), we have $\mathbf{Su}(o) = \{B\}$. Using $(*1)$, we have

$$\to \mathbf{an}(o) \rhd (B \vee \Diamond B) \in \mathbf{GK4} + J.$$

Using Lemma 2.4(5),

$$\to \mathbf{an}(o) \rhd B \in \mathbf{GK4} + J.$$

Using $(T \to)$, possibly several times, we obtain the lemma.

Suppose that $d_{\mathbf{SEQ}}(o) > 1$ and the lemma holds for any occurrence o' of a sequent such that $d_{\mathbf{SEQ}}(o') < d_{\mathbf{SEQ}}(o)$. Let C be a formula in $\mathbf{Su}(o)_B$. Since $\mathbf{SEQ}$ is essential, there exists a pseudo-onto mapping f on $\mathbf{SEQ}/\Sigma \to A \rhd B$ that satisfies

$$C \rhd \mathbf{an}(f(o, C)) \in \Sigma \text{ and } d_{\mathbf{SEQ}}(f(o, C)) < d_{\mathbf{SEQ}}(o).$$

By $C \rhd \mathbf{an}(f(o, C)) \in \Sigma$, we have

$$\Sigma \to C \rhd \mathbf{an}(f(o, C)) \in \mathbf{GK4} + J.$$

Also by $d_{\mathbf{SEQ}}(f(o, C)) < d_{\mathbf{SEQ}}(o)$ and the induction hypothesis,

$$\Sigma \to \mathbf{an}(f(o, C)) \rhd B \in \mathbf{GK4} + J.$$

Using Lemma 2.4(2),

$$\Sigma \to C \rhd B \in \mathbf{GK4} + J.$$

Using Lemma 2.4(6),

$$\Sigma \to (C \vee \Diamond C) \rhd B \in \mathbf{GK4} + J.$$

By Lemma 2.4(4), we also have

$$\Sigma \to (B \vee \Diamond B) \rhd B \in \mathbf{GK4} + J.$$

Using Lemma 2.4(3), possibly several times,

$$\Sigma \to (\bigvee_{D \in \mathbf{Su}(o)} (D \vee \Diamond D)) \rhd B \in \mathbf{GK4} + J.$$

Using $(*1)$ and Lemma 2.4(2),

$$\Sigma \to \mathbf{an}(o) \rhd B \in \mathbf{GK4} + J.$$

$\dashv$

Corollary 2.6. *Let* **SEQ** *be an essential sequence on* $\Sigma \to A \rhd B$ *such that each sequent occurring in* $\mathbf{SEQ}^*$ *is provable in* $\mathbf{GK4} + J$*. Then*

$$\Sigma \to A \rhd B \in \mathbf{GK4} + J.$$

Proof. Since **SEQ** is on $\Sigma \to A \rhd B$, there exists an occurrence o of a sequent in **SEQ** such that $\mathbf{an}(o) = A$. So, using Lemma 2.5, we obtain the corollary. $\dashv$

Lemma 2.7. $\to A \in \mathbf{GIK4}$ implies $\to A \in \mathbf{GK4} + J$.

Proof. From Corollary 2.6, we can see that the inference rule $(\rhd_{IK4})$ holds in $\mathbf{GIK4} + J$. Hence, we obtain the lemma. $\dashv$

From Lemma 2.3, Lemma 2.2 and Lemma 2.7, we obtain Theorem 2.1.

3 Cut-elimination theorem for GIK4

In this section, we prove cut-elimination theorem for **GIK4**.

Theorem 3.1. *If* $\Gamma \to \Delta \in \mathbf{GIK4}$*, then there exists a cut-free proof figure for* $\Gamma \to \Delta$ *in* **GIK4***.*

It is easily seen that Theorem 3.1 follows from the following lemma.

Lemma 3.2. *Let* P *be a proof figure satisfying the following two conditions:*
(1) *the inference rule* I *introducing the end sequent of* P *is a cut,*
(2) I *is the only one cut in* P*,*
Then there exists a cut-free proof figure for the end sequent of P*.*

To prove the lemma above, we use the usual way, which originated with Gentzen [Gen35]. We have only to show the case concerning the operator $\rhd$ since the other cases can be shown in the usual way. In order to show the case, we provide some preparations.

Lemma 3.3. Let **SEQ** be a sequence on $\Sigma \to A \rhd B$ and let it be that $\Sigma' \subseteq \Sigma$. If there exists a pseudo-onto mapping f on $\mathbf{SEQ}/\Sigma' \to A \rhd B$, then **SEQ** is also on $\Sigma' \to A \rhd B$.

Proof. By Definition 1.5, it is sufficient to show the following two:
(1) each sequent S in **SEQ** is on $\Sigma' \to A \rhd B$,
(2) $\mathbf{an}(S) = A$ for some sequent S in **SEQ**.
Immediately, we obtain (2) since **SEQ** is on $\Sigma \to A \rhd B$. So, we show (1). Let o be an occurrence of S. Since S is on $\Sigma \to A \rhd B$, $B \in \mathbf{Su}(o) = \mathbf{Su}(S)$. Then we have only to show the following two:
(1.1) $\mathbf{Su}(o)_B = \mathbf{Su}(S)_B \subseteq \{C \mid C \rhd D \in \Sigma'\}$.
(1.2) $\mathbf{an}(o) = \mathbf{an}(S) \in \{A\} \cup \{D \mid C \rhd D \in \Sigma'\}$,
For (1.1): Let E be a formula in $\mathbf{Su}(o)_B$. Since f is on $\mathbf{SEQ}/\Sigma'$, $E \rhd \mathbf{an}(f(o, E)) \in \Sigma'$. Hence, we obtain (1.1).
For (1.2): If $\mathbf{an}(o) = A$, then (1.2) is trivial. Let it be that $\mathbf{an}(o) \neq A$. Since f is pseudo-onto, there exist an occurrence o' and a formula $E \in \mathbf{Su}(o')_B$ such that $f(o', E) = o$. Since f is on $\mathbf{SEQ}/\Sigma'$, we have

$$E \rhd \mathbf{an}(o) = E \rhd \mathbf{an}(f(o', E)) \in \Sigma'.$$

Hence, we obtain (1.2). $\dashv$

A sequence obtained from a sequence **SEQ** of sequents by removing some occurrences of sequents, or **SEQ** itself, is called *a subsequence* of **SEQ**.

