A Note on Iterated Hechler Forcing along Templates

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January 4th 2002

Abstract

We present a note on iterated Hechler forcing based on [S], [B] and [F].

Introduction

We are interested in constructions of iterated forcing. We have gained an access to a construction in [S] due to [B]. We are encouraged by a series of lectures by [F] at the set theory seminar, Nagoya university, October through December 2001. We present a note on the very basic part of the construction.

§1. Fundamentals on Complete Sub-preorders, Reductions and Quotients

We review complete sub-preorders, reductions and quotient preorders.

1.1 Definition. Let P be a subset of a preorder Q. We consider P as a *sub-preorder* of Q. Namely, $1 \in P$ and the order on P is $\leq_Q \lceil P$. We call P is a *complete sub-preorder* of Q, if every maximal antichain A in P remains a maximal antichain in Q. We write $P \lessdot Q$ to express P is a complete sub-preorder of Q.

For $q \in Q$, $p \in P$ is called a *reduction* of q, if every $p' \leq p$ in P is compatible with q in Q.

Let $P \lessdot Q$ and G_P be a P-generic filter over V. The *quotient* preorder, denoted by Q/G_P, is defined in $V[G_P]$ as follows; $Q/G_P = \{q \in Q \mid$ for any $a \in G_P$, we demand a and q are compatible in $Q\}$. For $q, r \in Q/G_P$, we define $r \leq q$, if $r \leq q$ in Q. So Q/G_P is a sub-preorder of Q defined in $V[G_P]$.

The following are from [K].

1.2 Lemma. *(1) Let $P \lessdot Q$ and G_Q be a Q-generic filter over V. Then $G_P = G_Q \cap P$ is a P-generic filter over V.*

(2) $P \lessdot Q$ iff (i); incompatibility preserved between P and Q and (ii); for any $q \in Q$, q has a reduction $p \in P$.

(3) Let $P \lessdot Q$ and G_P be a P-generic filter over V. Then Q/G_P is a sub-preorder of Q with $1 \in Q/G_P$. And even if Q is separative, it is not clear whether so is Q/G_P.

(4) Let $q \in Q$ and $p \in P$. Then p is a reduction of q iff $p \Vdash_P$ "$q \in Q/\dot{G}_P$". So being a reduction is a witness and this should not be confused with $q \Vdash_Q$ "$p \in \dot{G}_Q \cap P$".

(5) Let $P \lessdot Q$, G_P be a P-generic filter over V and H be a Q/G_P-generic filter over $V[G_P]$. Then H is a Q-generic filter over V with $G_P = H \cap P = (Q/G_P) \cap P$.

(6) Let $P \lessdot Q$ and G_Q be a Q-generic filter over V. Then $G_P = G_Q \cap P$ is a P-generic filter over V and G_Q is a Q/G_P-generic filter over $V[G_P]$.

Proof. For (1): (Directed): Let $p_0, p_1 \in G_P$. Let $D = \{p \in P \mid$ either (p and p_0 are incompatible in P) or (p and p_1 are incompatible in P) or ($p \leq p_0, p_1$ in P)$\}$. Then D is dense in P. Let A be a maximal antichain in D and so A is a maximal antichain in P. Since we assume $P \lessdot Q$, we have $A \cap G_Q \neq \emptyset$. Take $p_3 \in A \cap G_Q$. Since $p_0, p_1 \in G_Q$, we conclude $p_3 \leq p_0, p_1$ in P. Since $p_3 \in G_P$, we are done.

(Upward closed): Let $p_0 \in G_P$ and $p_0 \leq p$ in P. Then $p \in G_Q \cap P = G_P$.

(Dense): Let A be a maximal antichain in P. Then A remains so in Q and so $A \cap G_Q \neq \emptyset$. Hence $A \cap G_P \neq \emptyset$.

For (2): Suppose $P \lessdot Q$. Then since every two element incompatible set can be extended to a maximal one, the preservation of incompatibility is immediate. Let $q \in Q$. We want to find a reduction $p \in P$ of q. Suppse to the contrary every $p \in P$ failed to be a reduction of q. So for any $p \in P$, there is $r \leq p$ in P such that r and q are incompatible in Q. This means $D = \{r \in P \mid r$ and q are incompatible in $Q\}$ is dense in P.

Let A be a maximal antichain in D. Then A is a maximal antichain in P. Hence by assumption, A remains a maximal antichain in Q. So there is some $r \in A$ such that r and q are compatible. But this is contradicts to $q \in D$.

Conversely, suppose incompatibility preserved between P and Q and every $q \in Q$ has a reduction $p \in P$. Take a maximal antichain A in P. We want to show that A remains a maximal antichain in Q. It remains to show the maximality of A in Q. To this end take $q \in Q$ and its reduction $p \in P$. Since A is a maximal antichain in P, we have $r \in A$ such that p and r are compatible in P. Say, $t \leq p, r$ in P. Then since p is a reduction of q, it holds that t and q are compatible in Q. Hence we find $r \in A$ which is compatible with q in Q.

For (3): For any $a \in G_P$, we know $a \leq 1$ in Q. Hence a and 1 are certainly compatible in Q. Hence $1 \in Q/G_P$.

For (4): Suppose $p \in P$ is a reduction of $q \in Q$. Let G_P be a P-generic filter with $p \in G_P$. Take any $a \in G_P$. Since $a, p \in G_P$, we have $b \in G_P$ with $b \leq a, p$ in P. So b and q are compatible in Q. Hence a and q are compatible in Q.

Conversely, suppose $p \Vdash_P$ "$q \in Q/\dot{G}_P$". We want to show p is a reduction of q. To this end take $r \leq p$ in P. Then r and q are compatible in Q. Thus p is a reduction of q.

For (5): (Directed): Since $H \subseteq Q/G_P \subseteq Q$, we know that $H \subseteq Q$ is directed.

(Upward closed): Let $q \in H$ and $q \leq r$ in Q. Then for any $a \in G_P$, a and q are compatible in Q. And so are a and r. Hence $r \in Q/G_P$ and so $r \in H$.

(Dense): Let D be a dense subset of Q in V. We want to show $D \cap H \neq \emptyset$. Let $D' = D \cap Q/G_P$. It suffices to show D' is dense in Q/G_P. To this end let $q \in Q/G_P$. Take $p \in G_P$ with $p \Vdash_P$ "$q \in Q/\dot{G}_P$". Take any $p' \leq p$ in P. Since p is a reduction of q, we know p' and q are compatible in Q. Hence there is $d \in D$ with $d \leq p', q$ in Q. Take a reduction $p_d \in P$ of d. Then p_d and d are compatible in Q. So are p_d and p' in Q and so in P. Take $a' \in P$ with $a' \leq p_d, p'$ in P. Then since $a' \leq p_d$ in P, we have $a' \Vdash_P$ "$d \in Q/\dot{G}_P$" and $d \leq q$ in Q. Since we find a' dense below p, we conclude that there is $d \in D \cap Q/G_P$ with $d \leq q$. Hence $D \cap Q/G_P$ is dense in Q/G_P.

It is clear that $H \cap P \subseteq (Q/G_P) \cap P$. To show $(Q/G_P) \cap P \subseteq G_P$, let $p \in (Q/G_P) \cap P$. Then $p \in Q/G_P$. Hence p is compatible with every element in G_P in Q and so in P. Therefore we have $p \in G_P$. Since $H \cap P \subseteq G_P$, we conclude these two P-generic filters are equal. So these relevant three sets are all equal.

For (6): $G_Q \subseteq Q/G_P$ holds. And so G_Q is a filter in Q/G_P. Let D be a dense subset of Q/G_P. We want to show that $G_Q \cap D \neq \emptyset$. Let $\dot{D}$ be a P-name with $\dot{D}[G_P] = D$. Take $p \in G_P$ such that $p \Vdash_P$ "$\dot{D}$ is dense in $Q/\dot{G}_P$". Let $D' = \{q \in Q \mid \exists a, d \in P$ such that $q \leq a, d$ and $a \Vdash_P$ "$d \in \dot{D}$"$\}$. It suffices to show D' is dense below p in Q. Take any $r \leq p$ with $r \in Q$. Take a reduction $p_r \in P$ of r. Then p_r and p are compatible in P. Take $b \leq p_r, p$ in P. Since $b \Vdash_P$ "$r \in Q/\dot{G}_P$ and $\dot{D}$ is dense in $Q/\dot{G}_P$", we have $a \in P$ and $d \in Q$ such that $a \leq b$ in P, $a \Vdash_P$ "$d \in \dot{D}$" and $d \leq r$ in Q. Since a is a reduction of d, we have $q \in Q$ such that $q \leq a, d$ in Q. Since $q \in D'$ with $q \leq r$ in Q, we are done.

□

§2. **The Hechler Forcing**

We begin by introducing the Hechler Forcing $\mathcal{D}$. This notion of focing is one of those which adds a dominating real.

2.1 Definition. $\mathcal{D} = \{(s, f) \mid s$ is an initial segment of $f \in {}^{\omega}\omega\}$. We put the order $(t, g) \leq (s, f)$, if $t \supseteq s$ and $g \geq f$ pointwise.

We describe a picture behind this definition. Given (s, f), s stands for the initial segment of the Hechler real $\dot{h} : \omega \longrightarrow \omega$ and f is a commitment that $f \leq \dot{h}$ pointwise. $(t, g) \leq (s, f)$ means that we first end-extend s to t while dominating f. Then we cook up a new stronger commitment $g \lceil [l(t), \omega)$, where $l(t)$ denotes the length of t. We set $g = t \cup g\lceil [l(t), \omega)$ and so we would have $g \geq f$ pointwise.

2.2 Lemma. *(1) $\mathcal{D}$ is a p.o. set with the greatest element $(\emptyset, c_0)$, where c_0 denotes the constant function with the only value* 0.

(2) Let $(t,g),(s,f) \in \mathcal{D}$ with $l(t) \geq l(s)$. Then (t,g) and (s,f) are incompatible iff $s \not\subseteq t$ or not ($t \geq f$ on $[l(s), l(t))$). In particular, (t,g) and (s,f) are once incompatible, then so are they in any bigger universe.

(3) $\mathcal{D}$ is separative.

(4) $\mathcal{D}$ has the c.c.c.

Proof. For (1): This is easy by the definition of $\leq$.

For (2): Suppose (t,g) and (s,f) are compatible, then take $(u,h) \leq (t,g),(s,f)$. Since $u \supseteq t, s$, we have $s \subseteq t$. Since $u \geq f$ on $[l(s), l(u))$, we have $t \geq f$ on $[l(s), l(t))$. Conversely, suppose $s \subseteq t$ and $t \geq f$ on $[l(s), l(t))$. Then we have $(t, \mathrm{Max}\{f,g\}) \leq (t,g),(s,f)$.

For (3): Suppose $(t,g) \not\leq (s,f)$. We have two cases.

Case 1. $l(t) \geq l(s)$: Then $s \not\subseteq t$ or not ($g \leq f$ pointwise). If the first alternative holds, then (t,g) and (s,f) are incompatible. So suppose $s \subseteq t$ and the second one holds, then take a sufficiently large integer n so that not ($g \lceil n \geq f$ on $[l(s), n)$). Hence $(g\lceil n, g) \leq (t,g)$ and $(g\lceil n, g)$ and (s,f) are incompatible.

Case 2. $l(t) < l(s)$: Take an integer m so that $s(l(t)), g(l(t)) < m$. Then $(t \cup \{(l(t),m)\}, t \cup \{(l(t),m)\} \cup g[\lceil l(t)+1, \omega)) \leq (t,g)$ and $(t \cup \{(l(t),m)\}, t \cup \{(l(t),m)\} \cup g[\lceil l(t)+1, \omega))$ and (s,f) are incompatible, as $t \cup \{(l(t),m)\} \not\subseteq s$.

For (4): We know $(s, \mathrm{Max}\{f,g\}) \leq (s,f),(s,g)$. Therefore $\mathcal{D}$ satisfies a strong form of c.c.c.

□

We introduce the Hechler real.

