A sequent system for the interpretability logic with the persistence axiom

Katsumi Sasaki

Abstract. In [Sas01], it was given a cut-free sequent system for the smallest interpretability logic **IL**. He first gave a cut-free system for **IK4**, a sublogic of **IL**, whose $\rhd$-free fragment is the modal logic **K4**. Here, using the method in [Sas01], we give sequent systems for the interpretability logic **ILP** obtained by adding the persistence axiom $P : (p \rhd q) \supset \Box(p \rhd q)$ to **IL** and for the logic **IK4**+P obtained by adding P to **IK4**. We also prove a cut-elimination theorem for the system for **IK4P**.

1 Introduction

The idea of interpretability logics arose in Visser [Vis90]. He introduced the logics as extensions of the provability logic **GL** with a binary modality $\rhd$. The arithmetic realization of $A \rhd B$ in a theory T will be that T plus the realization of B is interpretable in T plus the realization of A ($T + A$ *interprets* $T + B$). More precisely, there exists a function f (the relative interpretation) on the formulas of the language of T such that $T + B \vdash C$ implies $T + A \vdash f(C)$.

The interpretability logics were considered in several papers. An arithmetic completeness of the interpretability logic **ILM**, obtained by adding Montagna's axiom to the smallest interpretability logic **IL**, was proved in Berarducci [Ber90] and Shavrukov [Sha88] (see also Hájek and Montagna [HM90] and Hájek and Montagna [HM92]). [Vis90] proved that the interpretability logic **ILP**, obtained by adding the persistence axiom to **IL**, is also complete for another arithmetic interpretation. The completeness with respect to Kripke semantics due to Veltman was, for **IL**, **ILM** and **ILP**, proved in de Jongh and Veltman [JV90]. The fixed point theorem of **GL** can be extended to **IL** and hence **ILM** and **ILP** (cf. de Jongh and Visser [JV91]). The unary pendant "T interprets $T + A$" is much less expressive and was studied in de Rijke [Rij92]. For an overview of interpretability logic, see Visser [Vis97], and Japaridze and de Jongh [JJ98].

The language of interpretability logics contain a unary modal operator $\Box$ and a binary modal operator $\rhd$. However, we can show the equivalence between $\Box A$ and $\neg A \rhd \bot$ in sublogic **IK4**, which is the smallest among the logics treated here (cf. [JJ98]). Hence, we do not have to treat $\Box$ as a primary operator. Systems for interpretability logics with two primary modal operators are much more complicated than the ones with one primary modal operator. So, in this paper, we treat $\Box A$ as an abbreviation of $\neg A \rhd \bot$.

We use lower case Latin letters p, q, r, possibly with suffixes, for propositional variables. Formulas are defined, as usual, from the propositional variables and the logical constant $\bot$ (contradiction) by using binary logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and $\rhd$ (interpretation). We use upper case Latin letters $A, B, C, \cdots$, possibly with suffixes, for formulas. A formula of the form $A \rhd B$ is said to be a $\rhd$-formula. The expressions $\neg A$, $\Box A$ and $\Diamond A$ are abbreviations for $A \supset \bot, \neg A \rhd \bot$ and $\neg(A \rhd \bot)$, respectively.

Definition 1.1. The degree $d(A)$ of a formula A is defined inductively as follows:
(1) $d(p) = 1$,
(2) $d(\bot) = 0$,
(3) $d(A \wedge B) = d(A \vee B) = d(A \supset B) = d(A \rhd B) = d(A) + d(B) + 1$.

Note that $d(A \rhd \bot) < d(A \rhd B)$ for each $B \neq \bot$.

An interpretability logic is a set of formulas containing all the tautologies and axioms
$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$,
$L : \Box(\Box p \supset p) \supset \Box p$,
$J1 : \Box(p \supset q) \supset (p \rhd q)$,
$J2 : (p \rhd q) \wedge (q \rhd r) \supset (p \rhd r)$,
$J3 : (p \rhd r) \wedge (q \rhd r) \supset ((p \vee q) \rhd r)$,

$J5 : (\Diamond p) \rhd p$,
and closed under modus ponens, substitution and necessitation. By **IL**, we mean the smallest interpretability logic. By **ILP**, we mean the smallest set of formulas containing all the theorems in **IL** and the axiom
$P : (p \rhd q) \supset \Box(p \rhd q)$
and closed under modus ponens, substitution and necessitation.

If we use $\Box$ as a primary operator, then we need one more axiom
$J4 : (p \rhd q) \supset (\Diamond p \supset \Diamond q)$
to define interpretability logics. Here $\Box$ is not primary and we can prove $(J4)$ in the logics defined in this paper (see [Sas01]).

The aim of this paper is to give a cut-free sequent system for **ILP** using the method in [Sas01]. [Sas01] first gave a cut-free system for a sublogic **IK4** of **IL**, whose $\rhd$-free fragment is the normal modal logic **K4** in a sense that $\Box$ is a primary. Using the system for a **IK4** and a property of Löb's axiom, a cut-free system for **IL** was given.

Here, as in [Sas01], we first give a cut-free system for a sublogic **IK4** $+ P$ of **ILP**, whose $\rhd$-free fragment is **K4**. The precise definitions of the logic **IK4** and **IK4** $+ P$ we need here are given as follows.