Let **SEQ** and $\mathbf{SEQ}'$ be a sequences of sequents, possibly empty, and let o be an occurrence of a sequent in **SEQ**. Let $\mathbf{SEQ}''$ be the sequence obtained from **SEQ** by replacing o by $\mathbf{SEQ}'$. If an occurrence o_1 occurs in **SEQ** and is different from o, there exists the corresponding occurrence in $\mathbf{SEQ}''$ and the sequent at the occurrence in $\mathbf{SEQ}''$ also occurs at o_1. So, we use the same expression o_1 for the corresponding occurrence in $\mathbf{SEQ}''$. Similarly, we use the same expression for an occurrence o' in $\mathbf{SEQ}'$ and for its corresponding occurrence in $\mathbf{SEQ}''$.

Remark 3.4. *Let* **SEQ** *and* $\mathbf{SEQ}'$ *be a sequences of sequents, possibly empty, and let o be an occurrence of a sequent in* **SEQ**. *Let* $\mathbf{SEQ}''$ *be the sequence obtained from* **SEQ** *by replacing o by* $\mathbf{SEQ}'$. *Then for any occurrences o_1 and o_2 in* $\mathbf{SEQ}''$, *we have the following:*

(1) *The case that neither o_1 nor o_2 occurs in* $\mathbf{SEQ}'$*:*

$$d_{\mathbf{SEQ}}(o_1) > d_{\mathbf{SEQ}}(o_2) \text{ if and only if } d_{\mathbf{SEQ}''}(o_1) > d_{\mathbf{SEQ}''}(o_2).$$

(2) *The case that o_2 occurs in* $\mathbf{SEQ}'$ *but o_1 does not:*

$$d_{\mathbf{SEQ}}(o_1) > d_{\mathbf{SEQ}}(o) \text{ if and only if } d_{\mathbf{SEQ}''}(o_1) > d_{\mathbf{SEQ}''}(o_2).$$

(3) *The case that o_1 occurs in* $\mathbf{SEQ}'$ *but o_2 does not:*

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(o_2) \text{ if and only if } d_{\mathbf{SEQ}''}(o_1) > d_{\mathbf{SEQ}''}(o_2).$$

(4) *The case that both of o_1 and o_2 occur in* $\mathbf{SEQ}'$*:*

$$d_{\mathbf{SEQ}'}(o_1) > d_{\mathbf{SEQ}'}(o_2) \text{ if and only if } d_{\mathbf{SEQ}''}(o_1) > d_{\mathbf{SEQ}''}(o_2).$$

Lemma 3.5. *Let* **SEQ** *be a sequence on $\Sigma \to A \rhd B$ and let it be that $\Sigma' \subseteq \Sigma$. If there exists a mapping f on* $\mathbf{SEQ}/\Sigma'$, *then there exist a subsequence* $\mathbf{SEQ}'$ *of* **SEQ** *on $\Sigma \to A \rhd B$ and a pseudo-onto mapping on* $\mathbf{SEQ}'/\Sigma' \to A \rhd B$.

Proof. We put

$$\#(\mathbf{SEQ}, f) = \max(\{d_{\mathbf{SEQ}}(o) \mid o \notin Im(f) \text{ and } \mathbf{an}(o) \neq A\} \cup \{0\})$$

and we use an induction on $\#(\mathbf{SEQ}, f)$.

If $\#(\mathbf{SEQ}, f) = 0$, then f is a pseudo-onto mapping on $\mathbf{SEQ}/\Sigma' \to A \rhd B$ and **SEQ** is a subsequence of **SEQ** on $\Sigma \to A \rhd B$.

Suppose that $\#(\mathbf{SEQ}, f) > 0$ and the lemma holds for any pair $(\mathbf{SEQ}', f')$ such that $\#(\mathbf{SEQ}', f') < \#(\mathbf{SEQ}, f)$. Then there exists an occurrence $o \in \{o \mid o \notin Im(f) \text{ and } \mathbf{an}(o) \neq A\}$ such that

$$\#(\mathbf{SEQ}, f) = d_{\mathbf{SEQ}}(o).$$

We note that for any o' in **SEQ**,

$$d_{\mathbf{SEQ}}(o') > d_{\mathbf{SEQ}}(o) \text{ implies } o' \notin \{o \mid o \notin Im(f) \text{ and } \mathbf{an}(o) \neq A\}. \cdots (*1)$$

Let $\mathbf{SEQ}'$ be the sequence obtained from **SEQ** by removing o and let f' be a mapping obtained from f by restricting its domain into the set of pairs for $\mathbf{SEQ}'$. We show the following three:

(1) $\mathbf{SEQ}'$ is on $\Sigma \to A \rhd B$,

(2) f' is a mapping on $\mathbf{SEQ}'/\Sigma'$,

(3) $\#(\mathbf{SEQ}', f') < d_{\mathbf{SEQ}}(o) = \#(\mathbf{SEQ}, f)$.

By these conditions and the induction hypothesis, there exist a subsequence $\mathbf{SEQ}''$ of $\mathbf{SEQ}'$ on $\Sigma \to A \rhd B$ and a pseudo-onto mapping on $\mathbf{SEQ}''/\Sigma' \to A \rhd B$. Since $\mathbf{SEQ}'$ is a subsequence of **SEQ**, so is $\mathbf{SEQ}''$, and so, we can obtain the lemma.

For (1): It is sufficient to show the following two:

(1.1) each sequent in $\mathbf{SEQ}'$ is on $\Sigma \to A \rhd B$,

(1.2) $\mathbf{an}(S) = A$ for some sequent S in $\mathbf{SEQ}$.

We show (1.1). Let S be a sequent in $\mathbf{SEQ}'$. Since $\mathbf{SEQ}'$ is a subsequence of $\mathbf{SEQ}$, S also occurs in $\mathbf{SEQ}$. Since $\mathbf{SEQ}$ is on $\Sigma \to A \rhd B$, S is on $\Sigma \to A \rhd B$. Hence we obtain (1.1).

We show (1.2). Since $\mathbf{SEQ}$ is on $\Sigma \to A \rhd B$, we have $\mathbf{an}(S) = A$ for some S in $\mathbf{SEQ}$. Using $\mathbf{an}(o) \neq A$, S does not occur at o, and so, S occurs in $\mathbf{SEQ}'$. Hence we obtain (1.2).

For (2): Note that f' does not have to be pseudo-onto. So, by $o \notin Im(f)$, (2) can be shown.