2.3 Lemma. *(1) Let G be a $\mathcal{D}$-generic filter over V. Let $h = \bigcup\{s \mid (s,f) \in G\}$. Then $h : \omega \longrightarrow \omega$.*

(2) Let $G(h) = \{(s,f) \in \mathcal{D} \mid s \subset h$ and $f \leq h$ pointwise $\}$. Then $G(h) = G$ holds.

(3) In particular, if $(s,f) \in G$, then we have $s \subset h$ and $f \leq h$ pointwise.

(4) For any $f : \omega \longrightarrow \omega$ in V, we have $f \leq^* h$.

Proof. For (1): By an easy density argument.

For (2): We first observe $G \subseteq G(h)$. Let $(s,f) = p \in G$. Then $s \subset h$. For any integer n, take $(t,g) \in G$ such that $(t,g) \leq p$ and $n \in \mathrm{dom}(t)$. Then $f(n) \leq t(n) = h(n)$. Hence $p \in G(h)$. We then show that $G(h)$ is directed. Let $(s,f),(t,g) \in G(h)$. We may assume that $l(t) \geq l(s)$. We know that $(t, \mathrm{Max}\{f,g\}) \leq (s,f),(t,g)$ and $(t, \mathrm{Max}\{f,g\}) \in G(h)$. Hence $G(h)$ is directed. We lastly observe that $G(h)$ is upward closed in $\mathcal{D}$. Let $(t,g) \in G(h)$ and $(t,g) \leq (s,f)$. Then $s \subseteq t \subset h$ and $f \leq g \leq h$ pointwise. Hence $(s,f) \in G(h)$. Therefore we conclude $G = G(h)$.

For (3): This is immediate from (2).

For (4): Given $(t,g) \in \mathcal{D}$, we have $(t, t \cup (\mathrm{Max}\{f,g\}\lceil[l(t), \omega))) \leq (t,g)$ and so $f \leq^* h$ holds.

□

§3. The Hechler Forcing $\mathcal{D}$ vs. The preorder $P * \dot{\mathcal{D}}$

3.1 Definition. Let P be a notion of forcing. We set $P * \dot{\mathcal{D}} = \{(p,(s,\dot{f})) \mid p \in P$ and $p \Vdash_P$ "$(s,\dot{f}) \in \dot{\mathcal{D}}$"$\}$. Hence we are considering a dense subset of the two stage iteration of P followed by $\mathcal{D}^{V[G_P]}$.

We introduce the *canonical* reduction.

3.2 Lemma. *Let P be a preorder and $\dot{\mathcal{D}}$ denote the Hechler forcing in V^P.*

(1) Let $p \Vdash_P$ "$(s,\dot{f}) \in \dot{\mathcal{D}}$" and prepare any sequence $\langle p_n \mid n < \omega\rangle$ and $g : \omega \longrightarrow \omega$ such that

- $p \geq p_n \geq p_{n+1}$,
- $p_n \Vdash_P$ "$g\lceil n \subset \dot{f}$".

Then (s, g) *is a reduction of* $(p, (s, \dot{f}))$. *More precisely, for any* $(t, k) \leq (s, g)$, *we have* $(1, (t, k))$ *and* $(p, (s, \dot{f}))$ *are compatible in* $P * \dot{\mathcal{D}}$.

(2) And so $\mathcal{D} \simeq \{1\} * \check{\mathcal{D}} = \{(1, (\check{s}, \check{f})) \mid (s, f) \in \mathcal{D}\} \lessdot P * \dot{\mathcal{D}}$. *This should not be confused with* $\mathcal{D}^V \lessdot \mathcal{D}^{V[G_P]}$ *in* $V[G_P]$.

(3) If G_P *is* P*-generic over* V *and* h *is a Hechler real over* $V[G_P]$, *then* $G(h)$ *with respect* $\mathcal{D}^V$ *(i.e.* $G(h)^{\mathcal{D}^V} = \{(s, f) \in \mathcal{D}^V \mid s \subset h$ *and* $h \geq f$ *pointwise* $\}$*) is* $\mathcal{D}^V$*-generic over* V *and so* h *is simultaneously Hechler over both* V *and* $V[G_P]$.

(4) Hence we do have if G *is* $\mathcal{D}^{V[G_P]}$*-generic over* $V[G_P]$ *, then* $G \cap \mathcal{D}^V$ *is* $\mathcal{D}^V$*-generic over* V. *(not over* $V[G_P]$*)*

Proof. For (1): Let $\langle p_n \mid n < \omega \rangle$ and g be as in the statement. Since we simply keep deciding the initial values of $\dot{f}$, there are many choices for them. Let $(t, k) \leq (s, g)$ in $\mathcal{D}$. We want to show $(1, (t, k))$ and $(p, (s, \dot{f}))$ are compatible in $P * \mathcal{D}$. Let $n = l(t)$. Then we have $p_n \Vdash_P$"$g\lceil n \subset \dot{f}$". Hence $(p_n, (t, \text{Max}\{k, \dot{f}\})) \leq (1, (t, k)), (p, (s, \dot{f}))$, as for $i < l(s)$, we have $p_n \Vdash_P$"$k(i) = t(i) = s(i) = g(i) = \dot{f}(i)$" and for i with $l(s) \leq i < n$, we have $p_n \Vdash_P$"$t(i) = k(i) \geq g(i) = \dot{f}(i)$". Hence $p_n \Vdash_P$"$\text{Max}\{k, \dot{f}\} = t$ on n".

For (2): Notice that $P * \dot{\mathcal{D}}$ is dealt by its dense subset. We never go outside of this dense subset. Namely, the first coodinate of the second entry is actually a finite sequence. We make sure that $(1, (s_1, f_1))$ and $(1, (s_2, f_2))$ are incompatible in $\{1\} * \check{\mathcal{D}}$ iff so are they in $P * \dot{\mathcal{D}}$. Suppose $(1, (s_1, f_1))$ and $(1, (s_2, f_2))$ are compatible in $\{1\} * \check{\mathcal{D}}$. Take $(1, (s_3, f_3)) \leq (1, (s_1, f_1)), (1, (s_2, f_2))$. Then $\Vdash_P$"$(s_3, f_3)) \leq (s_1, f_1), (s_2, f_2)$ in $\dot{\mathcal{D}}$". Hence $(1, (s_1, f_1))$ and $(1, (s_2, f_2))$ are compatible in $P * \dot{\mathcal{D}}$.

Conversely, suppose $(1, (s_1, f_1))$ and $(1, (s_2, f_2))$ are compatible in $P * \dot{\mathcal{D}}$. So there is $p \in P$ such that $p \Vdash_P$"(s_1, f_1) and (s_2, f_2) are compatible in $\dot{\mathcal{D}}$". If (s_1, f_1) and (s_2, f_2) were incompatible in $\mathcal{D}$, then they would remain so in $V[G_P]$. Hence (s_1, f_1) and (s_2, f_2) are compatible in $\mathcal{D}$. And so $(1, (s_1, f_1))$ and $(1, (s_2, f_2))$ are compatible in $\{1\} * \check{\mathcal{D}}$.

For (3): $\{(p, (s, \dot{f})) \in P * \dot{\mathcal{D}} \mid p \in G_P,\ s \subset h$ and $h \geq \dot{f}[G_P]$ pointwise $\}$ is $P * \dot{\mathcal{D}}$-generic over V. Hence $\{(s, f) \in \mathcal{D}^V \mid s \subset h$ and $h \geq f$ pointwise $\}$ is $\mathcal{D}^V$-generic over V.

For (4): Let $h = \bigcup\{s \mid (s, f) \in G\}$. Then $G = G(h)$ holds and so $G \cap \mathcal{D}^V = \{(s, f) \in \mathcal{D}^V \mid s \subset h$ and $h \geq f$ pointwise $\}$ which is $\mathcal{D}^V$-generic over V by (3).

□

We repeat the above in a more general setting.

3.3 Lemma. *Let* $P \lessdot Q$. *Then*

(0) If G_Q *is* Q*-generic over* V *and* $\dot{f}$ *is any* P*-name in* V, *then* $G_P = G_Q \cap P$ *is* P*-generic over* V *and* $\dot{f}$ *is a* Q*-name as well. However we have* $\dot{f}[G_Q] = \check{\dot{f}}[G_P] = \dot{f}[G_P]$ *in* $V[G_Q]$, *where* ˇ *is taken with respect to* Q. *Hence there are no real ambiguities. In particular, we have* $P * \dot{\mathcal{D}}^{V[G_P]} \subseteq Q * \dot{\mathcal{D}}^{V[G_Q]}$.

(1) Let $q \in Q$ *and* $p \in P$ *be any reduction of* q. *And let* $q \Vdash_Q$ *"*$(s, \dot{f}) \in \mathcal{D}^{V[G_Q]}$*" and prepare any sequence of* P*-names* $\langle \dot{q}_n \mid n < \omega \rangle$ *and a* P*-name* $\dot{g}$ *such that*

- $p \Vdash_P$ *"*$\dot{q}_n$ *is a descending sequence in the quotient* $Q/\dot{G}_P$ *with* $\dot{q}_0 = q$*". Namely,* $p \Vdash_P$ *"*$\forall a \in \dot{G}_P$, *we have* a *and* $\dot{q}_n$ *are compatible in* Q *and* $\dot{q}_n \geq \dot{q}_{n+1}$ *in* Q*" and* $p \Vdash_P$ *"*$\dot{g} : \omega \longrightarrow \omega$*".
- $\Vdash_Q$ *"If* $p \in G_P$ *and* $\dot{q}_n[G_P] \in \dot{G}_Q$, *then* $\dot{g}[G_P]\lceil n \subset \dot{f}$, *where* $G_P = \dot{G}_Q \cap V$*".

Then $(p, (s, \dot{g})) \in P * \dot{\mathcal{D}}^{V[G_P]}$ *is a reduction of* $(q, (s, \dot{f})) \in Q * \mathcal{D}^{V[G_Q]}$. *Namely, for any* $(r, (t, \dot{k})) \leq (p, (s, \dot{g}))$ *in* $P * \dot{\mathcal{D}}^{V[G_P]}$, *we have that* $(r, (t, \dot{k}))$ *and* $(q, (s, \dot{f}))$ *are compatible in* $Q * \dot{\mathcal{D}}^{V[G_Q]}$.

(2) And so $P * \dot{\mathcal{D}}^{V[G_P]} \lessdot Q * \dot{\mathcal{D}}^{V[G_Q]}$. *This should not be confused with* $\mathcal{D}^{V[G_P]} \lessdot \mathcal{D}^{V[G_Q]}$ *in* $V[G_Q]$.

(3) If G_Q *is* Q*-generic over* V *and* h *is Hechler over* $V[G_Q]$, *then* $G_P = G_Q \cap V$ *is* P*-generic over* V *and* $G(h)$ *with respect to* $\mathcal{D}^{V[G_P]}$ *(i.e.* $G(h)^{V[G_P]} = \{(s, f) \in \mathcal{D}^{V[G_P]} \mid s \subset h$ *and* $h \geq f$ *pointwise* $\}$*) is* $\mathcal{D}^{V[G_P]}$*-generic over* $V[G_P]$. *So* h *is simultaneously Hechler over both* $V[G_P]$ *and* $V[G_Q]$.

(4) If G is $\mathcal{D}^{V[G_Q]}$-generic over $V[G_Q]$, then $G\cap\mathcal{D}^{V[G_P]}$ is $\mathcal{D}^{V[G_P]}$-generic over $V[G_P]$, where $G_P = G_Q\cap P$.

Proof. For (0): Since $P \subseteq Q$, we have $\dot f$ is a Q-name as well. We show by induction on $\in^V$ in $V[G_Q]$ that $\dot f[G_Q] = \dot f[G_P]$. $\dot f[G_Q] = \{\tau[G_Q] \mid \exists p \in G_Q$ such that $(\tau, p) \in \dot f\}$. But by induction, $\tau[G_Q] = \tau[G_P]$ for any $\tau \in \mathrm{dom}(\dot f)$. Hence $\dot f[G_Q] = \{\tau[G_P] \mid \exists p \in G_P$ such that $(\tau, p) \in \dot f\} = \dot f[G_P]$.