By **IK4**, we mean the smallest set of formulas containing all the tautologies and axioms K, $J1$, $J2$, $J3$, $J5$ and
$4 : \Box p \supset \Box\Box p$,
and closed under modus ponens, substitution and necessitation. For a formula A and a logic **L**, **L** $+ A$ is the smallest set of formulas including $\mathbf{L} \cup \{A\}$ and closed under modus ponens, substitution and necessitation.

Lemma 1.2.
(1) **IL**=**IK4**+L,
(2) **ILP**=**IL**+P=**IK4**+P+L.

In the next section we give a sequent system for **IK4** $+ P$. Cut-elimination theorem is shown in Section 3. In Section 4, we give a sequent system for **ILP**.

2 A sequent system for IK4+P

In this section we introduce a sequent system **GIK4P** for **IK4**+P. We use Greek letters, possibly with suffixes, for finite sets of formulas. The expression Γ_A denotes the set $\Gamma - \{A\}$. In this paper, we often use finite sets of $\rhd$-formulas. So, it is useful to prepare symbols for them and we use Σ, possibly with suffixes, for finite sets of $\rhd$-formulas. For each prefix $\odot \in \{\Box, \Diamond, \neg\}$, the expression $\odot\Gamma$ denotes the set $\{\odot A \mid A \in \Gamma\}$. Similarly, $\Gamma \rhd \bot$ denotes $\{A \rhd \bot \mid A \in \Gamma\}$. By a sequent, we mean the expression

$$\Gamma \to \Delta.$$

For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma^\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

By $\mathsf{Sub}(A)$, we mean the set of subformulas of A. By $\mathsf{Sub}(\Gamma \to \Delta)$, we mean the set of subformulas of each formula occurring in $\Gamma \cup \Delta$.

Our system **GIK4P** is defined from the following axioms and inference rules in the usual way.

Axioms of GIK4P

$$A \to A$$

$$\bot \rightarrow$$

Inference rules of GIK4P

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}(T \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}(\rightarrow T)$$

$$\frac{\Gamma \rightarrow \Delta, A \quad A, \Pi \rightarrow \Lambda}{\Gamma, \Pi_A \rightarrow \Delta_A, \Lambda}(\mathrm{cut})$$

$$\frac{A_i, \Gamma \rightarrow \Delta}{A_1 \wedge A_2, \Gamma \rightarrow \Delta}(\wedge \rightarrow_i) \qquad \frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}(\rightarrow \wedge)$$

$$\frac{A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}(\vee \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, A_i}{\Gamma \rightarrow \Delta, A_1 \vee A_2}(\rightarrow \vee_i)$$

$$\frac{\Gamma \rightarrow \Delta, A \quad B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta}(\supset\rightarrow) \qquad \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B}(\rightarrow\supset)$$

$$\frac{A, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \rightarrow B, X_1, \cdots, X_n \quad \Sigma \rightarrow Y_1 \rhd B \quad \cdots \quad \Sigma \rightarrow Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow A \rhd B}(\rhd_{K4P})$$

where $n = 0, 1, 2, \cdots$.

Note that in $(\rhd_{K4P})$, Σ might contain $X_i \rhd Y_i$.

Definition 2.1. A proof figure in **GIK4P** for a sequent $\Gamma \rightarrow \Delta$ is defined as follows:
(1) if a sequent S is an axiom in **GIK4P**, then S is a proof figure for S,
(2) if $\mathcal{P}_1, \cdots, \mathcal{P}_n$ are proof figures for sequents $S_1, \cdots, S_n$, and $\dfrac{S_1 \quad \cdots \quad S_n}{S}$ is an inference rule in **GIK4P**, then $\dfrac{\mathcal{P}_1 \quad \cdots \quad \mathcal{P}_n}{S}$ is a proof figure for S.

We say that a sequent S is provable in **GIK4P**, and write $S \in$ **GIK4P**, if there exists a proof figure for S. We use $\mathcal{P}, \mathcal{Q}$, possibly with suffixes, for proof figures.
Let $\mathcal{P}$ be a proof figure for $\Gamma \rightarrow \Delta$. In order to emphasize the end sequent of $\mathcal{P}$, we also use the expressions

$$\mathcal{P}\left\{\begin{array}{c} \vdots \\ \Gamma \rightarrow \Delta \end{array}\right. \qquad \text{and} \qquad \left.\begin{array}{c} \vdots \\ \Gamma \rightarrow \Delta \end{array}\right\}\mathcal{P}$$

instead of $\mathcal{P}$.

Definition 2.2. A set $\mathsf{SubFig}(\mathcal{P})$ of a proof figure $\mathcal{P}$ is defined as follows:
(1) $\mathsf{SubFig}(\mathcal{P}) = \{\mathcal{P}\}$ if $\mathcal{P}$ is an axiom,
(2) $\mathsf{SubFig}(\dfrac{\mathcal{P}_1 \quad \cdots \quad \mathcal{P}_n}{\Gamma \rightarrow \Delta}) = \mathsf{SubFig}(\mathcal{P}_1) \cup \cdots \mathsf{SubFig}(\mathcal{P}_n) \cup \{\mathcal{P}\}$.