For (3): It is sufficient to show that for any occurrence o' in $\mathbf{SEQ}'$,

$$d_{\mathbf{SEQ}'}(o') \geq d_{\mathbf{SEQ}}(o) \text{ implies } o' \notin \{o \mid o \notin Im(f') \text{ and } \mathbf{an}(o) \neq A\}.$$

In other words,

$$d_{\mathbf{SEQ}'}(o') \geq d_{\mathbf{SEQ}}(o) \text{ implies either } o' \in Im(f') \text{ or } \mathbf{an}(o') = A.$$

Let o' be an occurrence in $\mathbf{SEQ}'$ and let it be that

$$d_{\mathbf{SEQ}'}(o') \geq d_{\mathbf{SEQ}}(o).$$

We note that o' also occurs in $\mathbf{SEQ}$ and

$$d_{\mathbf{SEQ}}(o') = d_{\mathbf{SEQ}'}(o') + 1.$$

Hence

$$d_{\mathbf{SEQ}}(o') = d_{\mathbf{SEQ}'}(o') + 1 \geq d_{\mathbf{SEQ}}(o) + 1,$$

and so,

$$d_{\mathbf{SEQ}}(o') > d_{\mathbf{SEQ}}(o).$$

Using $(*1)$,

$$o' \notin \{o \mid o \notin Im(f) \text{ and } \mathbf{an}(o) \neq A\}.$$

In other words, either $o' \in Im(f)$ or $\mathbf{an}(o) = A$. If $\mathbf{an}(o) = A$, then we obtain (3). So, we assume that $o' \in Im(f)$. Then there exist o'' in $\mathbf{SEQ}$ and $E \in \mathbf{Su}(o'')_B$ such that $f(o'', E) = o'$. Since f is on $\mathbf{SEQ}/\Sigma'$, we have

$$d_{\mathbf{SEQ}}(o'') > d_{\mathbf{SEQ}}(f(o'', E)) = d_{\mathbf{SEQ}}(o') > d_{\mathbf{SEQ}}(o),$$

and so, o'' also occurs in $\mathbf{SEQ}'$. Hence $f'(o'', E) = o'$, i.e., $o' \in Im(f')$, we obtain (3). ⊣

Lemma 3.6. *Let* $\mathbf{SEQ}$ *be a sequence on* $\Sigma \to A \rhd B$ *and let* f *be a pseudo-onto mapping on* $\mathbf{SEQ}/\Sigma \to A \rhd B$. *Then there exists an essential subsequence* $\mathbf{SEQ}'$ *of* $\mathbf{SEQ}$ *on* $\Sigma \to A \rhd B$.

Proof. We use an induction of the number $\#(\mathbf{SEQ})$ of pairs (o_1, o_2) of different occurrences in $\mathbf{SEQ}$ such that $\mathbf{an}(o_1) = \mathbf{an}(o_2)$.

If $\#(\mathbf{SEQ}) = 0$, then $\mathbf{SEQ}$ is an essential subsequence of $\mathbf{SEQ}$.

Suppose that $\#(\mathbf{SEQ}) > 0$ and the lemma holds for any $\mathbf{SEQ}'$ such that $\#(\mathbf{SEQ}') < \#(\mathbf{SEQ})$. Then there exists a pair (o_1, o_2) of different occurrences of sequents such that $\mathbf{an}(o_1) = \mathbf{an}(o_2)$. Without loss of generality, we assume that $d_{\mathbf{SEQ}}(o_1) > d_{\mathbf{SEQ}}(o_2)$.

Let $\mathbf{SEQ}'$ be the sequence obtained from $\mathbf{SEQ}$ by removing o_1. We show the following three:

(1) $\#(\mathbf{SEQ}') < \#(\mathbf{SEQ})$,

(2) $\mathbf{SEQ}'$ is on $\Sigma \to A \rhd B$,

(3) there exists a mapping f' on $\mathbf{SEQ}'/\Sigma$.

(1) is trivial. We show (2). Since $\mathbf{SEQ}'$ is a subsequence of $\mathbf{SEQ}$, we have only to show that $\mathbf{an}(o') = A$ for some occurrence o' in $\mathbf{SEQ}'$. Since $\mathbf{SEQ}$ is on the sequent $\Sigma \to A \rhd B$, $\mathbf{an}(o) = A$ for some occurrence o in $\mathbf{SEQ}$. If $o = o_1$, then $A = \mathbf{an}(o) = \mathbf{an}(o_1) = \mathbf{an}(o_2)$ and o_2 occurs in $\mathbf{SEQ}'$; if not, o occurs in $\mathbf{SEQ}'$.

We show (3). We define a mapping f' on $\mathbf{SEQ}'/\Sigma$. A mapping f' is defined as follows.

$$f'(o, E) = \begin{cases} f(o, E) & \text{if } f(o, E) \neq o_1 \\ o_2 & \text{if } f(o, E) = o_1 \end{cases}$$

In order to show that f' on $\mathbf{SEQ}'/\Sigma$, it is sufficient to show $E \rhd \mathbf{an}(f'(o,E)) \in \Sigma$ and $d_{\mathbf{SEQ}'}(o) > d_{\mathbf{SEQ}'}(f'(o,E))$.

If $f(o,E) \neq o_1$, then

$$E \rhd \mathbf{an}(f'(o,E)) = E \rhd \mathbf{an}(f(o,E)) \in \Sigma;$$

if not,

$$E \rhd \mathbf{an}(f'(o,E)) = E \rhd \mathbf{an}(o_2) = E \rhd \mathbf{an}(o_1) = E \rhd \mathbf{an}(f(o,X)) \in \Sigma.$$

If $f(o,E) \neq o_1$, then

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f(o,E)) = d_{\mathbf{SEQ}}(f'(o,E));$$

if not,

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f(o,E)) = d_{\mathbf{SEQ}}(o_1) > d_{\mathbf{SEQ}}(o_2) = d_{\mathbf{SEQ}}(f'(o,E)).$$

Hence in any case, $d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f'(o,E))$, and so, we have $d_{\mathbf{SEQ}'}(o) > d_{\mathbf{SEQ}'}(f'(o,E))$.

Hence f' is on $\mathbf{SEQ}'/\Sigma$.

By (2),(3) and Lemma 3.5, there exist a subsequence $\mathbf{SEQ}''$ of $\mathbf{SEQ}'$ on $\Sigma \to A \rhd B$ and a pseudo-onto mapping f'' on $\mathbf{SEQ}''/\Sigma \to A \rhd B$. Since $\mathbf{SEQ}''$ is a subsequence of $\mathbf{SEQ}'$, we have $\#(\mathbf{SEQ}'') \leq \#(\mathbf{SEQ}')$ Using (1), $\#(\mathbf{SEQ}'') < \#(\mathbf{SEQ})$. So, by the induction hypothesis, there exists an essential subsequence $\mathbf{SEQ}'''$ of $\mathbf{SEQ}''$, which is also a subsequence of $\mathbf{SEQ}$, on $\Sigma \to A \rhd B$. Hence, we obtain the lemma. $\dashv$

Lemma 3.7. *Let* $\mathbf{SEQ}$ *be a sequence on* $\Sigma \to A \rhd B$ *and let it be that* $\Sigma' \subseteq \Sigma$. *If there exists a mapping* f *on* $\mathbf{SEQ}/\Sigma'$, *then there exists an essential subsequence* $\mathbf{SEQ}'$ *of* $\mathbf{SEQ}$ *on* $\Sigma' \to A \rhd B$.