For (1): Take $\langle \dot q_n \mid n < \omega\rangle$ and $\dot g$ as in the statement. We give some explanation to this. Since we may simply keep deciding the values of the intial segments of $\dot f$ in Q/G_P in $V[G_P]$ with $p \in G_P$, we may construct a descending sequence $\langle q_n \mid n < \omega\rangle$ in Q/G_P and $g : \omega \longrightarrow \omega$ such that $q_n \Vdash^{V[G_P]}_{Q/G_P}$ "$g\lceil n \subset \dot f[G_P * \dot G_{Q/G_P}]$", where $\dot G_{Q/G_P}$ denotes the canonical P-name for the Q/G_P-generic filter over $V[G_P]$. Now back in V, we may fix P-names for those q_n's and g. Notice that since p is a reduction of q, we certainly have $p \Vdash_P$ "$q \in Q/\dot G_P$". So we may start deciding from $q \in Q/G_P$. Therefore for any Q-generic filter G_Q, if $p \in G_P = G_Q \cap P$ and $q_n = \dot q_n[G_P] \in Q_G$ which is Q/G_P-generic over $V[G_P]$, we have $\dot g[G_P]\lceil n \subset \dot f[G_Q]$.

Suppose $(r, (t, \dot k)) \leq (p, (s, \dot g))$ in $P * \dot{\mathcal{D}}^{V[G_P]}$. Let $n = l(t)$. Since $r \Vdash_P$ "$\dot q_n \in Q/\dot G_P$", we may take $a \leq r$ in P and $d \in Q$ so that $a \Vdash_P$ "$\dot q_n = d$". Since a is a reduction of d, we have $b \leq a, d$ in Q. We have $(b, (t, \mathrm{Max}\{\dot k, \dot f\})) \leq (r, (t, \dot k)), (q, (s, \dot f))$ in $Q * \dot{\mathcal{D}}^{V[G_Q]}$. To see this we need $b \Vdash_Q$ "$(t, \mathrm{Max}\{\dot k, \dot f\}) \leq (t, \dot k), (s, \dot f)$ in $\dot{\mathcal{D}}^{V[\dot G_Q]}$". To this end let G_Q be any Q-generic filter with $b \in G_Q$. Let $G_P = G_Q \cap P$. Since $r, q \in G_Q$, we have $p \in G_P$ and $d = \dot q_n[G_P] \in G_Q$. Hence $g\lceil n = \dot g[G_P]\lceil n \subset f = \dot f[G_Q]$, $(t, k) = (t, \dot k[G_P]) \leq (s, g)$ in $\mathcal{D}^{V[G_P]}$. Then for $i < l(s)$, we have $k(i) = t(i) = s(i) = g(i) = f(i)$. For i with $l(s) \leq i < n$, we have $t(i) = k(i) \geq g(i) = f(i)$. Hence we have $t \subset \mathrm{Max}\{k, f\}$ and so $(t, \mathrm{Max}\{k, f\}) \leq (t, k), (s, f)$ in $\mathcal{D}^{V[G_Q]}$ follows.

For (2): In view of (1), it remains to show that $(p_1, (s_1, \dot g_1))$ and $(p_2, (s_2, \dot g_2))$ are incompatible in $P * \dot{\mathcal{D}}^{V[G_P]}$ iff so they are in $Q * \dot{\mathcal{D}}^{V[G_Q]}$. Suppose $(p_1, (s_1, \dot g_1))$ and $(p_2, (s_2, \dot g_2))$ are compatible in $P * \dot{\mathcal{D}}^{V[G_P]}$. Then we have $(p_3, (s_3, \dot g_3)) \leq (p_1, (s_1, \dot g_1)), (p_2, (s_2, \dot g_2))$ in $P * \dot{\mathcal{D}}^{V[G_P]}$. Then $p_3 \leq p_1, p_2$ in Q as well and $p_3 \Vdash_P$ "$(s_3, \dot g_3) \leq (s_1, \dot g_1), (s_2, \dot g_2)$ in $\mathcal{D}^{V[\dot G_P]}$". Then by absoluteness we have $p_3 \Vdash_Q$ "$(s_3, \dot g_3) \leq (s_1, \dot g_1), (s_2, \dot g_2)$ in $\mathcal{D}^{V[\dot G_Q]}$". Hence $(p_3, (s_3, \dot g_3)) \leq (p_1, (s_1, \dot g_1)), (p_2, (s_2, \dot g_2))$ in $Q * \dot{\mathcal{D}}^{V[G_Q]}$.

Conversely, suppose $(p_3, (s_3, \dot g_3)) \leq (p_1, (s_1, \dot g_1)), (p_2, (s_2, \dot g_2))$ in $Q * \dot{\mathcal{D}}^{V[G_Q]}$. Let G_Q be Q-generic over V with $p_3 \in G_Q$ and set $G_P = G_Q \cap P$. Then $(s_3, \dot g_3[G_Q]) \leq (s_1, \dot g_1[G_P]), (s_2, \dot g_2[G_P])$ with respect to $\mathcal{D}^{V[G_Q]}$ in $V[G_Q]$. Hence $(s_1, \dot g_1[G_P])$ and $(s_2, \dot g_2[G_P])$ are compatible with respect to $\mathcal{D}^{V[G_Q]}$ in $V[G_Q]$. If they were incompatible in $\mathcal{D}^{V[G_P]}$ in $V[G_P]$, they would remain so in $\mathcal{D}^{V[G_Q]}$ in $V[G_Q]$ and this would be a contradiction. Hence $(s_1, \dot g_1[G_P])$ and $(s_2, \dot g_2[G_P])$ are compatible with respect to $\mathcal{D}^{V[G_P]}$ in $V[G_P]$. Say, $(s, k) \leq (s_1, \dot g_1[G_P]), (s_2, \dot g_2[G_P])$. Take $a \leq p_1, p_2$ such that $a \in G_P$ and $a \Vdash_P$ "$(s, \dot k) \leq (s_1, \dot g_1), (s_2, \dot g_2)$". Hence $(a, (s, \dot k)) \leq (p_1, (s_1, \dot g_1)), (p_2, (s_2, \dot g_2))$. We are done.

For (3): We know that $G(h) = \{(s, f) \in \mathcal{D}^{V[G_Q]} \mid s \subset h$ and $h \geq f$ pointwise $\}$ is $\mathcal{D}^{V[G_Q]}$-generic over $V[G_Q]$. Hence $\{(q, (s, \dot f)) \in Q * \dot{\mathcal{D}}^{V[G_Q]} \mid q \in G_Q$ and $(s, \dot f[G_Q]) \in G(h)\}$ is $Q * \dot{\mathcal{D}}^{V[G_Q]}$-generic over V. So $\{(p, (s, \dot f)) \in P * \mathcal{D}^{V[G_P]} \mid p \in G_P,\ s \subset h$ and $h \geq \dot f[G_P]\}$ is $P * \dot{\mathcal{D}}^{V[G_P]}$-generic over V. So $\{(s, f) \in \mathcal{D}^{V[G_P]} \mid s \subset h$ and $h \geq f$ pointwise$\}$ is $\mathcal{D}^{V[G_P]}$-generic over $V[G_P]$.

For (4): We know $G = G(h)^{\mathcal{D}^{V[G_Q]}}$ and $G(h)^{\mathcal{D}^{V[G_P]}} = G(h)^{\mathcal{D}^{V[G_Q]}} \cap \mathcal{D}^{V[G_P]}$ is $\mathcal{D}^{V[G_P]}$-generic over $V[G_P]$. Hence $G \cap \mathcal{D}^{V[G_P]}$ is $\mathcal{D}^{V[G_P]}$-generic over $V[G_P]$.

□

§4. **Introduction of Templates**

4.1 Definition. Let $(L, <)$ be a linear order. A *template* on $(L, <)$ is a collection of subsets of L such that

(1) $\emptyset$, $L \in \mathcal{I}$,

(2) $\mathcal{I}$ is closed under finite ($\neq \emptyset$) intersections and unions,

(3) For all $y \in L$, we have $L_y = \{x \in L \mid x < y\} = \bigcup\{B \in \mathcal{I} \mid B \subseteq L_y\}$,

(4) If $A \in \mathcal{I}$ and $x \in L \setminus A$, then $A \cap L_x \in \mathcal{I}$.

(5) There exists no strictly descending infinite sequence $A_0 \supset A_1 \supset \cdots \supset A_n \supset \cdots$ through $\mathcal{I}$.

□

We state a preview on how we use templates. We understand (1), (5) are for the start, the goal and the process. We take (2) and (3) for granted. We use (4) in an inductive proof to show a strong form of reductions exists.

4.2 Notation. Let $\mathcal{I}$ be a template on $(L, <)$. For any $A \in \mathcal{I}$ and $x \in A$, we write

$$\mathcal{I}_x^A = \{B \in \mathcal{I} \mid B \subseteq L_x \cap A\}.$$

Since $\mathcal{I}$ is well-founded, we have the *depth function* $\langle A \mapsto \ \mathrm{Dp}(A) \mid A \in \mathcal{I}\rangle$ such that each $\mathrm{Dp}(A)$ is an ordinal and if B is a proper subset of A, then $\mathrm{Dp}(B)$ <$\mathrm{Dp}(A)$ holds. Notice that if $x \in A \in \mathcal{I}$ and $B \in \mathcal{I}_x^A$, then $\mathrm{Dp}(B) < \mathrm{Dp}(A)$ holds.

□

4.3 Definition. Let $\mathcal{I}$ be a template on $(L, <)$. We plan to build a collection of preorders $\langle P\lceil A \mid A \in \mathcal{I}\rangle$ by recursion on $\mathrm{Dp}(A)$. Each $P\lceil A$ will consist of finite sequences whose domains are subsets of A. For any $x \in L$ with $A \in \mathcal{I}_x^L$, we write

$$P\lceil A * \mathcal{D}_x^{V[G_A]} = \{p \cup \{(x, (s, \dot{f}))\} \mid p \in P\lceil A,\ p \Vdash_{P\lceil A} \text{“}(\check{s}, \dot{f}) \in \mathcal{D}^{V[G_A]}\text{”}\},$$

where G_A denotes the canonical $P\lceil A$-name for the $P\lceil A$-generic filters over V and $\mathcal{D}^{V[G_A]}$ names the Hechler forcing in the generic extensions $V[G_A]$.

For $\{p \cup \{(x, (s, \dot{f}))\} \in P\lceil A * \mathcal{D}_x^{V[G_A]}$, we may simply write

$$p^\frown\langle(s, \dot{f})\rangle = \{p \cup \{(x, (s, \dot{f}))\}$$

For $q_2 = p_2{}^\frown\langle(s_2, \dot{f}_2)\rangle$, $q_1 = p_1{}^\frown\langle(s_1, \dot{f}_1)\rangle$, we define

$$q_2 \le q_1 \text{ in } P\lceil A * \mathcal{D}_x^{V[G_A]} \text{ iff } p_2 \le p_1 \text{ in } P\lceil A \text{ and } p_2 \Vdash_{P\lceil A} \text{“}(\check{s_2}, \dot{f}_2) \le (\check{s_1}, \dot{f}_1) \text{ in } \mathcal{D}^{V[G_A]}\text{”}$$

□

4.5 Proposition. *$P\lceil A * \mathcal{D}_x^{V[G_A]}$ is dense embeddable into the two stage itertaion $P\lceil A$ followed by the Hechler forcing $\mathcal{D}^{V[G_A]}$.*

□

§5. The Recursive Construction of $P\lceil A$ and The Induction Hypothesis

5.1 Definition. Let $\mathcal{I}$ be a template on $(L, <)$. We construct a collection of preorders $\langle P\lceil A \mid A \in \mathcal{I}\rangle$ by recursion on α for all $\mathrm{Dp}(A) \le \alpha$ such that

(1) $P\lceil A$ is a preorder with the greatest element $\emptyset$. $P\lceil A$ consists of a collection of finite sequences whose domains are finite subsetes of A.

$$P\lceil A = \{\emptyset\} \cup \bigcup\{P\lceil B * \mathcal{D}_x^{V[G_B]} \mid x \in A,\ B \in \mathcal{I}_x^A\},$$

For $p \in P\lceil A$ and $x \in$dom(p), we write $p(x) = (s_x^p, \dot{f}_x^p)$.