We call an element of $\mathsf{SubFig}(\mathcal{P})$ a subfigure of $\mathcal{P}$ and an element of $\mathsf{SubFig}(\mathcal{P}) - \{\mathcal{P}\}$ a proper subfigure of $\mathcal{P}$. As to the other terminology concerning the system, we mainly follow Gentzen [Gen35].

If $n = 0$, the inference rule $(\rhd_{K4P})$ has only one upper sequent and is of the following form:

$$\frac{A, B \rhd \bot, \Sigma \rightarrow B}{\Sigma \rightarrow A \rhd B}$$

Hence

Lemma 2.3. *There exist cut-free proof figures for* $\rightarrow \bot \rhd A$ *and* $\rightarrow A \rhd A$ *in* **GIK4P**.

The main theorem in this section is

Theorem 2.4. $A \in \mathbf{IK4} + P$ *iff* $\rightarrow A \in \mathbf{GIK4P}$.

To prove the theorem above, we need some preparations.

By **GIK4**, we mean the system obtained from **GIK4P** by replacing $(\rhd_{K4P})$ by

$$\frac{A, \{B, X_1, \cdots, X_n\} \rhd \bot \rightarrow B, X_1, \cdots, X_n \quad \Sigma \rightarrow Y_1 \rhd B \quad \cdots \quad \Sigma \rightarrow Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \rightarrow A \rhd B}(\rhd_{K4})$$

By $\mathbf{GIK4} + P$, we mean the system obtained by adding the axiom
$GP : A \rhd B \rightarrow \Box(A \rhd B)$
to **GIK4**.

[Sas01] proved cut-elimination theorem of **GIK4** and the following two lemmas.

Lemma 2.5. *There exist cut-free proof figures for* $\rightarrow \bot \rhd A$ *and* $\rightarrow A \rhd A$ *in* **GIK4**.

Lemma 2.6. $A \in \mathbf{IK4}$ *iff* $\rightarrow A \in \mathbf{GIK4}$.

Corollary 2.7. $A \in \mathbf{IK4} + P$ *iff* $\rightarrow A \in \mathbf{GIK4} + P$.

Lemma 2.8. $\rightarrow A \in \mathbf{GIK4} + P$ *implies* $\rightarrow A \in \mathbf{GIK4P}$.

Proof. It is sufficient to show that the axiom GP is provable in **GIK4P** and the inference rule $(\rhd_{K4})$ holds in **GIK4P**. Using $(T \rightarrow)$ and $(\rhd_{K4P})$, we can easily see that $(\rhd_{K4})$ holds in **GIK4P**. The following is the proof figure for GP:

$$\dfrac{\dfrac{\dfrac{B \rhd C \rightarrow B \rhd C}{B \rhd C \rightarrow B \rhd C, \bot} \quad \dfrac{\bot \rightarrow \bot}{B \rhd C, \bot \rightarrow \bot}}{\neg(B \rhd C), B \rhd C \rightarrow \bot}}{B \rhd C \rightarrow \Box(B \rhd C)}.$$

⊣

Lemma 2.9. $\rightarrow A \in \mathbf{GIK4P}$ *implies* $\rightarrow A \in \mathbf{GIK4} + P$.

Proof. It is sufficient to show that the rule $(\rhd_{K4P})$ holds in $\mathbf{GIK4} + P$. We can see it by using the following inference rule, the axiom $X \rhd Y \rightarrow \Box(X \rhd Y)$ for $X \rhd Y \in \Sigma$, Lemma 2.5 and cut, possibly several times.

$$\frac{A, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \rightarrow B, X_1, \cdots, X_n, \neg\Sigma \quad \Sigma \rightarrow Y_1 \rhd B \quad \cdots \quad \Sigma \rightarrow Y_n \rhd B \quad \Sigma \rightarrow \bot \rhd B \quad \cdots \quad \Sigma \rightarrow \bot \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Box\Sigma, \Sigma \rightarrow A \rhd B}.$$

⊣

From Corollary 2.7, Lemma 2.8 and Lemma 2.9, we obtain Theorem 2.4.

3 Cut-elimination theorem for GIK4P

In this section, we prove cut-elimination theorem for **GIK4P**.

Theorem 3.1. *If* $\Gamma \rightarrow \Delta \in \mathbf{GIK4P}$, *then there exists a cut-free proof figure for* $\Gamma \rightarrow \Delta$ *in* **GIK4P**.

To prove the theorem, we need some lemmas.

Lemma 3.2. *Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be cut-free proof figures for* $\Sigma_1 \rightarrow A \rhd B$ *and* $\Sigma_2 \rightarrow B \rhd C$, *respectively. Then there exists a cut-free proof figure for* $\Sigma_1, \Sigma_2 \rightarrow A \rhd C$.