Proof. By Lemma 3.5, there exist a subsequence $\mathbf{SEQ}'$ of $\mathbf{SEQ}$ on $\Sigma \to A \rhd B$ and a pseudo-onto mapping on $\mathbf{SEQ}'/\Sigma' \to A \rhd B$. Using Lemma 3.3, $\mathbf{SEQ}'$ is also on $\Sigma' \to A \rhd B$. Using Lemma 3.6, there exists an essential subsequence $\mathbf{SEQ}''$ of $\mathbf{SEQ}'$ on $\Sigma' \to A \rhd B$. $\dashv$

Lemma 3.8. *If there exists a cut-free proof figure for* $A, \Delta \rhd \bot \to \Delta$, *then there exists a cut-free proof figure for* $\Delta \rhd \bot \to A \rhd \bot$ *in* **GIK4** *such that* $A \rhd \bot$ *does not occur in any succedent of upper sequents of the inference rule introducing the end sequent.*

Proof. It is easily seen that the sequence

$$A \to \Delta, \bot \qquad\qquad \bot \to \bot$$

is essential on $\Delta \rhd \bot \to A \rhd \bot$. So, the following figure convinces us that we can construct a cut-free proof figure for $\Delta \rhd \bot \to A \rhd \bot$ satisfying the condition using a cut-free proof figure for $A, \Delta \rhd \bot \to \Delta$.

$$\dfrac{\dfrac{\dfrac{A, \Delta \rhd \bot \to \Delta}{A, \Delta \rhd \bot \to \Delta, \bot}}{A, \bot \rhd \bot, \Delta \rhd \bot \to \Delta, \bot} \qquad \dfrac{\dfrac{\bot \to}{\bot \to \bot}}{\bot, \bot \rhd \bot \to \bot}}{\Delta \rhd \bot \to A \rhd \bot}$$

$\dashv$

Definition 3.9. The degree $d(A)$ of a formula A is redefined inductively as follows:
(1) $d(p) = 1$,
(2) $d(\bot) = 0$,
(3) $d(A \wedge B) = d(A \vee B) = d(A \supset B) = d(A \rhd B) = d(A) + d(B) + 1$.

Proof of Lemma 3.2. The degree $d(P)$, the left rank $R_l(P)$ and the right rank $R_r(P)$ of P are defined as usual. We use an induction on $R_l(P) + R_r(P) + \omega d(P)$. We only treat the case that P is of the form:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ (\mathbf{SEQ}^{\ell})^*\end{matrix}}{\Sigma^{\ell} \to C \rhd D} \qquad \dfrac{\begin{matrix}\vdots\\ (\mathbf{SEQ}^{r})^*\end{matrix}}{C \rhd D, \Sigma^r \to A \rhd B}}{\Sigma^{\ell}, \Sigma^r_{C \rhd D} \to A \rhd B}$$

where $C \rhd D \notin \Sigma^{\ell}$ and two sequences $\mathbf{SEQ}^{\ell}$ and $\mathbf{SEQ}^r$ are essential on

$$\Sigma^\ell \to C \rhd D \text{ and } C \rhd D, \Sigma^r \to A \rhd B,$$

respectively. Without loss of generality, we assume that P is of the following form if $C = \perp$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\perp \to}{\perp \to D}}{\perp, D \rhd \perp \to D}}{\Sigma^\ell \to \perp \rhd D} \qquad \dfrac{\begin{array}{c}\vdots \\ (\mathbf{SEQ}^r)^*\end{array}}{\perp \rhd D, \Sigma^r \to A \rhd B}}{\Sigma^\ell, \Sigma^r_{\perp \rhd D} \to A \rhd B}$$

Since $\mathbf{SEQ}^\ell$ and $\mathbf{SEQ}^r$ are essential, there exist pseudo-onto mappings f^ℓ and f^r on $\mathbf{SEQ}^\ell/\Sigma^\ell \to C \rhd D$ and $\mathbf{SEQ}^r/C \supset D, \Sigma^r \to A \rhd B$, respectively. Using f^r, we define a sequence $\mathbf{SEQ}$ as follows.

Let $o^r_1, \cdots, o^r_n$ be an enumeration of the occurrences of sequents in $\mathbf{SEQ}^r$ such that

$$C \in \mathbf{Su}(o^r_i)_B \text{ and } \mathbf{an}(f^r(o^r_i, C)) = D.$$

So, $\mathbf{an}(f^r(o^r_1, C)) = \cdots = \mathbf{an}(f^r(o^r_n, C)) = D$. Since $\mathbf{SEQ}^r$ is essential (i.e., by Definition 1.7(3)), we have $f^r(o^r_1, C) = \cdots = f^r(o^r_n, C)$. We put

$$o^r_d = f^r(o^r_1, C) = \cdots = f^r(o^r_n, C).$$

Here we note that for any occurrence o in $\mathbf{SEQ}$ and for any $E \in \mathbf{Su}(o)_B$,

$$E \rhd \mathbf{an}(f^r(o, E)) = C \rhd D \text{ implies } o \in \{o^r_1, \cdots, o^r_n\}. \cdots (*1)$$

Let o^ℓ_0 be the occurrence of the sequent at the top of $\mathbf{SEQ}^\ell$ and let S^ℓ_0 be the sequent at o^ℓ_0. For a sequent S^ℓ in $\mathbf{SEQ}^\ell$ and $i \in \{1, \cdots, n\}$, we define $g_i(S^\ell)$ as follows.

$$g_i(S^\ell) = \begin{cases} \mathbf{an}(o^r_i) \to \mathbf{Su}(o^r_i)_C, \mathbf{Su}(o^r_d), \mathbf{Su}(S^\ell)_D & \text{if } S = S^\ell_0 \\ \mathbf{an}(S^\ell) \to \mathbf{Su}(o^r_d), \mathbf{Su}(S^\ell)_D & \text{otherwise} \end{cases}$$

Let $g_i(\mathbf{SEQ}^\ell)$ be the sequence obtained from $\mathbf{SEQ}^\ell$ by replacing each sequent S^ℓ by $g_i(S^\ell)$, respectively. For the occurrence o^ℓ of S^ℓ in $\mathbf{SEQ}^\ell$, $g_i(o^\ell)$ denotes the occurrence of $g_i(S^\ell)$ in $g_i(\mathbf{SEQ}^\ell)$. The sequence $\mathbf{SEQ}$ denotes the sequence obtained from $\mathbf{SEQ}^r$ by replacing each occurrence o^r_i by the sequence $g_i(\mathbf{SEQ}^\ell)$, respectively.