We set $q \le p$ in $P\lceil A$, if (1); $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and (2); either (*Type 0*), (*Type 1*) or (*Type 2*) holds exclusively, where

(*Type 0*): $p = \emptyset$,

(*Type 1*): $\exists x \in A \, \exists B \in \mathcal{I}^A_x$ such that $q, p \in P\lceil B * \mathcal{D}^{V[G_B]}_x$ and so $x = \text{Max dom}(q) = \text{Max dom}(p)$ and $q \leq p$ in $P\lceil B * \mathcal{D}^{V[G_B]}_x$,

(*Type 2*): $\exists y \in A \, \exists C \in \mathcal{I}^A_y$ such that $y = \text{Max dom}(q)$, $q\lceil L_y, p \in P\lceil C$ and $q\lceil L_y \leq p$ in $P\lceil C$.

(2) If $C \subseteq B$, then $P\lceil C$ is a complete sub-preorder of $P\lceil B$. Denoted simply as $P\lceil C \lessdot P\lceil B$.

(3) For $C \subseteq B$ and $p \in P\lceil B$, we associate $p_0 = p_0(p, B, C) \in P\lceil C$ such that

- $\text{dom}(p_0) = \text{dom}(p) \cap C$,
- For any $y \in \text{dom}(p_0)$, we have $s^{p_0}_y = s^p_y$,
- p_0 is a reduction of p in $P\lceil C \lessdot P\lceil B$,
- For any $E, F \in \mathcal{I}$ with $C = B \cap E \subseteq B \cup E \subseteq F$, p_0 remains a reduction of p in $P\lceil E \lessdot P\lceil F$. Namely,

$$\text{For any } q_0 \leq p_0 \text{ in } P\lceil E,\ q_0 \text{ and } p \text{ are compatible in } P\lceil F.$$

□

§6. **A Crutial Inductive Preparation with** $P\lceil B * \mathcal{D}^{V[G_B]}_x$ **before Going into** $P\lceil A$

6.1 Lemma. *Let $C, B \in \mathcal{I}$ such that $Dp(B)$, $Dp(C) < \alpha$ and $C \subseteq B$. Then we have not only $P\lceil C \lessdot P\lceil B$ but also $P\lceil C * \mathcal{D}^{V[G_C]} \lessdot P\lceil B * \mathcal{D}^{V[G_B]}$.*

Proof. We inductively assume $P\lceil C \lessdot P\lceil B$. This in turn entails $P\lceil C * \mathcal{D}^{V[G_C]} \lessdot P\lceil B * \mathcal{D}^{V[G_B]}$ by 3.3 Lemma.

□

6.2 Lemma. *Let $x \in L$ and $B, C \in \mathcal{I}^L_x$ such that $Dp(B)$, $Dp(C) < \alpha$ and $C \subseteq B$. For $p \in P\lceil B * \mathcal{D}^{V[G_B]}_x$, we may associate $p_0 = p_0(p, B, C, x) \in P\lceil C * \mathcal{D}^{V[G_C]}_x$ such that*

(1) $x \in dom(p_0) \cap dom(p)$ and $dom(p_0) \cap B = dom(p) \cap C$,

(2) For any $y \in dom(p_0)$, we have $s^{p_0}_y = s^p_y$,

(3) p_0 is a reduction of p in $P\lceil C * \mathcal{D}^{V[G_C]}_x \lessdot P\lceil B * \mathcal{D}^{V[G_B]}_x$,

(4) For any $E, F \in \mathcal{I}^L_x$ such that $Dp(E)$, $Dp(F) < \alpha$ and $C = B \cap E \subseteq B \cup E \subseteq F$, p_0 remains a reduction of p in $P\lceil E * \mathcal{D}^{V[G_E]}_x \lessdot P\lceil F * \mathcal{D}^{V[G_F]}_x$.

Proof. We have several steps.

Step 1: Let $p \in P\lceil B * \mathcal{D}^{V[G_B]}_x$. We wrire $\bar{p} = p\lceil L_x$. So we have $\bar{p} \in P\lceil B$ and

$$p = \bar{p}^\frown \langle (s^p_x, \dot{f}^p_x) \rangle.$$

By induction we have associated a reduction $\bar{p}_0 = p_0(\bar{p}, B, C) \in P\lceil C$ of $p \in P\lceil B$ such that it remains a reduction in $P\lceil E \lessdot P\lceil F$. By applying Lemma, we may fix $\langle \dot{q}_n \mid n < \omega \rangle$ and $\dot{g}$ such that

- $\bar{p}_0 \Vdash_{P\lceil C}$ "$\dot{q}_n$ are descending sequence in P_B/G_C with $\bar{p} = \dot{q}_0$ and $\dot{g} : \omega \longrightarrow \omega$",
- $\Vdash_{P\lceil B}$ "If $\bar{p}_0 \in G_C$ and $\dot{q}_n[G_C] \in G_B$, then $\dot{g}\lceil n = \dot{g}[G_C]\lceil n = \dot{f}^p_x\lceil n$, where $G_C = G_B \cap P\lceil C$",

Then we may show [Let $\bar{p}_0 \in G_C$ and $n = \mid s^p_x \mid$. Take G_B such that $\dot{q}_n[G_C] \in G_B$ with $G_B \cap P\lceil C = G_C$. Then $(\dot{q}_n[G_C] \leq) \bar{p} \in G_B$ and so $s^p_x = \dot{f}^p_x[G_B]\lceil n$. But $\dot{g}[G_C]\lceil n = \dot{f}^p_x[G_B]\lceil n$. So $s^p_x \subset \dot{g}[G_C]$.]

- $\bar{p}_0 \Vdash_{P\lceil C}$ "$s^p_x \subset \dot{g}$".

Let us define

$$p_0 = p_0(p, B, C, x) = \bar{p}_0 {}^\frown \langle (s^p_x, \dot{g}) \rangle \in P\lceil C * \mathcal{D}^{V[G_C]}_x.$$

We claim this p_0 works. But (1), (2) are immediate by definition and (3) implied by (4). So we may concentrate on (4).

Step 2: Let us take

$$q_0 = \bar{q}_0 {}^\frown \langle (s^{q_0}_x, \dot{f}^{q_0}_x) \rangle \leq p_0 \text{ in } P\lceil E * \mathcal{D}^{V[G_E]}_x.$$

And so

$$\bar{q}_0 \leq \bar{p}_0 \text{ in } P\lceil E.$$

Let $m = \mid s^{q_0}_x \mid$ and $\bar{p}^*_0$ be a reduction of $\bar{q}_0$ in $P\lceil C \lessdot P\lceil E$.

Step 3: Since we may show $\bar{p}^*_0$ and $\bar{p}_0$ are compatible in $P\lceil E$ [Let $\bar{p}^*_0 \in G_C$ and $\bar{q}_0 \in G_E$ with $G_C = G_E \cap P\lceil C$. Then $(\bar{q}_0 \leq)\bar{p}_0 \in G_E \cap P\lceil C = G_C$], so are they in $P\lceil C$. Hence wlog, we may assume

$$\bar{p}^*_0 \leq \bar{p}_0 \text{ in } P\lceil C.$$

Step 4: We may further assume that $\bar{p}^*_0$ decides the value of $\dot{q}_m$ to, say, $b \in P\lceil B$. Hence we have

- $\bar{p}^*_0 \Vdash_{P\lceil C}$ "$\dot{q}_m = \check{b} \in P_B/G_C$",
- $\Vdash_{P\lceil B}$ "If $\bar{p}^*_0 \in G_C$ and $\check{b} \in G_B$, then $\dot{g}\lceil m = \dot{g}[G_C]\lceil m = \dot{f}^p_x\lceil m$".

Step 5: Notice that $\bar{p}^*_0$ is a reduction of $b \in P\lceil B$. So we may take $\bar{p}^+ \in P\lceil B$ such that

$$\bar{p}^+ \leq \bar{p}^*_0,\, b \text{ in } P\lceil B.$$

Step 6: Let $\bar{p}^+_0 = p_0(\bar{p}^+, B, C) \in P\lceil C$. Since $\bar{p}^+_0$ is a reduction of $\bar{p}^+$ in $P\lceil C \lessdot P\lceil B$, we may show that $\bar{p}^+_0$ and $\bar{p}^*_0$ are compatible in $P\lceil C$. [Let $\bar{p}^+_0 \in G_C$ and $\bar{p}^+ \in G_B$ with $G_C = G_B \cap P\lceil C$. Then $(\bar{p}^+ \leq)\bar{p}^*_0 \in G_B \cap P\lceil C = G_C$.] So we may fix $a \in P\lceil C$ with

$$a \leq \bar{p}^+_0,\, \bar{p}^*_0 \text{ in } P\lceil C.$$

Step 7: Since $a \leq \bar{p}^*_0$ in $P\lceil C$ and $\bar{p}^*_0$ is a reduction of $\bar{q}_0$, we know a and $\bar{q}_0$ are compatible in $P\lceil E$. So we may fix $\bar{q}^+_0 \in P\lceil E$ such that

$$\bar{q}^+_0 \leq a,\, \bar{q}_0 \text{ in } P\lceil E.$$

Step 8: So we have $\bar{q}^+_0(\leq a) \leq \bar{p}^+_0$ in $P\lceil E$. But $\bar{p}^+_0 = p_0(\bar{p}^+, B, C)$. By applying induction hypothesis, it holds that $\bar{p}^+$ and $\bar{q}^+_0$ are compatible in $P\lceil F$. So we may fix $\bar{q}^+ \in P\lceil F$ such that

$$\bar{q}^+ \leq \bar{p}^+,\, \bar{q}^+_0 \text{ in } P\lceil F.$$

Let us define

$$q = \bar{q}^+ {}^\frown \langle (s^{q_0}_x, \mathrm{Max}\{\dot{f}^{q_0}_x, \dot{f}^p_x\}) \rangle.$$

Then we may show that $q \in P\lceil F * \mathcal{D}^{V[G_F]}_x$ and that

$$q \leq p, q_0 \text{ in } P\lceil F * \mathcal{D}^{V[G_F]}_x.$$

To see these, we observe $\bar{q}^+ \leq \bar{q}^+_0 \leq a \leq \bar{p}^*_0$ and $\bar{q}^+ \leq \bar{p}^+ \leq b$ in $P\lceil F$. Hence we conclude

- $\bar{q}^+ \Vdash_{P\lceil F}$ "$\dot{g}\lceil m = \dot{f}^p_x\lceil m$".

We also have $\bar{q}^+ \leq \bar{q}^+_0 \leq \bar{q}_0$ and so

- $\bar{q}^+ \Vdash_{P\lceil F}$ "$(s_x^{q_0}, \dot{f}_x^{q_0}) \leq (s_x^p, \dot{g})$".

  And so

- $\bar{q}^+ \Vdash_{P\lceil F}$ "$s_x^{q_0} = \mathrm{Max}\{\dot{f}_x^{q_0}, \dot{f}_x^p\}\lceil m$".

  Hence we have $q \in P\lceil F * \mathcal{D}^{V[G_F]}$.

  Since $\bar{q}^+(\leq b) \leq \bar{p}$ and $\bar{q}^+(\leq \bar{q}_0^+) \leq \bar{q}_0$, we conclude

$$q \leq \bar{p}^\frown\langle(s_x^p, \dot{f}_x^p)\rangle,\ \bar{q}_0^\frown\langle(s_x^{q_0}, \dot{f}_x^{q_0})\rangle \quad \text{in } P\lceil F * \mathcal{D}_x^{V[G_F]}.$$

□

§7. Basic Properties of $P\lceil A$ with $\mathrm{Dp}(A) = \alpha$

7.1 Lemma. *$P\lceil A$ is a preorder for all $A \in \mathcal{I}$ with $Dp(A) = \alpha$.*

Proof. (reflexive): Let $p \in P\lceil A$.