Proof. We use an induction on $\mathcal{P}_1$. If $\mathcal{P}_1$ is an axiom, then $\Sigma_1 = \{A \rhd B\}$, and hence we have the following cut-free proof figure for $\Sigma_1, \Sigma_2 \to A \rhd C$.

$$\dfrac{\dfrac{\dfrac{A \to A}{\text{using } (T \to) \text{ twice, and } (\to T)}}{A, C \rhd \bot, A \rhd \bot, \Sigma_2 \to C, A} \qquad \left.\begin{matrix} \vdots \\ \Sigma_2 \to B \rhd C \end{matrix}\right\} \mathcal{P}_2}{A \rhd B, \Sigma_2 \to A \rhd C}$$

If $\mathcal{P}_1$ is not axiom, then there exists an inference rule I that introduces the end sequent of $\mathcal{P}_1$. We only show the case that I is $(\rhd_{K4P})$ since the other cases can be shown easily. The inference rule I is of the form

$$\frac{A, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma_1' \to B, X_1, \cdots, X_n \qquad \Sigma_1' \to Y_1 \rhd B \quad \cdots \quad \Sigma_1' \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma_1' \to A \rhd B}$$

where $\Sigma_1 = \Sigma_1' \cup \{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n\}$. Clearly, there exist cut-free proof figures for the upper sequents of I. Using the induction hypothesis and $\mathcal{P}_2$, there exists a cut-free proof figure for $\Sigma_1', \Sigma_2 \to Y_i \rhd C$ for each $i = 1, \cdots, n$. Using $(\rhd_{K4})$ below, we obtain the lemma.

$$\frac{A, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma_1' \to B, X_1, \cdots, X_n \qquad \Sigma_1', \Sigma_2 \to Y_1 \rhd C \quad \cdots \quad \Sigma_1', \Sigma_2 \to Y_n \rhd C}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma_1', \Sigma_2 \to A \rhd C}$$

$\dashv$

Lemma 3.3. *If there exists a cut-free proof figure for $\Sigma \to A \rhd B$, then either one of the following two holds:*

(1) *there exists a cut-free proof figure for $\Sigma \to$,*

(2) *for some subsets Σ_1 and Σ_2 of Σ, there exist cut-free proof figures for*

$$A, B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\}, \Sigma_2 \to \{X \mid X \rhd Y \in \Sigma_1\}, B$$

and

$$\Sigma_2 \to Y \rhd B, \textit{ for each } Y \in \{Y' \mid X \rhd Y' \in \Sigma_1\}.$$

Proof. We use an induction on the cut-free proof figure $\mathcal{P}$ for $\Sigma \to A \rhd B$. If $\mathcal{P}$ is an axiom, then $\{A \rhd B\} = \Sigma$ and by Lemma 2.3, there exist cut-free proof figures for

$$A, B \rhd \bot, A \rhd \bot \to A, B \text{ and } \to B \rhd B.$$

Hence (2) holds.

If $\mathcal{P}$ is not axiom, then there exists an inference rule I that introduces the end sequent of $\mathcal{P}$. If I is $(\to T)$, then (1) holds. If I is $(T \to)$, then by the induction hypothesis, we obtain the lemma. If I is $(\rhd_{K4P})$, then (2) holds. $\dashv$

It is known that Theorem 3.1 follows from the following lemma.

Lemma 3.4. *Let $\mathcal{P}^\ell$ be a cut-free proof figure for $\Gamma \to \Delta, X$ and $\mathcal{P}^r$ be a cut-free proof figure for $X, \Pi \to \Lambda$. Let $\mathcal{P}$ be the proof figure*

$$\frac{\mathcal{P}^\ell \left\{\begin{matrix} \vdots \\ \Gamma \to \Delta, X \end{matrix}\right. \qquad \left.\begin{matrix} \vdots \\ X, \Pi \to \Lambda \end{matrix}\right\} \mathcal{P}^r}{\Gamma, \Pi_X \to \Delta_X, \Lambda}.$$

Then there exists a cut-free proof figure for the end sequent of $\mathcal{P}$.

Proof. The degree $d(\mathcal{P})$ of $\mathcal{P}$ is defined as $d(X)$. The left rank $R^\ell(\mathcal{P})$ and the right rank $R^r(\mathcal{P})$ of P are defined as usual. We use an induction on $R^\ell(\mathcal{P}) + R^r(\mathcal{P}) + \omega d(\mathcal{P})$. We only treat the case that $\mathcal{P}$, $\mathcal{P}^\ell$ and $\mathcal{P}^r$ are of the following forms.

$\mathcal{P}^\ell$:

$$\dfrac{\mathcal{P}_0^\ell\left\{\begin{array}{c}\vdots\\ C,\mathbf{X}^\ell\rhd\bot,\Sigma^L\to\mathbf{X}^\ell\end{array}\right.\qquad\left.\begin{array}{c}\vdots\\ \Sigma^L\to Y_1^\ell\rhd D\end{array}\right\}\mathcal{P}_1^\ell\cdots\qquad\left.\begin{array}{c}\vdots\\ \Sigma^L\to Y_m^\ell\rhd D\end{array}\right\}\mathcal{P}_m^\ell}{\Sigma^\ell,\Sigma^L\to C\rhd D}$$

$\mathcal{P}^r$:

$$\dfrac{\mathcal{P}_0^r\left\{\begin{array}{c}\vdots\\ A,\mathbf{X}^r\rhd\bot,\Sigma^R\to\mathbf{X}^r\end{array}\right.\qquad\left.\begin{array}{c}\vdots\\ \Sigma^R\to Y_1^r\rhd B\end{array}\right\}\mathcal{P}_1^r\cdots\qquad\left.\begin{array}{c}\vdots\\ \Sigma^R\to Y_n^r\rhd B\end{array}\right\}\mathcal{P}_n^r}{C\rhd D,\Sigma^r,\Sigma^R\to A\rhd B}$$