Now, we show the following:
(3) $\mathbf{SEQ}$ is on $C \rhd D, \Sigma^\ell, \Sigma^r \to A \rhd B$,
(4) there exists a mapping f on $\mathbf{SEQ}/\Sigma^\ell \cup \Sigma^r_{C \rhd D}$,
(5) there exists a cut-free proof figure for each sequent in $\mathbf{SEQ}^*$.
By (3), (4) and Lemma 3.7, we can obtain an essential subsequence of $\mathbf{SEQ}$ on the end sequent of P and using (5), we can obtain the lemma.

For (3): It is sufficient to show the following two:
(3.1) each sequent S in $\mathbf{SEQ}$ is on $C \rhd D, \Sigma^\ell, \Sigma^r \to A \rhd B$,
(3.2) $\mathbf{an}(o) = A$, for some occurrence o in $\mathbf{SEQ}$.
To show (3.1), we divide into the following three cases.

(i) The case that S does not occur in any $g_i(\mathbf{SEQ}^\ell)$: Then S occurs in $\mathbf{SEQ}^r$, and so, on $C \rhd D, \Sigma^r \to A \rhd B$. Using Remark 1.4(1), we obtain (3.2).

(ii) The case that S occurs at $g_i(o^\ell_0)$ for some i: Then $S = g_i(S^\ell_0)$, and it is of the form

$$\mathbf{an}(o^r_i) \to \mathbf{Su}(o^r_d), \mathbf{Su}(S^\ell_0)_D, \mathbf{Su}(o^r_i)_C.$$

Since sequents occurring at o^r_i and o^r_d are on $C \rhd D, \Sigma^r \to A \rhd B$, we have

$$\mathbf{an}(o^r_i) \in \{A\} \cup \{F \mid E \rhd F \in \{C \rhd D\} \cup \Sigma^r\},$$

$$\mathbf{Su}(o^r_d) \subseteq \{E \mid E \rhd F \in \{C \rhd D\} \cup \Sigma^r\} \cup \{B\},$$

$$\mathbf{Su}(o_i^r) \subseteq \{E \mid E \rhd F \in \{C \rhd D\} \cup \Sigma^r\} \cup \{B\}.$$

Similarly, since S_0^ℓ is on $\Sigma^\ell \to C \rhd D$, we have

$$\mathbf{Su}(S_0^\ell) \subseteq \{E \mid E \rhd F \in \Sigma^\ell\} \cup \{D\},$$

and so,

$$\mathbf{Su}(S_0^\ell)_D \subseteq \{E \mid E \rhd F \in \Sigma^\ell\}.$$

Hence

$$\mathbf{an}(S) = \mathbf{an}(o_i^r) \in \{A\} \cup \{F \mid E \rhd F \in \{C \rhd D\} \cup \Sigma^\ell \cup \Sigma^r\}$$

and

$$\mathbf{Su}(S) = \mathbf{Su}(o_d^r) \cup \mathbf{Su}(S_0^\ell)_D \cup \mathbf{Su}(o_i^r)_C \subseteq \{E \mid E \rhd F \in \{C \rhd D\} \cup \Sigma^\ell \cup \Sigma^r\} \cup \{B\}.$$

Hence we obtain (3.1).

(iii) The case that S does not occur at $g_i(o_0^\ell)$ for any i, but occurs in $g_i(\mathbf{SEQ}^\ell)$ for some i: In a similar way to the case above, we obtain (3.1).

We show (3.2). Since $\mathbf{SEQ}^r$ is essential, $\mathbf{an}(o^r) = A$ for some occurrence o^r of a sequent in $\mathbf{SEQ}^r$. If $o^r \notin \{o_1^r, \cdots, o_n^r\}$, then o^r also occurs in $\mathbf{SEQ}$, and $\mathbf{an}(o^r) = A$. If $o^r = o_i^r$ for some i, then $A = \mathbf{an}(o^r) = \mathbf{an}(o_i^r) = \mathbf{an}(g_i(o_0^\ell))$, and $g_i(o_0^\ell)$ occurs in $\mathbf{SEQ}$.

For (4): We define a mapping f on $\mathbf{SEQ}/\Sigma^\ell \cup \Sigma^r_{C \rhd D}$. Let o be an occurrence in $\mathbf{SEQ}$ and let E be a formula in $\mathbf{Su}(o)_B$. We divide into the same three cases as the ones we used to show (3.1). In each case, we define $f(o, E)$ in $\mathbf{SEQ}$ and show the following two:

(4.1) $E \rhd \mathbf{an}(f(o, E)) \in \Sigma^\ell \cup \Sigma^r_{C \rhd D}$,

(4.2) $d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f(o, E))$.

From these two conditions, we will see that f is a mapping on $\mathbf{SEQ}/\Sigma^\ell \cup \Sigma^r_{C \rhd D}$.

(i) The case that o does not occur in any $g_i(\mathbf{SEQ}^\ell)$: We note that o occurs in $\mathbf{SEQ}^r$, and

$$o \notin \{o_1^r, \cdots, o_n^r\}. \cdots (*2)$$

Now, we define $f(o, E)$ as follows.

$$f(o, E) = \begin{cases} f^r(o, E) & \text{if } f^r(o, E) \notin \{o_1^r, \cdots, o_n^r\} \\ g_i(o_0^\ell) & \text{if } f^r(o, E) = o_i^r \text{ for some } i \end{cases}$$

We show (4.1). If $f^r(o, E) = o_i^r$ for some i, then

$$\mathbf{an}(f(o, E)) = \mathbf{an}(g_i(o_0^\ell)) = \mathbf{an}(o_i^r) = \mathbf{an}(f^r(o, E));$$

if not, we also have

$$\mathbf{an}(f(o, E)) = \mathbf{an}(f^r(o, E)).$$

Since f^r is on $\mathbf{SEQ}^r/\{C \supset D\} \cup \Sigma^r$,

$$E \rhd \mathbf{an}(f(o, E)) = E \rhd \mathbf{an}(f^r(o, E)) \in \{C \supset D\} \cup \Sigma^r.$$

On the other hand, by $(*1)$ and $(*2)$, we have

$$E \rhd \mathbf{an}(f^r(o, E)) \neq C \rhd D.$$

Hence

$$E \rhd \mathbf{an}(f(o, E)) = E \rhd \mathbf{an}(f^r(o, E)) \in \Sigma^r_{C \supset D}.$$