Case 0: $p = \emptyset$: $\emptyset$ is a greatest element. So $p \leq p$.

Case 1: $p \in P\lceil B * \mathcal{D}_x^{V[G_B]}$ for some $x \in A$ and $B \in \mathcal{I}_x^A$: Then $p\lceil L_x \in P\lceil B$ and so $p\lceil L_x \leq p\lceil L_x$ in $P\lceil B$. Since $p\lceil L_x \Vdash_{P\lceil B}$ "$(s_x^p, \dot{f}_x^p) \leq (s_x^p, \dot{f}_x^p)$ in $\mathcal{D}^{V[G_B]}$", we have $p \leq p$ in $P\lceil A$.

(transitive): Let $r \leq q \leq p$ in $P\lceil A$.

Case 0: $p = \emptyset$: Then $r \leq p$ in $P\lceil A$.

Case 1: $p \in P\lceil B * \mathcal{D}_x^{V[G_B]}$ for some $x \in A$ and $B \in \mathcal{I}_x^A$: Since $\mathrm{dom}(r) \supseteq \mathrm{dom}(q) \supseteq \mathrm{dom}(p)$, we have $B_1, B_2 \in \mathcal{I}_x^A$ such that $q \leq p$ in $P\lceil B_1 * \mathcal{D}_x^{V[G_{B_1}]}$ and $r \leq q$ in $P\lceil B_2 * \mathcal{D}_x^{V[G_{B_2}]}$. Let $C = B_1 \cup B_2$. Then $C \in \mathcal{I}_x^A$, $q \leq p$ in $P\lceil C * \mathcal{D}_x^{V[G_C]}$ and $r \leq q$ in $P\lceil C * \mathcal{D}_x^{V[G_C]}$. Hence $r \leq p$ in $P\lceil C * \mathcal{D}_x^{V[G_C]}$ and so $r \leq p$ in $P\lceil A$.

□

7.2 Lemma. *Let $x \in A \in \mathcal{I}$ with $Dp(A) = \alpha$ and $B \in \mathcal{I}_x^A$. For any $p = \bar{p}^\frown\langle(s_x^p, \dot{f}_x^p)\rangle \in P\lceil B * \mathcal{D}_x^{V[G_B]}$, we have $\bar{p} \in P\lceil A$ and $p \leq \bar{p}$ in $P\lceil A$.*

Proof. Since $\bar{p} \in P\lceil B$, we have either (1); $\bar{p} = \emptyset$ or (2); There exists $z \in B$ and $C \in \mathcal{I}_z^B$ such that $\bar{p} \in P\lceil C * \mathcal{D}_z^{V[G_C]}$. Since $z \in (B \subset)A$ and $\mathcal{I}_z^B \subset \mathcal{I}_z^A$, we have $\bar{p} \in P\lceil A$ in either case.

Since $x = \mathrm{Max}\ \mathrm{dom}(p) \in A$, $B \in \mathcal{I}_x^A$, $\bar{p} \in P\lceil B$ and $p\lceil L_x = \bar{p} \leq \bar{p}$ in $P\lceil B$, we have $p \leq \bar{p}$ in $P\lceil A$.

□

7.3 Lemma. *Let $D \subset A$ such that $D, A \in \mathcal{I}$ and $Dp(D) < Dp(A) = \alpha$. Then $P\lceil D \subset P\lceil A$.*

Proof. Let $p \in P\lceil D$.

Case 0: $p = \emptyset$: Then $p \in P\lceil A$.

Case 1: $p \in P\lceil E * \mathcal{D}_z^{V[G_E]}$ for some $z \in D$ and $E \in \mathcal{I}_z^D$: Since $z \in A$ and $E \in \mathcal{I}_z^A$, we have $P\lceil E * \mathcal{D}_z^{V[G_E]} \subset P\lceil A$.

□

7.4 Lemma. *Let $D \subset A$ such that $D, A \in \mathcal{I}$ and $Dp(D) < Dp(A) = \alpha$. Then $P\lceil D$ is a sub-preorder of $P\lceil A$. Namely, for $p, q \in P\lceil D$, we have $q \leq p$ in $P\lceil D$ iff $q \leq p$ in $P\lceil A$.*

Proof. Suppose $q \leq p$ in $P\lceil D$.

Case 0: $p = \emptyset$: Then $q \leq p$ in $P\lceil A$.

Case 1: $p \in P\lceil E * \mathcal{D}_x^{V[G_E]}$ for some $x \in D$ and $E \in \mathcal{I}_x^D$: We may assume that $q \leq p$ in $P\lceil E * \mathcal{D}_x^{V[G_E]}$. Since $x \in A$ and $E \in \mathcal{I}_x^A$, we conclude $q \leq p$ in $P\lceil A$.

Case 2: $q\lceil L_y \leq p$ in $P\lceil E$ for $y = \text{Max dom}(q) \in D$ and some $E \in \mathcal{I}_y^D$: Since $y \in A$ and $\mathcal{I}_y^D \subset \mathcal{I}_y^A$, we conclude $q \leq p$ in $P\lceil A$.

Conversely, suppose $q \leq p$ in $P\lceil A$.

Case 0: $p = \emptyset$: Then $q \leq p$ in $P\lceil D$.

Case 1: $q \leq p$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$ for some $x \in A$ and $B \in \mathcal{I}_x^A$: Since $p, q \in P\lceil D$, we have $E_1, E_2 \in \mathcal{I}_x^D$ such that $x \in D$, $p \in P\lceil E_1 * \mathcal{D}_x^{V[G_{E_1}]}$ and $q \in P\lceil E_2 * \mathcal{D}_x^{V[G_{E_2}]}$. Let $E = E_1 \cup E_2$ and $C = B \cup E_1 \cup E_2$. Then $E \in \mathcal{I}_x^D$ and $C \in \mathcal{I}_x^A$. Since $P\lceil B * \mathcal{D}_x^{V[G_B]} \lessdot P\lceil C * \mathcal{D}_x^{V[G_C]}$, we have $q \leq p$ in $P\lceil C * \mathcal{D}_x^{V[G_C]}$. Since $P\lceil E * \mathcal{D}_x^{V[G_E]} \lessdot P\lceil C * \mathcal{D}_x^{V[G_C]}$, we conclude $q \leq p$ in $P\lceil E * \mathcal{D}_x^{V[G_E]}$ and so in $P\lceil D$.

Case 2: $q\lceil L_y \leq p$ in $P\lceil B$ for $y = \text{Max dom}(q) \in A$ and some $B \in \mathcal{I}_x^A$: Let $x = \text{Max dom}(p)$. Take $E_1 \in \mathcal{I}_x^D$ and $E_2 \in \mathcal{I}_y^D$ such that $p \in P\lceil E_1 * \mathcal{D}_x^{V[G_{E_1}]}$ and $q \in P\lceil E_2 * \mathcal{D}_y^{V[G_{E_2}]}$. Since $\text{dom}(p) \subseteq \text{dom}(q) \cap L_y \subseteq E_2$, we have $x \in E_2 \subseteq E_1 \cup E_2$. So $E_1 \in \mathcal{I}_x^{E_1 \cup E_2}$. Since $E_1 \cup E_2 \in \mathcal{I}_y^D$, we have $P\lceil(E_1 \cup E_2)$. Since $p \in P\lceil E_1 * \mathcal{D}_x^{V[G_{E_1}]}$, we have $p \in P\lceil(E_1 \cup E_2)$. Since $B \cup E_1 \cup E_2 \in \mathcal{I}_y^A$, we have $P\lceil(B \cup E_1 \cup E_2)$. Since $P\lceil B \lessdot P\lceil(B \cup E_1 \cup E_2)$, we have $q\lceil L_y \leq p$ in $P\lceil(B \cup E_1 \cup E_2)$. Since $q\lceil L_y, p \in P\lceil(E_1 \cup E_2)$ and $P\lceil(E_1 \cup E_2) \lessdot P\lceil(B \cup E_1 \cup E_2)$, we have $q\lceil L_y \leq p$ in $P\lceil(E_1 \cup E_2)$. Since $E_1 \cup E_2 \in \mathcal{I}_y^D$, we conclude $q \leq p$ in $P\lceil D$.

□

7.5 Lemma. *Let $D \subset A$ such that $D, A \in \mathcal{I}$ and $Dp(D) < Dp(A) = \alpha$. Then for $p, q \in P\lceil D$, we have q and p are incompatible in $P\lceil D$ iff q and p are incompatible in $P\lceil A$.*

Proof. Suppose $r \leq q, p$ in $P\lceil D$, then so they are in $P\lceil A$.

Conversely, suppose $r \leq q, p$ in $P\lceil A$ for some $r \in P\lceil A$. We may assume that $\mid r \mid$ is the least among those r.

Case 0: Either $p = \emptyset$ or $q = \emptyset$: Then it is immediate that p and q are compatible in $P\lceil D$.

Case 1: $p \in P\lceil E_1 * \mathcal{D}_x^{V[G_{E_1}]}$ and $q \in P\lceil E_2 * \mathcal{D}_x^{V[G_{E_2}]}$ for some $x \in D$ and $E_1, E_2 \in \mathcal{I}_x^D$: So $x = \text{Max dom}(p) = \text{Max dom}(q)$. Since $r \leq p$ in $P\lceil A$, we have

Subcase 1: $r \leq p$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$ for some $B \in \mathcal{I}_x^A$: Then by taking a union of two B's in $\mathcal{I}_x^A$, we may assume that $r \leq q$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$ as well. Since

$$P\lceil B * \mathcal{D}_x^{V[G_B]}, P\lceil(E_1 \cup E_2) * \mathcal{D}_x^{V[G_{E_1 \cup E_2}]} \lessdot P\lceil(B \cup E_1 \cup E_2) * \mathcal{D}_x^{V[G_{B \cup E_1 \cup E_2}]},$$

we may conclude p and q are compatible in $P\lceil(E_1 \cup E_2) * \mathcal{D}_x^{V[G_{E_1 \cup E_2}]}$. Hence so they are in $P\lceil D$.

Subcase 2: $r\lceil L_y \leq p$ in $P\lceil B$ for $y = \text{Max dom}(r)$ and some $B \in \mathcal{I}_y^A$: Then by taking a union of two B's in $\mathcal{I}_y^A$, we may assume that $r\lceil L_y \leq q$ in $P\lceil B$ as well. Then $r\lceil L_y \leq q, p$ in $P\lceil A$. This contradicts the least choice of r. Hence this subcase does not occure.

Case 2: $p \in P\lceil E_1 * \mathcal{D}_x^{V[G_{E_1}]}$ and $q \in P\lceil E_2 * \mathcal{D}_y^{V[G_{E_2}]}$ for some $x, y \in D$ with $x < y$ and some E_1 and E_2 such that $E_1 \in \mathcal{I}_x^D$, $E_2 \in \mathcal{I}_y^D$: We may assume that $x \in E_2$ as $\mathcal{I}$ is a template. So $E_1 \in \mathcal{I}_x^{E_1 \cup E_2}$ and $E_1 \cup E_2 \in \mathcal{I}_y^D$. We have $q\lceil L_y, p \in P\lceil(E_1 \cup E_2)$.

Subcase 1: $y = \text{Max dom}(r)$ and $r \leq q$ in $P\lceil C * \mathcal{D}_y^{V[G_C]}$ for some $C \in \mathcal{I}_y^A$: Then we may assume $r\lceil L_y \leq p$ in $P\lceil C$. We have $C \cup E_1 \cup E_2 \in \mathcal{I}_y^A$. Since $P\lceil C, P\lceil(E_1 \cup E_2) \lessdot P\lceil(C \cup E_1 \cup E_2)$, we know $q\lceil L_y$ and p are compatible in $P\lceil(E_1 \cup E_2)$, say, $\bar{r}' \leq q\lceil L_y, p$. Let $r' = \bar{r}' {}^\frown \langle q(y) \rangle$. Then $r' \in P\lceil D$ and $r' \leq q, p$ in $P\lceil D$ holds.