$\mathcal{P}$:

$$\dfrac{\mathcal{P}^\ell\left\{\begin{array}{c}\vdots\\ \Sigma^\ell,\Sigma^L\to C\rhd D\end{array}\right.\qquad\left.\begin{array}{c}\vdots\\ C\rhd D,\Sigma^r,\Sigma^R\to A\rhd B\end{array}\right\}\mathcal{P}^r}{\Sigma^\ell,\Sigma^L,\Sigma^r_{C\rhd D},\Sigma^R_{C\rhd D}\to A\rhd B}$$

where
$\Sigma^\ell=\{X_1^\ell\rhd Y_1^\ell,\cdots,X_m^\ell\rhd Y_m^\ell\}$,
$\Sigma^r=\{X_1^r\rhd Y_1^r,\cdots,X_n^r\rhd Y_n^r\}$,
$\mathbf{X}^\ell=\{X_1^\ell,\cdots,X_m^\ell,D\}$,
$\mathbf{X}^r=\{X_1^r,\cdots,X_n^r,B\}$
and $C\rhd D\in\Sigma^r\cup\Sigma^R$.

By $\mathcal{P}^\ell$ and $\mathcal{P}_0^r$, we have the following proof figure for each $j=1,\cdots,n$:

$$\dfrac{\mathcal{P}^\ell\left\{\begin{array}{c}\vdots\\ \Sigma^\ell,\Sigma^L\to C\rhd D\end{array}\right.\qquad\left.\begin{array}{c}\vdots\\ A,\mathbf{X}^r\rhd\bot,\Sigma^R\to\mathbf{X}^r\end{array}\right\}\mathcal{P}_0^r}{\Sigma^\ell,\Sigma^L,(A,\mathbf{X}^r\rhd\bot,\Sigma^R)_{C\rhd D}\to\mathbf{X}^r}$$

We note the degree and the left rank of the figure above are the same as those of $\mathcal{P}$ and the right rank is smaller. Using the induction hypothesis and $(T\to)$, possibly several times, we obtain a cut-free proof figure $\mathcal{Q}_0^r$ for

$$A,\mathbf{X}^r\rhd\bot,\Sigma^\ell,\Sigma^L,\Sigma^R_{C\rhd D}\to\mathbf{X}^r.$$

Similarly, by $\mathcal{P}^\ell$ and $\mathcal{P}_j^r$, we have the following proof figure for each $j=1,\cdots,n$:

$$\dfrac{\mathcal{P}^\ell\left\{\begin{array}{c}\vdots\\ \Sigma^\ell,\Sigma^L\to C\rhd D\end{array}\right.\qquad\left.\begin{array}{c}\vdots\\ \Sigma^R\to Y_j^r\rhd B\end{array}\right\}\mathcal{P}_j^r}{\Sigma^\ell,\Sigma^L,\Sigma^R_{C\rhd D}\to Y_j^r\rhd B}$$

and using the induction hypothesis, we obtain a cut-free proof figure $\mathcal{Q}_j^r$ for the end sequent of the figure above.

If $C\rhd D\notin\Sigma^r$, then by $\mathcal{Q}_0^r$, $\mathcal{Q}_j^r$ and $(\rhd_{K4P})$, we obtain the cut-free proof figure for the end sequent of $\mathcal{P}$.

Assume that $C\rhd D\in\Sigma^r=\{X_1^r\rhd Y_1^r,\cdots,X_n^r\rhd Y_n^r\}$. Without loss of generality, we also assume that $C\rhd D=X_1^r\rhd Y_1^r\notin\Sigma^r-\{X_1^r\rhd Y_1^r\}$. We divide into the cases.

The case that $C=D=\bot$: By $\mathcal{P}_0^r$, we have the following proof figure $\mathcal{Q}_1$:

$$\dfrac{\dfrac{\dfrac{\bot\to\bot}{\bot,\bot\rhd\bot\to\bot}}{\to\bot\rhd\bot}\qquad\left.\begin{array}{c}\vdots\\ A,\{B,\bot,X_2^r,\cdots,X_n^r\}\rhd\bot,\Sigma^R\to B,\bot,X_2^r,\cdots,X_n^r\end{array}\right\}\mathcal{P}_0^r}{(A,\{B,X_2^r,\cdots,X_n^r\}\rhd\bot,\Sigma^R)_{\bot\rhd\bot}\to B,\bot,X_2^r,\cdots,X_n^r}$$

We note that $d(\mathcal{Q}_1)=d(\bot\rhd\bot)=d(\bot\rhd D)=d(\mathcal{P})$, $1=R^\ell(\mathcal{Q}_1)=R^\ell(\mathcal{P})$ and $R^r(\mathcal{Q}_1)<R^r(\mathcal{P})$. Using the induction hypothesis, we obtain a cut-free proof figure for the end sequent of the figure above. Using the axiom $\bot\to$, (cut) and the induction hypothesis, we obtain a cut-free proof figure

for $(A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^R)_{\bot\rhd\bot} \to (B, X_2^r, \cdots, X_n^r)_\bot$. Using $(T \to)$ and $(\to T)$, possibly several times, we obtain a cut-free proof figure for

$$A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D} \to B, X_2^r, \cdots, X_n^r.$$

Using $\mathcal{Q}_2^r, \cdots, \mathcal{Q}_n^r$ and $(\rhd_{K4P})$, we have a cut-free proof figure for the end sequent of $\mathcal{P}$.