We show (4.2). Since f^r is on $\mathbf{SEQ}^r/\{C \supset D\} \cup \Sigma^r$,

$$d_{\mathbf{SEQ}^r}(o) > d_{\mathbf{SEQ}^r}(f^r(o, E)).$$

If $f^r(o,E) \notin \{o^r_1, \cdots, o^r_n\}$, then $f^r(o,E)$ occurs in **SEQ**, and so,

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f^r(o,E)) = d_{\mathbf{SEQ}}(f(o,E)).$$

If $f^r(o,E) = o^r_i$ for some i, then

$$d_{\mathbf{SEQ}^r}(o) > d_{\mathbf{SEQ}^r}(f^r(o,E)) = d_{\mathbf{SEQ}^r}(o^r_i),$$

and so,

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(g_i(o^\ell_0)) = d_{\mathbf{SEQ}}(f(o,E)).$$

(ii) The case that o occurs at $g_i(o^\ell_0)$ for some i: Then

$$\mathbf{Su}(o)_B = \mathbf{Su}(g_i(o^\ell_0))_B = (\mathbf{Su}(o^r_i)_C \cup \mathbf{Su}(o^r_d) \cup \mathbf{Su}(o^\ell_0)_D)_B.$$

Note that $f^r(o^r_d, E)$ occurs in **SEQ** if $E \in \mathbf{Su}(o^r_d)_B$, and $g_i(f^\ell(o^\ell, E))$ occurs in **SEQ** if $E \in \mathbf{Su}(o^\ell)_D$. Now, $f(o,E)$ is defined as follows.

$$f(o,E) = \begin{cases} f^r(o^r_i, E) & \text{if } E \in (\mathbf{Su}(o^r_i)_C)_B \text{ and } f^r(o^r_i, E) \notin \{o^r_1, \cdots, o^r_n\} \\ g_j(o^\ell_0) & \text{if } E \in (\mathbf{Su}(o^r_i)_C)_B \text{ and } f^r(o^r_i, E) = o^r_j \text{ for some } j \\ f^r(o^r_d, E) & \text{if } E \in \mathbf{Su}(o^r_d)_B \\ g_i(f^\ell(o^\ell, E)) & \text{if } E \in (\mathbf{Su}(o^\ell)_D)_B \end{cases}$$

We show (4.1).

If $E \in (\mathbf{Su}(o^r_i)_C)_B$ and $f^r(o^r_i, E) \notin \{o^r_1, \cdots, o^r_n\}$, then

$$E \rhd \mathbf{an}(f(o,E)) = E \rhd \mathbf{an}(f^r(o^r_i, E)) \in \{C \supset D\} \cup \Sigma^r.$$

Since $E \in (\mathbf{Su}(o^r_i)_C)_B$, we have $E \neq C$, and so, $E \rhd \mathbf{an}(f(o,E)) \in \Sigma^r_{C \rhd D}$.

If $E \in (\mathbf{Su}(o^r_i)_C)_B$ and $f^r(o^r_i, E) = o^r_j$, then

$$E \rhd \mathbf{an}(f(o,E)) = E \rhd \mathbf{an}(g_j(o^\ell_0)) = E \rhd \mathbf{an}(o^r_j) = E \rhd \mathbf{an}(f^r(o^r_i, E)) \in \{C \supset D\} \cup \Sigma^r.$$

Since $E \in (\mathbf{Su}(o^r_i)_C)_B$, we have $E \rhd \mathbf{an}(f(o,E)) \in \Sigma^r_{C \rhd D}$, similarly.

If $E \in \mathbf{Su}(o^r_d)_B$, then

$$E \rhd \mathbf{an}(f(o,E)) = E \rhd \mathbf{an}(f^r(o^r_d, E)) \in \{C \supset D\} \cup \Sigma^r.$$

By $(*1)$ and $o^r_d \notin \{o^r_1, \cdots, o^r_n\}$, we have $E \rhd \mathbf{an}(f^r(o^r_d, E)) \neq C \rhd D$. Hence, $E \rhd \mathbf{an}(f(o,E)) \in \Sigma_{C \rhd D}$.

If $E \in (\mathbf{Su}(o^\ell)_D)_B$, then

$$E \rhd \mathbf{an}(f(o,E)) = E \rhd \mathbf{an}(g_i(f^\ell(o^\ell, E))) = E \rhd \mathbf{an}(f^\ell(o^\ell, E)) \in \Sigma^\ell.$$

We show (4.2).

If $E \in (\mathbf{Su}(o^r_i)_C)_B$ and $f^r(o^r_i, E) \notin \{o^r_1, \cdots, o^r_n\}$, then

$$d_{\mathbf{SEQ}^r}(o^r_i) > d_{\mathbf{SEQ}^r}(f^r(o^r_i, E)),$$

and so,

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f^r(o^r_i, E)) = d_{\mathbf{SEQ}}(f(o,E)).$$

If $E \in (\mathbf{Su}(o^r_i)_C)_B$ and $f^r(o^r_i, E) = o^r_j$, then

$$d_{\mathbf{SEQ}^r}(o^r_i) > d_{\mathbf{SEQ}^r}(o^r_j),$$

and so,

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(g_j(o^\ell_o)) = d_{\mathbf{SEQ}}(f(o,E)).$$

If $E \in \mathbf{Su}(o^r_d)_B$, then

$$d_{\mathbf{SEQ}^r}(o^r_i) > d_{\mathbf{SEQ}^r}(o^r_d) > d_{\mathbf{SEQ}^r}(f(o^r_d, E)),$$

and so,

$$d_{\mathbf{SEQ}}(o) > d_{\mathbf{SEQ}}(f(o^r_d, E)) = d_{\mathbf{SEQ}}(f(o, E)).$$

If $E \in (\mathbf{Su}(o^\ell)_D)_B$, then

$$d_{\mathbf{SEQ}^\ell}(o^\ell) > d_{\mathbf{SEQ}^\ell}(f(o^\ell, E)),$$

and so,

$$d_{\mathbf{SEQ}}(o) = d_{\mathbf{SEQ}}(g_i(o^\ell)) > d_{\mathbf{SEQ}}(g_i(f(o^\ell, E))) = d_{\mathbf{SEQ}}(f(o, E)).$$

(iii) The case that S does not occur at $g_i(o^\ell_0)$ for any i, but occurs in $g_i(\mathbf{SEQ}^\ell)$ for some i: By the definition of $g_i(\mathbf{SEQ}^\ell)$, there exists an occurrence o^ℓ, which is different from o^ℓ_0, in $\mathbf{SEQ}^\ell$ such that $o = g_i(o^\ell)$. Also

$$\mathbf{Su}(o)_B = \mathbf{Su}(g_i(o^\ell))_B = (\mathbf{Su}(o^r_d) \cup \mathbf{Su}(o^\ell)_D)_B.$$

$f(o, E)$ is defined as follows:

$$f(o, E) = \begin{cases} f^r(o^r_d, E) & \text{if } E \in \mathbf{Su}(o^r_d)_B \\ g_i(f^\ell(o^\ell, E)) & \text{if } E \in (\mathbf{Su}(o^\ell)_D)_B \end{cases}$$

(4.1) and (4.2) can be shown in a way similar to the case (ii) above.