Subcase 2: $y < z = \text{Max dom}(r)$ and $r\lceil L_z \leq q$ in $P\lceil E$ for some $E \in \mathcal{I}_z^A$: We may assume that $r\lceil L_z \leq p$ in $P\lceil E$ as well. So $r\lceil L_z \leq q, p$ in $P\lceil A$. But this contradicts the least choice of r. Hence this case does not occur.

□

§8. **Technical Lemmas**

8.1 Lemma. (*Initial segments are again conditions*) *Let $A \in \mathcal{I}$, $Dp(A) \leq \alpha$, $p \in P\lceil A$ and $x \in dom(p)$. Then there exists $B \in \mathcal{I}_x^A$ such that $p\lceil L_x \in P\lceil B$ and $p\lceil L_x \Vdash_{P\lceil B}$ "$(s_x^p, \dot{f}_x^p) \in \mathcal{D}^{V[G_B]}$".*

Proof. By induction on $|\text{ dom}(p) \,|$. Let $x_0 = \text{Max dom}(p)$. So $x \leq x_0$ in L. There exists $C \in \mathcal{I}_{x_0}^A$ such that $\bar{p} = p\lceil L_{x_0} \in P\lceil C$ and $\bar{p} \Vdash_{P\lceil C}$ "$(s_{x_0}^p, \dot{f}_{x_0}^p) \in \mathcal{D}^{V[G_C]}$".

Case 1: $x = x_0$: We are done.

Case 2: $x < x_0$: Since $x \in \text{dom}(\bar{p})$, we may apply induction hypothesis to this shorter $\bar{p}$. There is $B \in \mathcal{I}_x^C$ such that $\bar{p}\lceil L_x = p\lceil L_x \in P\lceil B$ and $p\lceil L_x \Vdash_{P\lceil B}$ "$(s_x^p, \dot{f}_x^p) \in \mathcal{D}^{V[G_B]}$". Since $\mathcal{I}_x^C \subset \mathcal{I}_x^A$, we are done.

□

8.2 Lemma. (*Intitial segments are weaker than their mother*) *Let $A \in \mathcal{I}$, $Dp(A) \leq \alpha$, $p \in P\lceil A$ and $x \in dom(p)$. Then $p\lceil L_x$, $p\lceil L_x {}^\frown \langle p(x) \rangle \in P\lceil A$ and $p \leq p\lceil L_x {}^\frown \langle p(x) \rangle \leq p\lceil L_x$ in $P\lceil A$.*

Proof. By induction on $|\text{ dom}(p) \,|$. Let $y = \text{Max dom}(x)$. Take $B \in \mathcal{I}_y^A$ such that $p \in P\lceil B * \mathcal{D}_y^{V[G_B]}$.

Case 1: $x = y$: Since $p\lceil L_y \leq p\lceil L_x$ in $P\lceil B$, we have $p = p\lceil L_x {}^\frown \langle p(x) \rangle \leq p\lceil L_x$ in $P\lceil A$.

Case 2: $x < y$: Notice $p\lceil L_y \in P\lceil A$ and $x \in\text{dom}(p\lceil L_y)$. So by induction, $p\lceil L_y \leq (p\lceil L_y)\lceil L_x {}^\frown \langle p(x) \rangle \leq (p\lceil L_y)\lceil L_x$ in $P\lceil A$ and so $p\lceil L_y \leq p\lceil L_x {}^\frown \langle p(x) \rangle \leq p\lceil L_x$ in $P\lceil A$. But $p \leq p\lceil L_y$ in $P\lceil A$. Hence $p \leq p\lceil L_x {}^\frown \langle p(x) \rangle \leq p\lceil L_x$ in $P\lceil A$.

□

8.3 Lemma. (*Tails are tails*) *Let $E \in \mathcal{I}$ with $Dp(E) \leq \alpha$, $q \leq p$ in $P\lceil E$ and $x = Max\ dom(p)$. Then there exists $\bar{E} \in \mathcal{I}_x^E$ such that $q\lceil L_x {}^\frown \langle q(x) \rangle \leq p$ in $P\lceil \bar{E} * \mathcal{D}_x^{V[G_{\bar{E}}]}$.*

Proof. By induction on the length of $q\lceil(x, E)$, where $q\lceil(x, E) = \{(z, q(z)) \mid x < z,\ z \in\text{dom}(q)\}$.

Case 1: $q\lceil(x, E) = \emptyset$: $q\lceil L_x {}^\frown \langle q(x) \rangle = q \leq p$ in $P\lceil E$. So we are done.

Case 2: $q\lceil(x, E) \neq \emptyset$: Let $y = \text{Max dom}(q)$. Then there exists $E_1 \in \mathcal{I}_y^E$ such that $q \in P\lceil E_1 * \mathcal{D}_y^{V[G_{E_1}]}$ and $q\lceil L_y \leq p$ in $P\lceil E_1$. By induction, we have $\bar{E} \in \mathcal{I}_x^{E_1} \subset \mathcal{I}_x^E$ such that $q\lceil L_x {}^\frown \langle q(x) \rangle = (q\lceil L_y)\lceil L_x {}^\frown \langle q(x) \rangle \leq p$ in $P\lceil \bar{E} * \mathcal{D}_x^{V[G_{\bar{E}}]}$.

□

8.4 Lemma. (*Tails are tails in end-extensions*) *Let $E, F \in \mathcal{I}$ such that $E \subseteq F$ end-extension, $Dp(E)$, $Dp(F) \leq \alpha$, $p, \bar{q} \in P\lceil E$, $q \in P\lceil F$, $\bar{q}$ is an initial segment of q and $q \leq p$ in $P\lceil F$. Then $\bar{q} \leq p$ in $P\lceil E$.*

Proof. By induction on the length of the tail $q \setminus \bar{q}$.

Case 1: $q = \bar{q}$: Then $\bar{q} = q \leq p$ in $P\lceil F$. Since $P\lceil E$ is a sub-preorder of $P\lceil F$, we have $\bar{q} = q \leq p$ in $P\lceil E$.

Case 2: $q \setminus \bar{q} \neq \emptyset$: Let $y = \text{Max dom}(q)$ and take $F_1 \in \mathcal{I}_y^F$ such that $q\lceil L_y \leq p$ in $P\lceil F_1$. We may assume that $E \subseteq F_1$ end-extension. By induction we have $\bar{q} \leq p$ in $P\lceil E$.

□

8.5 Lemma. (*Replacing an initail segment by a stronger condition*) *Let $A \in \mathcal{I}$, $Dp(A) \leq \alpha$, $p \in P\lceil A$, $x \in dom(p)$, $B \in \mathcal{I}_x^A$, $p\lceil L_x \Vdash_{P\lceil B}$ "$(s_x^p, \dot{f}_x^p) \in \mathcal{D}^{V[G_B]}$", and $a \leq p\lceil L_x \frown \langle(s_x^p, \dot{f}_x^p)\rangle$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$. Then*

(1) $a \frown p\lceil(x, A) \in P\lceil A$,

(2) $a \frown p\lceil(x, A) \leq p$ in $P\lceil A$,

where $p\lceil(x, A) = \{(s_y^p, \dot{f}_y^p) \mid y \in dom(p)$ and $x < y\}$.

Proof. By induction on $\mid$ dom$(p) \mid$. Let $x_0 =$ Max dom(p). So $x \leq x_0$ in L.

Case 1: $x = x_0$: Then $p\lceil(x, A) = \emptyset$ and $a \leq p$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$ implies $a \leq p$ in $P\lceil A$.

Case 2: $x < x_0$: Then $\bar{p} = p\lceil L_{x_0} \in P\lceil C$ and $\bar{p} \Vdash_{P\lceil C}$"$(s_x^p, \dot{f}_x^p) \in \mathcal{D}^{V[G_C]}$" for some $C \in \mathcal{I}_{x_0}^A$. We may assume that $B \subseteq C$.

We have $C \in \mathcal{I}$, Dp$(C) < \alpha$, $\bar{p} \in P\lceil C$, $x \in$ dom$(\bar{p})$, $B \in \mathcal{I}_x^C$, $\bar{p}\lceil L_x \Vdash_{P\lceil B}$"$(s_x^{\bar{p}}, \dot{f}_x^{\bar{p}}) \in \mathcal{D}^{V[G_B]}$", and $a \leq \bar{p}\lceil L_x \frown \langle(s_x^{\bar{p}}, \dot{f}_x^{\bar{p}})\rangle$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$. Hence by induction $a \frown \bar{p}\lceil(x, C) \in P\lceil C$ and $a \frown \bar{p}\lceil(x, C) \leq \bar{p}$ in $P\lceil C$. Hence $a \frown p\lceil(x, A) \in P\lceil C * \mathcal{D}_x^{V[G_C]} \subset P\lceil A$ and $a \frown p\lceil(x, A) \leq p$ in $P\lceil A$.

□

8.6 Lemma. (*Replacing an initial segment by a stronger condition with an added-value at x*) *Let $F \in \mathcal{I}$ with $Dp(F) \leq \alpha$, $x \in F$, $E \in \mathcal{I}_x^F$, $q \in P\lceil F$, $x \notin dom(q)$, $q\lceil L_x \in P\lceil E$, $r \in P\lceil E * \mathcal{D}_x^{V[G_E]}$ with $r\lceil L_x \leq q\lceil L_x$ in $P\lceil E$. Then $r \frown q\lceil(x, F) \in P\lceil F$ and $r \frown q\lceil(x, F) \leq q$ in $P\lceil F$.*

Proof. By induction on the length of the tail $q \setminus (q\lceil L_x)$.

Case 1: $q = q\lceil L_x$: Then $q\lceil(x, F) = \emptyset$. But $r \leq q$ in $P\lceil F$.

Case 2: $q \setminus (q\lceil L_x) \neq \emptyset$: Let $y =$ Max dom(q). Then $x < y$. Take $F_1 \in \mathcal{I}_y^F$ such that $q \in P\lceil F_1 * \mathcal{D}_y^{V[G_{F_1}]}$. We may assume that $\{x\} \cup E \subseteq F_1$ and so $E \in \mathcal{I}_x^{F_1}$. By induction we have $r \frown q\lceil(x, F_1) \leq q\lceil L_y$ in $P\lceil F_1$. Hence $r \frown q\lceil(x, F) = r \frown q\lceil(x, F_1) \frown \langle q(y)\rangle \leq q$ in $P\lceil F$.

□

§9. Reductions and Complete Sub-preorders of $P\lceil A$

9.1 Lemma. *Let $B, C \in \mathcal{I}$ such that $C \subseteq B$ and $Dp(B)$, $Dp(C) \leq \alpha$. For each $p \in P\lceil B$, we may associate $p_0 = p_0(p, B, C) \in P\lceil C$ such that*

- *$dom(p_0) = dom(p) \cap C$,*
- *For any $y \in dom(p_0)$, we have $s_y^{p_0} = s_y^p$,*
- *p_0 is a reduction of p in $P\lceil C \lessdot P\lceil B$,*
- *For any $E, F \in \mathcal{I}$ such that $C = B \cap E \subseteq B \cup E \subseteq F$ and $Dp(E)$, $Dp(F) \leq \alpha$, p_0 remains a reduction of p in $P\lceil E \lessdot P\lceil F$. Namely,*

For any $q_0 \leq p_0$ in $P\lceil E$, q_0 and p are compatible in $P\lceil F$.

Proof. We have 3 cases.

Case 0: $p = \emptyset$: Let $p_0 = p_0(p, B, C) = \emptyset$.

Since $\emptyset$ is a greatest element in every $P\lceil F$, we are done.

For the remaining two cases, let $p = \bar{p} \frown \langle(s_x^p, \dot{f}_x^p)\rangle \in P\lceil \bar{B} * \mathcal{D}_x^{V[G_B]}$, $x =$ Max dom$(p) \in B$ and $\bar{B} = \bar{B}(p, B) \in \mathcal{I}_x^B$.

Case 1: $x \notin C$: Let $p_0 = p_0(p, B, C,) = p_0(\bar{p}, \bar{B}, \bar{C}) \in P\lceil \bar{C}$, where $\bar{C} = C \cap \bar{B}$.