The case that $C = \bot$ and $D \neq \bot$: By $\mathcal{Q}_0^r$, we have the following proof figure $\mathcal{Q}_2$:

$$\dfrac{\dfrac{\dfrac{\bot \to \bot}{\bot, \bot \rhd \bot \to \bot}}{\to \bot \rhd \bot} \qquad \left.\begin{array}{c}\vdots\\ A, \{B, \bot, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D} \to B, \bot, X_2^r, \cdots, X_n^r\end{array}\right\} \mathcal{P}_0^r}{(A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D})_{\bot\rhd\bot} \to B, \bot, X_2^r, \cdots, X_n^r}$$

We note that $d(\mathcal{Q}_2) = d(\bot \rhd \bot) < d(\bot \rhd D) = d(\mathcal{P})$. Using the induction hypothesis, we obtain a cut-free proof figure for the end sequent of the figure above. Using the axiom $\bot \to$, (cut) and the induction hypothesis, we obtain a cut-free proof figure for $(A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R)_{\bot\rhd\bot} \to (B, X_2^r, \cdots, X_n^r)_\bot$. Using $(T \to)$ and $(\to T)$, possibly several times, we obtain a cut-free proof figure for

$$A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D} \to B, X_2^r, \cdots, X_n^r.$$

Using $\mathcal{Q}_2^r, \cdots, \mathcal{Q}_n^r$ and $(\rhd_{K4P})$, we have a cut-free proof figure for the end sequent of $\mathcal{P}$.

The case that $C \neq \bot$: By $\mathcal{P}_0^\ell$, Lemma 2.3 and $(\rhd_{K4P})$, we have the following cut-free proof figure:

$$\dfrac{\mathcal{P}_0^\ell \left\{\begin{array}{c}\vdots\\ C, \{D, X_1^\ell, \cdots, X_m^\ell\} \rhd \bot, \Sigma^L \to D, X_1^\ell, \cdots, X_m^\ell\end{array}\right. \quad \begin{array}{c}\vdots\\ \Sigma^L \to \bot \rhd \bot\end{array} \quad \cdots \quad \begin{array}{c}\vdots\\ \Sigma^L \to \bot \rhd \bot\end{array}}{\{D, X_1^\ell, \cdots, X_m^\ell\} \rhd \bot, \Sigma^L \to C \rhd \bot}$$

If $D = \bot$, then using $\mathcal{P}_0^r$, we have the following proof figure $\mathcal{P}_1$:

$$\dfrac{\dfrac{\mathcal{P}_0^\ell \quad \begin{array}{c}\vdots\\ \Sigma^L \to \bot \rhd \bot\end{array} \quad \cdots \quad \begin{array}{c}\vdots\\ \Sigma^L \to \bot \rhd \bot\end{array}}{\mathbf{X}^\ell \rhd \bot, \Sigma^L \to C \rhd \bot} \qquad \left.\begin{array}{c}\vdots\\ A, \{B, C, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^R \to B, C, X_2^r, \cdots, X_n^r\end{array}\right\} \mathcal{P}_0^r}{\{D, X_1^\ell, \cdots, X_m^\ell\} \rhd \bot, \Sigma^L, (A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^R)_{C\rhd\bot} \to B, C, X_2^r, \cdots, X_n^r}$$

and note that $d(\mathcal{P}_1) = d(C \rhd \bot) = d(C \rhd D) = d(\mathcal{P})$, $1 = R^\ell(\mathcal{P}_1) = R^\ell(\mathcal{P})$ and $R^r(\mathcal{P}_1) < R^r(\mathcal{P})$. Using the induction hypothesis, we obtain a cut-free proof figure for the end sequent of the figure above. Using $(T \to)$ and $(\to T)$, possibly several times, we obtain a cut-free proof figure $\mathcal{P}_2$ for

$$A, \{B, D, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D} \to B, C, X_2^r, \cdots, X_n^r.$$

If $D \neq \bot$, then using $\mathcal{Q}_0^r$, we have the following proof figure $\mathcal{P}_3$:

$$\dfrac{\dfrac{\mathcal{P}_0^\ell \quad \begin{array}{c}\vdots\\ \Sigma^L \to \bot \rhd \bot\end{array} \quad \cdots \quad \begin{array}{c}\vdots\\ \Sigma^L \to \bot \rhd \bot\end{array}}{\mathbf{X}^\ell \rhd \bot, \Sigma^L \to C \rhd \bot} \qquad \left.\begin{array}{c}\vdots\\ A, \{B, C, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D} \to B, C, X_2^r, \cdots, X_n^r\end{array}\right\} \mathcal{Q}_0^r}{\{D, X_1^\ell, \cdots, X_m^\ell\} \rhd \bot, \Sigma^L, (A, \{B, X_2^r, \cdots, X_n^r\} \rhd \bot, \Sigma^\ell, \Sigma^L, \Sigma^R_{C\rhd D})_{C\rhd\bot} \to B, C, X_2^r, \cdots, X_n^r}$$

and note that $d(\mathcal{P}_3) = d(C \rhd \bot) < d(C \rhd D) = d(\mathcal{P})$. Using the induction hypothesis, we obtain a cut-free proof figure for the end sequent of the figure above. Using $(T \to)$ and $(\to T)$, possibly several times, we obtain a cut-free proof figure $\mathcal{P}_4$ for the end sequent of $\mathcal{P}_2$.