For (5): Let S^r_i and S^r_d be the sequents occurring at o^r_i and o^r_d, respectively. Then $(S^r_i)^*$ and $(S^r_d)^*$ occurring in $\mathbf{SEQ}^r)^*$ are of the following forms:

$$(S^r_i)^* : \mathbf{an}(S^r_i), \mathbf{Su}(S^r_i) \rhd \bot \to \mathbf{Su}(S^r_i),$$

$$(S^r_d)^* : D, \mathbf{Su}(S^r_d) \rhd \bot \to \mathbf{Su}(S^r_d),$$

where $C \in \mathbf{Su}(S^r_i)$ and $B \in \mathbf{Su}(S^r_i) \cup \mathbf{Su}(S^r_d)$. Also $(S^\ell_0)^*$, which is the sequent occurring at the top of $(\mathbf{SEQ}^\ell)^*$, is of the form

$$(S^\ell_0)^* : C, \mathbf{Su}(S^\ell_0) \rhd \bot \to \mathbf{Su}(S^\ell_0),$$

where $D \in \mathbf{Su}(S^\ell_0)$. Hence there exist cut-free proof figures P^r_i, P^r_d and P^ℓ_0 for the sequents obtained from $(S^r_i)^*$, $(S^r_d)^*$ and $(S^\ell_0)^*$, respectively.

Using Lemma 3.8, there exist cut-free proof figures for

$$\mathbf{Su}(S^r_d) \rhd \bot \to D \rhd \bot$$

and

$$\mathbf{Su}(S^\ell_0) \rhd \bot \to C \rhd \bot.$$

Let Q^r_d and Q^ℓ_0 be the proof figures for the sequents above obtained by the way in Lemma 3.8.

Let T be a sequent in $\mathbf{SEQ}^*$. Then there exists a sequent S in $\mathbf{SEQ}$ such that $S^* = T$. We divide into the same three cases as the ones we used to show (3.1) and (4).

(i) The case that S does not occur in any $g_i(\mathbf{SEQ}^\ell)$: Then S occurs in $\mathbf{SEQ}^r$, and so, there exists a cut-free proof figure for $S^* = T$.

(ii) The case that S occurs at $g_i(o^\ell_0)$ for some i: S^* is of the form

$$\mathbf{an}(S^r_i), \mathbf{Su}(S^r_i)_C \rhd \bot, \mathbf{Su}(S^r_d) \rhd \bot, \mathbf{Su}(S^\ell_0)_D \rhd \bot \to \mathbf{Su}(S^r_i)_C, \mathbf{Su}(S^r_d), \mathbf{Su}(S^\ell_0)_D.$$

We divide into the following two subcases.

(ii.a) The subcase that $C \neq \bot$: By P^r_i, Q^ℓ_0 and cut, we have the following proof figure P_1:

$$\frac{Q^\ell_0 \qquad\qquad P^r_i}{\mathbf{Su}(S^\ell_0) \rhd \bot, (\{\mathbf{an}(S^r_i)\} \cup \mathbf{Su}(S^r_i) \rhd \bot)_{C \rhd \bot} \to \mathbf{Su}(S^r_i)}$$

We note that $d(P_1) = d(C \rhd \bot) \le d(C \rhd D) = d(P)$ and $R_r(P_1) < R_r(P)$. By Lemma 3.8, $R_l(P_1) = 1 = R_l(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of P_1. Using $(T \rightarrow)$, possibly several times, we obtain a cut-free proof figure P_2 for

$$\mathbf{an}(S_i^r), \mathbf{Su}(S_0^\ell) \rhd \bot, \mathbf{Su}(S_i^r)_C \rhd \bot \rightarrow \mathbf{Su}(S_i^r).$$

Using P_0^ℓ and cut, we have the following proof figure P_3:

$$\frac{P_2 \qquad\qquad\qquad\qquad P_0^\ell}{\mathbf{an}(S_i^r), \mathbf{Su}(S_0^\ell) \rhd \bot, \mathbf{Su}(S_i^r)_C \rhd \bot \rightarrow \mathbf{Su}(S_i^r)_C, \mathbf{Su}(S_0^\ell)}$$

We note that $d(P_3) = d(C) < d(C \rhd D) = d(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure P_4 for the end sequent of P_3. Using Q_d^r and cut, we have the following proof figure P_5:

$$\frac{Q_d^r \qquad\qquad\qquad\qquad P_4}{\mathbf{Su}(S_d^r) \rhd \bot, (\{\mathbf{an}(S_i^r)\} \cup \mathbf{Su}(S_0^\ell) \rhd \bot \cup \mathbf{Su}(S_i^r)_C \rhd \bot)_{D \rhd \bot} \rightarrow \mathbf{Su}(S_i^r)_C, \mathbf{Su}(S_0^\ell)}$$

Since $C \neq \bot$, we have $d(C) > 0$, and so, $d(P_5) = d(D \rhd \bot) = d(D) + 1 < d(C) + d(D) + 1 = d(C \rhd D) = d(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of P_5. Using $(T \rightarrow)$, possibly several times, we obtain a cut-free proof figure P_6 for

$$\mathbf{an}(S_i^r), \mathbf{Su}(S_d^r) \rhd \bot, \mathbf{Su}(S_0^\ell)_D \rhd \bot, \mathbf{Su}(S_i^r)_C \rhd \bot \rightarrow \mathbf{Su}(S_i^r)_C, \mathbf{Su}(S_0^\ell).$$

Using P_d^r and cut, we have the following proof figure P_7:

$$\frac{P_6 \qquad\qquad\qquad\qquad P_d^r}{\mathbf{an}(S_i^r), \mathbf{Su}(S_d^r) \rhd \bot, \mathbf{Su}(S_0^\ell)_D \rhd \bot, \mathbf{Su}(S_i^r)_C \rhd \bot \rightarrow (\mathbf{Su}(S_i^r)_C \cup \mathbf{Su}(S_0^\ell))_D, \mathbf{Su}(S_d^r)}$$

We note that $d(P_7) = d(D) < d(C \rhd D) = d(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of P_7. Using $(\rightarrow T)$, possibly several times, we obtain a cut-free proof figure for $S^* = T$.