We observe

$$\operatorname{dom}(p_0) = \operatorname{dom}(\bar{p}) \cap \bar{C} = \operatorname{dom}(\bar{p}) \cap \bar{B} \cap C = \operatorname{dom}(\bar{p}) \cap C = \operatorname{dom}(p) \cap C.$$

We show p_0 remains a reduction in $P\lceil E \lessdot P\lceil F$. To this end, let $q_0 \leq p_0$ in $P\lceil E$. Since $x \notin C$ and $C = B \cap E$, we have $x \notin E$. Since $\mathcal{I}$ is a template, we have

$$\bar{E} = E \cap L_x \in \mathcal{I} \text{ and so } \bar{E} \in \mathcal{I}_x^E.$$

$$\bar{C} = \bar{B} \cap C = \bar{B} \cap B \cap E = \bar{B} \cap L_x \cap E = \bar{B} \cap \bar{E}.$$

Let $\bar{q}_0 = q_0\lceil L_x$. Then

Claim. We have $\bar{q}_0 \leq p_0$ in $P\lceil \bar{E}$.

Proof. Let $y = \operatorname{Max} \operatorname{dom}(\bar{q}_0)$. Then $y \in \operatorname{dom}(q_0) \cap \bar{E}$. By Lemma (Initial segements are conditions), there exists $E_1 \in \mathcal{I}_y^E$ such that $\bar{q}_0 = q_0\lceil L_y \frown \langle q_0(y)\rangle \in P\lceil E_1 * \mathcal{D}_y^{V[G_{E_1}]}$. Since $\mathcal{I}_y^E = \mathcal{I}_y^{\bar{E}}$, we have $\bar{q}_0 \in P\lceil \bar{E}$.

Since $p_0 \in P\lceil \bar{C} \subseteq P\lceil \bar{E}$, we also have $p_0 \in P\lceil \bar{E}$. Since $\bar{E} \subset E$ end-extension, by Lemma (Tails are tails in end-extensions), we conclude $\bar{q}_0 \leq p_0$ in $P\lceil \bar{E}$.

□

Since $p_0 \in P\lceil \bar{E}$ remains a reduction of $\bar{p} \in P\lceil(\bar{B} \cup \bar{E})$, we have $\bar{r} \leq \bar{p}, \bar{q}_0$ in $P\lceil(\bar{B} \cup \bar{E})$. Let $r = \bar{r} \frown \langle p(x)\rangle$. Since $\bar{r} \leq \bar{p}$ in $P\lceil(\bar{B} \cup \bar{E})$, we have $r \in P\lceil(\bar{B} \cup \bar{E}) * \mathcal{D}_x^{V[G_{\bar{B} \cup \bar{E}}]}$ and so $r \leq p$ in $P\lceil F$.

Since $\bar{E} \subset E$ end-extension and $x \notin \operatorname{dom}(q_0)$, we may apply Lemma (Replacing an initial segment by a stronger condition with an added value at x). Hence we have $r \frown q_0\lceil(x, E) \leq q_0$ in $P\lceil E$. We have $r \frown q_0\lceil(x, E) \leq r$ in $P\lceil E$ by Lemma (Initial segments are weaker than their mother). Hence $r \frown q_0\lceil(x, E) \leq q_0, p$ in $P\lceil F$.

Case 2: $x \in C$: Let $p_0 = p_0(p, \bar{B}, \bar{C}, x) \in P\lceil \bar{C}$, where $\bar{C} = \bar{B} \cap C$.

We observe

$$\operatorname{dom}(p_0) = (\operatorname{dom}(\bar{p}) \cap \bar{C}) \cup \{x\} = (\operatorname{dom}(\bar{p}) \cap \bar{B} \cap C) \cup \{x\} = (\operatorname{dom}(\bar{p}) \cap C) \cup \{x\} = \operatorname{dom}(p) \cap C.$$

We show p_0 remains a reduction of p in $P\lceil E \lessdot P\lceil F$. To this end, let $q_0 \leq p_0$ in $P\lceil E$. By Lemma (Tails are tails), we may take $\bar{E} \in \mathcal{I}_x^E$ such that $q_0\lceil L_x \frown \langle q_0(x)\rangle \leq p_0 \in P\lceil \bar{E} * \mathcal{D}_x^{V[G_{\bar{E}}]}$ and so $q_0\lceil L_x \leq p_0\lceil L_x$ in $P\lceil \bar{E}$. Since $\bar{C} \in \mathcal{I}_x^E$, we may assume that $\bar{C} \subseteq \bar{E}$. And so

$$\bar{C} = \bar{B} \cap C = (\bar{B} \cap C) \cap \bar{E} = \bar{B} \cap B \cap E \cap \bar{E} = \bar{B} \cap \bar{E}.$$

Since p_0 remains a reduction of p in $P\lceil \bar{E} * \mathcal{D}_x^{V[G_{\bar{E}}]} \lessdot P\lceil(\bar{E} \cup \bar{B}) * \mathcal{D}_x^{V[G_{\bar{E} \cup \bar{B}}]}$, we have $r \leq p, q_0\lceil L_x \frown \langle q_0(x)\rangle$ in $P\lceil(\bar{B} \cup \bar{E}) * \mathcal{D}_x^{V[G_{\bar{B} \cup \bar{E}}]}$. Hence by Lemma (Replacing an initial segment by a stronger condition), we have $r \frown q_0\lceil(x, F) \leq q_0$ in $P\lceil F$. By Lemma (Initial segments are weaker than thier mother), we have $r \frown q_0\lceil(x, F) \leq r \leq p$ in $P\lceil F$. So q_0 and p are compatible in $P\lceil F$.

□

§10. **The Complete Sub-preorder** $P\lceil B * \mathcal{D}_x^{V[G_B]} \lessdot P\lceil A$

10.1 Lemma. (*Adding a value at x is dense*): *For any $p \in P\lceil A$ and any $x \in A$, there is $q \in P\lceil A$ such that $q \leq p$ in $P\lceil A$ and $x \in$ dom(q).*

Proof. By induction on the length of $p\lceil(x, A)$.

Case 0: $p = \emptyset$: Take any$q \in P\lceil \emptyset * \mathcal{D}_x^{V[G_\emptyset]}$. Then $q \le p$ in $P\lceil A$.

Case 1: $p\lceil(x, A) = \emptyset$: Let $z =$ Max dom$(p\lceil L_x)$. Then $z < x$. We may assume $p = p\lceil L_x$. Then by Lemma (initial segments are conditions), there is $B \in I_z^A$ such that $p \in P\lceil B * \mathcal{D}_z^{V[G_B]}$. Take $C \in \mathcal{I}_x^A$ with $\{z\} \cup B \subseteq C$. Then $p \in P\lceil C$. Take any $q \in P\lceil C * \mathcal{D}_x^{V[G_C]}$ with $q \le p$ in $P\lceil A$.

Case 2: $p\lceil(x, A) \neq \emptyset$: Let $y =$ Max dom(p). Then $x < y$. Take $C \in \mathcal{I}_y^A$ such that $p \in P\lceil C * \mathcal{D}_y^{V[G_C]}$. We may assume $x \in C$. By induction, we have $\bar{q} \le p\lceil L_y$ in $P\lceil C$ with $x \in \mathrm{dom}(\bar{q})$. Then $q = \bar{q}^\frown\langle p(y)\rangle \le p$ in $P\lceil C * \mathcal{D}_y^{V[G_C]}$. Hence $q \le p$ in $P\lceil A$.

□

10.2 Lemma. *Let $A \in \mathcal{I}$, $x \in A$, $B \in \mathcal{I}_x^A$. Then $P\lceil B * \mathcal{D}_x^{V[G_B]} \lessdot P\lceil A$ holds.*

Proof. We have several steps.

(subset): We know $P\lceil B * \mathcal{D}_x^{V[G_B]} \subset P\lceil A$.

(sub-preorder): Let $p_1, p_2 \in P\lceil B * \mathcal{D}_x^{V[G_B]}$. Let us write $p_1 = \bar{p}_1{}^\frown\langle s_1, \dot{f}_1)\rangle$ and $p_2 = \bar{p}_2{}^\frown\langle s_2, \dot{f}_2)\rangle$. We want to show $p_2 \le p_1$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$ iff $p_2 \le p_1$ in $P\lceil A$. Suppose $p_2 \le p_1$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$. Then it is immediate that $p_2 \le p_1$ in $P\lceil A$. Conversely suppose $p_2 \le p_1$ in $P\lceil A$. Then there exists $C \in \mathcal{I}_x^A$ such that $p_2 \le p_1$ in $P\lceil C * \mathcal{D}_x^{V[G_C]}$. Since

$$P\lceil B * \mathcal{D}_x^{V[G_B]}, P\lceil C * \mathcal{D}_x^{V[G_C]} \lessdot P\lceil(B \cup C) * \mathcal{D}_x^{V[G_{B\cup C}]},$$

we have $p_2 \le p_1$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$.

(incompatibilites): We want to show p_2 and p_1 are incompatible in $P\lceil B * \mathcal{D}_x^{V[G_B]}$ iff p_2 and p_1 are incompatible in $P\lceil A$. Suppose p_2 and p_1 are compatible in $P\lceil B * \mathcal{D}_x^{V[G_B]}$, say $p_3 \le p_1, p_2$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$. Then so they are in $P\lceil A$. Conversely, suppose p_2 and p_1 are compatible in $P\lceil A$, say $p \le p_1, p_2$ in $P\lceil A$. And so $p \in P\lceil A$. But by Lemma (Tails are tails), we know $p\lceil L_x{}^\frown\langle p(x)\rangle \le p_1, p_2$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$.

(reductions): For any $p \in P\lceil A$, there is a reduction $p_0 \in P\lceil B * \mathcal{D}_x^{V[G_B]}$ of p. Namely, for any $q_0 \le p_0$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$, q_0 and p are compatible in $P\lceil A$.

Case 1: $x \in \mathrm{dom}(p)$: By Lemma (initial segments are conditions), we may take $\bar{A} \in \mathcal{I}_x^A$ such that $p\lceil L_x{}^\frown\langle p(x)\rangle \in P\lceil \bar{A} * \mathcal{D}_x^{V[G_{\bar{A}}]}$. We may assume that $B \subseteq \bar{A}$. Let $p_0 = p_0(p\lceil L_x{}^\frown\langle p(x)\rangle, \bar{A}, B, x) \in P\lceil B * \mathcal{D}_x^{V[G_B]}$. Then this p_0 is a reduction of $p\lceil L_x{}^\frown\langle p(x)\rangle$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}P \lessdot P\lceil \bar{A} * \mathcal{D}_x^{V[G_{\bar{A}}]}$. Hence q_0 and $p\lceil L_x{}^\frown\langle p(x)\rangle$ are compatible in $P\lceil \bar{A} * \mathcal{D}_x^{V[G_{\bar{A}}]}$. Let $\bar{r} \le q_0, p\lceil L_x{}^\frown\langle p(x)\rangle$ in $P\lceil \bar{A} * \mathcal{D}_x^{V[G_{\bar{A}}]}$. Then by Lemma (Replacing an initial segment by a stronger condition), we have $\bar{r}^\frown p\lceil(x, A) \le p$ in $P\lceil A$. By Lemma (initial segments are weaker than mother), we have $\bar{r}^\frown p\lceil(x, A) \le \bar{r}$ in $P\lceil A$ and so q_0 and p are compatible in $P\lceil A$.

Case 2: $x \notin \mathrm{dom}(p)$: Take any extension $p' \le p$ in $P\lceil A$ with $x \in \mathrm{dom}(p')$. Then by Case 1, we have a reduction p_0 of p'. Since $p' \le p$ in $P\lceil A$, p_0 is a reduction of p as well.