By $\mathcal{P}_2$, $\mathcal{P}_4$ and $\mathcal{P}_0^\ell$, we have the following proof figure:

$$\dfrac{\mathcal{P}_2(\text{ or } \mathcal{P}_4) \qquad \left.\begin{array}{c}\vdots\\ C, \{D, X_1^\ell, \cdots, X_m^\ell\} \rhd \bot \to D, X_1^\ell, \cdots, X_m^\ell\end{array}\right\} \mathcal{P}_0^\ell}{A, \{B, D, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r\} \rhd \bot \to B, D, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r}$$

We note the degree of the figure above is smaller than that of $\mathcal{P}$. Using the induction hypothesis, we obtain a cut-free proof figure $\mathcal{P}_5$ for the end sequent of the figure above.

By $\mathcal{Q}_1^r$ and Lemma 3.3, either one of the following two holds:
(1) there exists a cut-free proof figure for $\Sigma^\ell, \Sigma^L, \Sigma^R_{C \rhd D} \to$,
(2) for some subsets Σ_1 and Σ_2 of $\Sigma^\ell \cup \Sigma^L \cup \Sigma^R_{C \rhd D}$, there exist cut-free proof figures for

$$D, B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\}, \Sigma_2 \to \{X \mid X \rhd Y \in \Sigma_1\}, B$$

and

$$\Sigma_2 \to Y \rhd B, \text{ for each } Y \in \{Y' \mid X \rhd Y' \in \Sigma_1\}.$$

If (1) holds, we obtain the lemma, immediately. Assume that (2) holds. Then by $\mathcal{P}_5$ and (cut) whose cut formula is D, we have the following proof figure:

$$\dfrac{\mathcal{P}_5 \qquad \begin{array}{c}\vdots\\ D, B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\}, \Sigma_2 \to \{X \mid X \rhd Y \in \Sigma_1\}, B\end{array}}{A, D \rhd \bot, \Delta \rhd \bot, \Sigma_2 \to B, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r, \{X \mid X \rhd Y \in \Sigma_1\}}$$

where Δ is the succedent of the end sequent. We note that the degree of the proof figure above is $d(D) < d(C \rhd D) = d(\mathcal{P})$. Using the induction hypothesis, we have a cut-free proof figure $\mathcal{P}_6$ for the end sequent of the figure above.

By (2), Lemma 2.3 and ($\rhd_{K4P}$), we have a cut-free proof figure for

$$B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\}, \Sigma_2 \to D \rhd \bot.$$

Using $\mathcal{P}_6$, we have the following proof figure:

$$\dfrac{\begin{array}{c}\vdots\\ B \rhd \bot, \{X \rhd \bot \mid X \rhd Y \in \Sigma_1\}, \Sigma_2 \to D \rhd \bot\end{array} \qquad \mathcal{P}_6}{A, \Delta \rhd \bot, \Sigma_2 \to B, X_1^\ell, \cdots, X_m^\ell, X_2^r, \cdots, X_n^r, \{X \mid X \rhd Y \in \Sigma_1\}}$$

Since $C \neq \bot$, the degree of the proof figure above is $d(D \rhd \bot) < d(C \rhd D) = d(\mathcal{P})$. Using the induction hypothesis, we have a cut-free proof figure $\mathcal{P}_7$ for the end sequent of the figure above.

On the other hand, by $\mathcal{P}_i^\ell$, $\mathcal{Q}_1^r$ and Lemma 4.2, we obtain a cut-free proof figure $\mathcal{Q}_i^\ell$ for $\Sigma^\ell, \Sigma^L, \Sigma^R_{C \rhd D} \to Y_i^\ell \rhd B$ for each $i = 1, \cdots, m$. Using $\mathcal{P}_7$, $\mathcal{Q}_2^r, \cdots, \mathcal{Q}_n^r$, (2), ($T \to$), possibly several times, and ($\rhd_{K4P}$), we obtain a cut-free proof figure for the end sequent of $\mathcal{P}$. ⊣

4 A sequent system for ILP

In this section, we introduce a sequent system **GILP** for **ILP**. A cut-elimination theorem for **GILP** is conjectured to be given by using the system **GIK4P** and a property of Löb's axiom. The method is used in [Sas01] to give a cut-elimination theorem for **IL**.

Definition 4.1. The system **GILP** is obtained from **GIK4P** by replacing ($\rhd_{K4P}$) by the following inference rule:

$$\dfrac{A, A \rhd \bot, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \to B, X_1, \cdots, X_n \quad \Sigma \to Y_1 \rhd B \ \cdots \ \Sigma \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \to A \rhd B}(\rhd_{LP})$$

where $n = 0, 1, 2, \cdots$.

Theorem 4.2. $A \in$ **ILP** *iff* $\to A \in$ **GILP**.

To prove the theorem above, we need some preparations.