(ii.b) The subcase that $C = \bot$: By P_i^r and cut, we have the following proof figure P_8:

$$\frac{\dfrac{\dfrac{\bot \rightarrow \bot}{\bot, \bot \rhd \bot \rightarrow \bot}}{\rightarrow \bot \rhd \bot} \qquad \left.\begin{matrix} \vdots \\ \mathbf{an}(S_i^r), \mathbf{Su}(S_i^r) \rhd \bot \rightarrow \mathbf{Su}(S_i^r) \end{matrix}\right\} P_i^r}{(\mathbf{an}(S_i^r), \mathbf{Su}(S_i^r) \rhd \bot)_{\bot \rhd \bot} \rightarrow \mathbf{Su}(S_i^r)}$$

We note that $d(P_8) = d(\bot \rhd \bot) \le d(\bot \rhd D) = d(P)$, $R_r(P_8) < R_r(P)$ and $R_l(P_8) = 1 = R_l(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of P_8. Using $(T \rightarrow)$, possibly several times, we obtain a cut-free proof figure P_9 for

$$\mathbf{an}(S_i^r), \mathbf{Su}(S_i^r)_\bot \rhd \bot \rightarrow \mathbf{Su}(S_i^r).$$

Using the axiom $\bot \rightarrow$ and cut, we have the following cut-free proof figure P_{10}:

$$\frac{P_9 \qquad\qquad\qquad\qquad \bot \rightarrow}{\mathbf{an}(S_i^r), \mathbf{Su}(S_i^r)_\bot \rhd \bot \rightarrow \mathbf{Su}(S_i^r)_\bot}$$

We note that $d(P_{10}) = d(\bot) < d(\bot \rhd D) = d(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of P_{10}. Using $(T \rightarrow)$ and $(\rightarrow T)$, possibly several times, we obtain a cut-free proof figure for $S^* = T$.

(iii) The case that S does not occur at $g_i(o_0^\ell)$ for any i, but occurs in $g_i(\mathbf{SEQ}^\ell)$ for some i: There exists a sequent S^ℓ in $\mathbf{SEQ}^\ell$ such that $g_i(S^\ell) = S$ and $S \neq S_0^\ell$. On the other hand, S^* is of the form

$$\mathbf{an}(S^\ell), \mathbf{Su}(S_d^r) \rhd \bot, \mathbf{Su}(S^\ell)_D \rhd \bot \rightarrow \mathbf{Su}(S_d^r), \mathbf{Su}(S^\ell)_D,$$

where $B \in \mathbf{Su}(S^r_d)$. Since S^ℓ occurs in $\mathbf{SEQ}^\ell$, $(S^\ell)^*$ occurs in $(\mathbf{SEQ}^\ell)^*$, and so, there exists a cut-free proof figure P^ℓ for

$$S^\ell : \mathbf{an}(S^\ell), \mathbf{Su}(S^\ell) \rhd \bot \to \mathbf{Su}(S^\ell).$$

Using Q^r_d and cuts, we have the following cut-free proof figure P_{11}:

$$\frac{Q^r_d \qquad\qquad\qquad P^\ell}{\mathbf{Su}(S^r_d) \rhd \bot, (\{\mathbf{an}(S^\ell)\} \cup \mathbf{Su}(S^\ell) \rhd \bot)_{D \rhd \bot} \to \mathbf{Su}(S^\ell)}$$

On the other hand, from our first assumption, $\mathbf{SEQ}^\ell$ consists of only one sequent if $C = \bot$. Here, however, $\mathbf{SEQ}^\ell$ has at least two sequents S^ℓ_0 and S^ℓ. Hence we have $C \neq \bot$ (i.e., $d(C) > 0$), and so, $d(P_{11}) = d(D \rhd \bot) = d(D) + 1 < d(C) + d(D) + 1 = d(C \rhd D) = d(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of P_{11}. Using P^r_d and cuts, we have the following cut-free proof figure P_{12}:

$$\frac{P_{11} \qquad\qquad\qquad P^r_d}{\mathbf{Su}(S^r_d) \rhd \bot, (\{\mathbf{an}(S^\ell)\} \cup \mathbf{Su}(S^\ell) \rhd \bot)_{D \rhd \bot} \to \mathbf{Su}(S^\ell)_D, \mathbf{Su}(S^r_d)}$$

We note $d(P_{12}) = d(D) < d(C \rhd D) = d(P)$. So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of P_{12}. Using $(T \to)$, possibly several times, we obtain a cut-free proof figure for $S^* = T$.

By (3),(4) and Lemma 3.7, there exists an essential subsequence $\mathbf{SEQ'}$ of $\mathbf{SEQ}$ on $\Sigma^\ell, \Sigma^r_{C \rhd D} \to A \rhd B$. So,

$$\frac{(\mathbf{SEQ'})^*}{\Sigma^\ell, \Sigma^r_{C \rhd D} \to A \rhd B}$$

is an inference rule. Since $\mathbf{SEQ'}$ is a subsequence of $\mathbf{SEQ}$, $(\mathbf{SEQ'})^*$ is of $\mathbf{SEQ}^*$, and so, every sequent in $(\mathbf{SEQ'})^*$ also occurs in $\mathbf{SEQ}^*$. Hence there exist a cut-free proof figure for each sequent in $(\mathbf{SEQ'})^*$. Using the inference rule above, we obtain a cut-free proof figure for

$$\Sigma^\ell, \Sigma^r_{C \rhd D} \to A \rhd B,$$

the end sequent of P. $\dashv$

Definition 3.10. The set $\mathsf{Sub}^\rhd(A)$ is defined inductively as follows:
(1) $\mathsf{Sub}^\rhd(B) = \mathsf{Sub}(B)$, for a non-modal formula B,
(2) $\mathsf{Sub}^\rhd(C \rhd D) = \mathsf{Sub}^\rhd(C) \cup \mathsf{Sub}^\rhd(D) \cup \{\bot, C \rhd \bot, D \rhd \bot\}$.

We note that $\mathsf{Sub}^\rhd(A)$ is finite.

Corollary 3.11. *Let it be that $\Gamma \to \Delta \in$ **GIK4**. Then there exists a cut-free proof figure P satisfying the following condition for any formula A:*

$$\textit{if } A \textit{ occurs in } P\textit{, then } A \in \mathsf{Sub}^\rhd(B) \textit{ for some } B \textit{ occurring in } \Gamma \to \Delta.$$

By Corollary 1.8 and Corollary 3.11, our system **GIK4** gives a decision procedure for **IK4**.

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