□

§11. Hechler Reals via $P\lceil L$

11.1 Definition. Let G_L be a $P\lceil L$-generic filter over V. For $x \in L$, we define

$$h_x = \bigcup\{s_x^p \mid p \in G_L,\ x \in \ \mathrm{dom}(p)\}.$$

□

11.2 Lemma. *Let $G_A = G_L \cap (P\lceil A)$ and $G_{B^\frown\langle x\rangle} = G_L \cap (P\lceil B * \mathcal{D}_x^{V[G_B]})$ for all $A \in \mathcal{I}$, $x \in A$ and $B \in \mathcal{I}_x^A$. Then G_A is a $P\lceil A$-generic filter over V and $G_{B^\frown\langle x\rangle}$ is a $P\lceil B * \mathcal{D}_x^{V[G_B]}$-generic filter over V.*

Proof. We know $P\lceil B * \mathcal{D}_x^{V[G_B]} \lessdot P\lceil A \lessdot P\lceil L$.

□

11.3 Lemma. *(1)* $h_x : \omega \longrightarrow \omega$,

(2) $h_x = \bigcup\{s_x^p \mid p \in G_A,\ x \in dom(p)\}$ *for any* $A \in \mathcal{I}$ *with* $x \in A$,

(3) $h_x = \bigcup\{s_x^p \mid p \in G_{B\frown\langle x\rangle}\}$ *for any* $B \in \mathcal{I}_x^L$,

(4) h_x *is simultaneously Hechler over all* $V[G_B]$ *with* $B \in \mathcal{I}_x^L$. *By this we mean that* $G(h_x)^{\mathcal{D}^{V[G_B]}} = \{(s,f) \in \mathcal{D}^{V[G_B]} \mid\ s \subset h_x$ *and* $h_x \geq f$ *pointwise* $\}$ *is a* $\mathcal{D}^{V[G_B]}$*-generic filter over* $V[G_B]$,

(5) For $x < y$ *in* L, *we have* $h_x(n) < h_y(n)$ *for all but finitely many* $n < \omega$.

Proof. For(1) and (2): By a density argument. For $A \in \mathcal{I}$, $x \in A$ and $n < \omega$, let $D_{Axn} = \{q \in P\lceil A \mid x \in \mathrm{dom}(q)$ and s_x^q is longer than $n\}$. We may show D_{Axn} is a dense subset of $P\lceil A$ by induction on $\mathrm{Dp}(A)$. Hence both h_x and $\bigcup\{s_x^p \mid p \in G_A,\ x \in \mathrm{dom}(p)\}$ are functions from ω into ω. By the definition of h_x, we have $h_x \supseteq \bigcup\{s_x^p \mid p \in G_A,\ x \in \mathrm{dom}(p)\}$. Hence they are equal.

For (3): By a density argument. For $n < \omega$, we may show $\{q \in P\lceil B * \mathcal{D}_x^{V[G_B]} \mid\ s_x^q$ is longer than $n\}$ is dense in $P\lceil B * \mathcal{D}_x^{V[G_B]}$. Hence $\bigcup\{s_x^p \mid p \in G_{B\frown\langle x\rangle}\}$ is a function from ω into ω. By the definition of h_x, we have $h_x \supseteq \bigcup\{s_x^p \mid p \in G_{B\frown\langle x\rangle}\}$. Hence they are equal.

For (4): $P\lceil B * \mathcal{I}_x^{V[G_B]}$ is a dense subset of two stage iteration $P\lceil B$ followed by the Hechler forcing $\mathcal{D}^{V[G_B]}$. Since $\bigcup\{s_x^p \mid p \in G_{B\frown\langle x\rangle}\}$ is a Hechler real over $V[G_B]$, we conclude so is h_x by (3).

For (5): Let $x < y$. Take any $A \in \mathcal{I}_y^L$ with $x \in A$. This is possible since $\mathcal{I}$ is a template. Then $h_x \in V[G_A]$ by (2). But by (3), h_y is Hechler over $V[G_A]$. Hence we are done.

□

§12. **Identifying What the** $P\lceil A$**'s add**

12.1 Lemma. *Let* $x \in L$, $B \in \mathcal{I}_x^L$ *and* $p \in P\lceil B * \mathcal{D}_x^{V[G_B]}$. *The following are equivalent.*

- $p \in G_{B\frown\langle x\rangle}$,
- $p\lceil L_x \in G_B$, $s_x^p \subset h_x$ *and* $h_x \geq \dot{f}_x^p[G_B]$ *pointwise.*

Proof. Suppose $p \in G_{B\frown\langle x\rangle}$. Then $p \leq p\lceil L_x$ in $P\lceil L$. Hence $p\lceil L_x \in G_L \cap (P\lceil B) = G_B$. Since h_x is Hechler over $V[G_B]$ and $p = p\lceil L_x \frown\langle(s_x^p, \dot{f}_x^p)\rangle \in G_{B\frown\langle x\rangle}$, we have $s_x^p \subset h_x$ and $h_x \geq \dot{f}_x^p[G_B]$ pointwise.

Conversely, suppose $p\lceil L_x \in G_B$, $s_x^p \subset h_x$ and $h_x \geq \dot{f}_x^p[G_B]$ pointwise. Since $h_x \in V[G_{B\frown\langle x\rangle}]$, this statement holds in $V[G_{B\frown\langle x\rangle}]$. Take $q \in G_{B\frown\langle x\rangle}$ such that $q \Vdash_{P\lceil B * \mathcal{D}_x^{V[G_B]}}$ "$p\lceil L_x \in G_B$, $s_x^p \subset \dot{h}_x$ and $\dot{h}_x \geq \dot{f}_x^p[G_B]$ pointwise". We may assume $q\lceil L_x \leq p\lceil L_x$ and s_x^q is at least as long as s_x^p. Since $q \Vdash_{P\lceil B * \mathcal{D}_x^{V[G_B]}}$ "$s_x^p, s_x^q \subset \dot{h}_x$", we have $q\lceil L_x \Vdash_{P\lceil B}$ "$s_x^p \subseteq s_x^q$". We claim $q \leq p$ in $P\lceil B * \mathcal{D}_x^{V[G_B]}$ and so $p \in G_{B\frown\langle x\rangle}$. To show the claim, we want $q\lceil L_x \Vdash_{P\lceil B}$ "$\dot{f}_x^q \geq \dot{f}_x^p$ pointwise". Let $a \leq q\lceil L_x$, $\mid s_x^p \mid \leq n < \omega$ and $m < \omega$ be sufficiently large, say, $\mid s_x^q \mid, n < m < \omega$. We may assume a decides the value of $\dot{f}_x^q\lceil m$, say, $a \Vdash_{P\lceil B}$ "$\dot{f}_x^q\lceil m = t$ and so $t \supset s_x^q$". Then $a\frown\langle(t, \dot{f}_x^q)\rangle \leq q$ and so $a\frown\langle(t, \dot{f}_x^q)\rangle \Vdash_{P\lceil B * \mathcal{D}_x^{V[G_B]}}$ "$\dot{f}_x^q[G_B]\lceil m = t = \dot{h}_x\lceil m \geq \dot{f}_x^p[G_B]\lceil m$". Hence $a \Vdash_{P\lceil B}$ "$\dot{f}_x^q(n) \geq \dot{f}_x^p(n)$" and so $q\lceil L_x \Vdash_{P\lceil B}$ "$\dot{f}_x^q \geq \dot{f}_x^p$ pointwise".

□

12.2 Lemma. *For all* $A \in \mathcal{I}$, *we have* $G_A = \bigcup\{G_{B\frown\langle x\rangle} \mid x \in A,\ B \in \mathcal{I}_x^A\}$.

Proof. Let $p \in G_A$. Then there is $x \in A$ and $B \in \mathcal{I}_x^A$ such that $p \in P\lceil B * \mathcal{D}_x^{B[G_B]}$. Hence $p \in G_L \cap (P\lceil B * \mathcal{D}_x^{B[G_B]}) = G_{B\frown\langle x\rangle}$. Conversely, $G_{B\frown\langle x\rangle} = G_L \cap (P\lceil B * \mathcal{D}_x^{B[G_B]}) \subseteq G_L \cap (P\lceil A) = G_A$.

□

12.3 Lemma. *Let $A \in \mathcal{I}$ and $p \in P\lceil A$. The following are equivalent.*

- $p \in G_A$,
- *For all $x \in dom(p)$, there exist $B \in \mathcal{I}_x^A$ such that $p\lceil L_x \in G_B$, $s_x^p \subset h_x$ and $h_x \geq \dot{f}_x^p[G_B]$ pointwise.*

Proof. Suppose $p \in G_A$. Then for any $x \in \mathrm{dom}(p)$, there is $B \in \mathcal{I}_x^A$ such that $p \leq p\lceil L_x{}^\frown\langle p(x)\rangle$ in $P\lceil A$. Hence $p\lceil L_x{}^\frown\langle p(x)\rangle \in G_A \cap (P\lceil B * \mathcal{D}_x^{V[G_B]}) = (G_L \cap P\lceil A) \cap (P\lceil B * \mathcal{D}_x^{V[G_B]}) = G_L \cap (P\lceil B * \mathcal{D}_x^{V[G_B]}) = G_{B^\frown\langle x\rangle}$. Therefore $p\lceil L_x \in G_B$, $s_x^p \subset h_x$ and $h_x \geq \dot{f}_x^p[G_B]$ pointwise.

Conversely, for all $x \in \mathrm{dom}(p)$, there exist $B \in \mathcal{I}_x^A$ such that $p\lceil L_x{}^\frown\langle p(x)\rangle \in G_{B^\frown\langle x\rangle}$. If $p = \emptyset$, then $p \in G_A$ trivially. If $p \neq \emptyset$, then at the largest $x \in \mathrm{dom}(p)$, we have $p = p\lceil L_x{}^\frown\langle p(x)\rangle \in G_{B^\frown\langle x\rangle} \subseteq G_A$.

□

12.4 Lemma. *For all $A \in \mathcal{I}$, we have $V[G_A] = V[\langle h_x \mid x \in A\rangle]$.*

Proof. It suffices to show $G_A \in V[\langle h_x \mid x \in A\rangle]$ by induction on $\mathrm{Dp}(A)$ in $V[G_L]$. We have seen that G_A is definable in terms of $\langle G_B \mid x \in A,\ B \in \mathcal{I}_x^A\rangle$ and $\langle h_x \mid x \in A\rangle$.

In $V[\langle h_x \mid x \in A\rangle]$, we define $\langle G(B) \mid x \in A,\ B \in \mathcal{I}_x^A\rangle$ by recursion on $\mathrm{Dp}(B)$ such that

$G(B)$ defined as G_B is in terms of $\langle G(C) \mid z \in B,\ C \in \mathcal{I}_z^B\rangle$ and $\langle h_z \mid z \in B\rangle$ when each $G(C)$ is a $P\lceil C$-generic filter over V. Otherwise, say, $G(B) = \emptyset$.

By induction on $\mathrm{Dp}(B)$, we may show $G(B) = G_B$. Hence $\langle G_B \mid x \in A,\ B \in \mathcal{I}_x^A\rangle \in V[\langle h_x \mid x \in A\rangle]$. Therefore $G_A \in V[\langle h_x \mid x \in A\rangle]$.

□

12.5 Lemma. *For any $x \in L$, h_x is simultaneously Hechler over $V[\langle h_z \mid z \in A\rangle]$ for all $A \in \mathcal{I}_x^L$.*

Proof. Immediate by previous Lemmas.

□

Note. In general we do not have h_x is Hechler over $V[\langle h_z \mid z < x\rangle]$. But we do have $\{h_z \mid z < x\} \subseteq \bigcup\{V[\langle h_z \mid z \in A\rangle] \mid A \in \mathcal{I}_x^L\}$.

References

[B]: J. Brendle, *Mad families and iteration theory*, Preprint, October 24, 2001.

[F]: S. Fuchino, A series of lectures, Set theory seminar, Nagoya University, October through December, 2001.

[K]: K. Kunen, *Set Theory, An Introduction to Independence Proofs*, studies in logic and the foundations of mathematics, Vol. 102, North-Holland, 1980.

[S]: S. Shelah, A lecture on forcing Construction, Godels Paradise with Saharon Shelah, Kobe, End of August through early September, 2000.

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