Definition 4.3. By $\mathbf{GIK4P}+L$, we mean the system obtained from **GIK4P** by adding Löb's axiom

$$\to \Box(\Box A \supset A) \supset \Box A.$$

Corollary 4.4. *$A \in \mathbf{ILP}$ iff $\to A \in \mathbf{GIK4P} + L$.*

Proof. From Theorem 2.4. ⊣

Lemma 4.5. *$\to A \in \mathbf{GIK4P} + L$ implies $\to A \in \mathbf{GILP}$.*

Proof. By the following figures, we can see that Löb's axiom $\to \Box(\Box A \supset A) \supset \Box A$ is provable in **GILP** and $(\rhd_{K4P})$ holds in **GILP**.

$$\dfrac{\neg A, \Box A, \bot \rhd \bot, \Box(\Box A \supset A) \to \bot, \neg(\Box A \supset A) \qquad \to \bot \rhd \bot}{\neg(\Box A \supset A) \rhd \bot \to \neg A \rhd \bot}(\rhd_{LP})$$

$$\dfrac{\dfrac{A, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \to B, X_1, \cdots, X_n}{A, A \rhd \bot, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \to B, X_1, \cdots, X_n} \quad \Sigma \to Y_1 \rhd B \cdots \Sigma \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \to A \rhd B}(\rhd_{LP})$$

⊣

Lemma 4.6. *$(A \wedge (A \rhd \bot)) \rhd B \to A \rhd B \in \mathbf{GIK4P} + L$.*

Proof. In [Sas01], it was proved that

$$(A \wedge (A \rhd \bot)) \rhd B \to A \rhd B \in \mathbf{GIK4} + L,$$

where $\mathbf{GIK4} + L$ is the system obtained by adding $\to \Box(\Box A \supset A) \supset \Box A$ to **GIK4**. On the other hand, in Lemma 2.8, we show that $(\rhd_{K4})$ holds in **GIK4P**. Hence we obtain the lemma. ⊣

Lemma 4.7. *$\to A \in \mathbf{GILP}$ implies $\to A \in \mathbf{GIK4P} + L$.*

Proof. By the following figure, Lemma 4.6 and cut, the inference rule $(\rhd_{LP})$ holds in $\mathbf{GIK4P} + L$.

$$\dfrac{\dfrac{\dfrac{A, A \rhd \bot, \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \to B, X_1, \cdots, X_n}{A, A \wedge (A \rhd \bot), \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \to B, X_1, \cdots, X_n}}{A \wedge (A \rhd \bot), \{B, X_1, \cdots, X_n\} \rhd \bot, \Sigma \to B, X_1, \cdots, X_n} \quad \Sigma \to Y_1 \rhd B \cdots \Sigma \to Y_n \rhd B}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Sigma \to (A \wedge (A \rhd \bot)) \rhd B}$$

⊣

From Corollary 4.4, Lemma 4.5 and Lemma 4.7, we obtain Theorem 4.2.

References

[Ber90] A. Berarducci, *The interpretability logic of Peano arithmetic*, The Journal of Symbolic Logic, 55, 1990, pp. 1059–1089.

[Gen35] G. Gentzen, *Untersuchungen über das logisch Schliessen*, Mathematische Zeitschrift, 39, 1934–35, pp. 176–210, 405–431.

[HM90] P. Hájek and F. Montagna, *The Logic of* Π_1*-conservativity*, Archiv for für Mathematische Logik und Grundlagenforschung, 30, 1990, pp. 113–123.

[HM92] P. Hájek and F. Montagna, *The Logic of* Π_1*-conservativity continued*, Archiv for für Mathematische Logik und Grundlagenforschung, 32, 1992, pp. 57–63.

[JJ98] G. Japaridze and D. H. J. de Jongh, *The Logic of Provability*, in *Handbook of proof theory*, edited by S. R. Buss, University of California, 1998, pp. 475–546.

[JV90] D. H. J. de Jongh and F. Veltman, *Provability logics for relative interpretability*, in Petkov [Pet90], pp. 31–42.

[JV91] D. H. J. de Jongh and A. Visser, *Explicit fixed points in interpretability logic*, Studia Logica, 50, 1991, pp. 39–50.

[Pet90] P. Petkov, editor, *Mathematical logic*, Proceedings of the Heyting 1988 Summer School, Plenum Press, 1990.

[Rij92] M. de Rijke, *Unary interpretability logic*, Notre Dame Journal of Formal Logic, 33, 1992 pp. 249–272.

[Sas01] K. Sasaki, *A cut-free system for the smallest interpretability logic*, Studia Logica, to appear.

[Sha88] V. Yu. Shavrukov, *The logic of relative interpretability over Peano arithmetic* (Russian), Technical Report No. 5, Stekhlov Mathematical Institute, Moscow, 1988.

[Vis90] A. Visser, *Interpretability logic*, In [Pet90], 1990, pp. 175–209.

[Vis97] A. Visser, *An overview of interpretability logic*, in *Advances in Modal Logic '96*, edited by M. Kraft, M. de Rijke, and H. Wansing, CSLI Publications, 1997.

Katsumi Sasaki
Department of Mathematical Sciences,
Nanzan University,
27 Seirei-Cho, Seto 489-0863, Japan
e-mail: sasaki@ms.nanzan-u.ac.jp.