Löb's axiom and cut-elimination theorem[1]

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Abstract

We consider Löb's axiom in modal logics. By adding it to the smallest normal modal logic **K**, we obtain provability logic **GL**, which is complete for the formal provability interpretation in Peano arithmetic **PA** (see Solovay [Sol76]). So, **GL** and Löb's axiom has been considered as one of the most important modal logics and axioms.

A cut-elimination theorem for **GL** was proved in Valentini [Val83]. Valentini uses an induction on *degree*, *rank* and *width*. The first two parameters are used in the standard proof of cut-elimination theorem presented in Gentzen [Gen35], but the proof for **GL** needs the third one *width*. The theorem was also proved semantically in Avron [Avr84]. Avron proved it by using completeness of **GL**. However, the completeness cannot be obtained by the standard method, i.e., the canonical model. Here we can see the difficulty to deal with **GL**.

The normal modal logic **K4** is a sublogic of **GL**, which is obtained from **K** by adding the transitivity axiom. **K4** is much easier to deal with than **GL**. A cut-elimination theorem and completeness for **K4** are given by the standard method mentioned above.

GL is also obtained by adding Löb's axiom to **K4**. So, the knowledge of **K4** is useful for the discussion of **GL**. In this paper, we give another proof of the cut-elimination theorem for **GL** using a cut-free system for **K4** and a property of Löb's axiom.

1 Introduction

We use lower case Latin letters for propositional variables. Formulas are defined, as usual, from the propositional variables and the logical constant $\perp$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction) $\supset$ (implication) and $\Box$ (necessity). We use upper case Latin letters, possibly with suffixes, for formulas. We use Greek letters for finite sets of formulas. By $\Box\Gamma$, we mean the set $\{\Box A \mid A \in \Gamma\}$.

Definition 1.1. The degree $d(A)$ of a formula A is defined inductively as follows:

(1) $d(p) = 1$, for each propositional variable p,
(2) $d(\perp) = 1$,
(3) $d(A \wedge B) = d(A \vee B) = d(A \supset B) = d(A) + d(B) + 1$,
(4) $d(\Box A) = d(A) + 1$.

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By **K4**, we mean the smallest set of formulas containing all the tautologies and axioms

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \text{ and } \Box p \supset \Box\Box p$$

and closed under modus ponens, substitution and necessitation, i.e., $A \in \mathbf{K4}$ implies $\Box A \in \mathbf{K4}$. By **GL**, we mean the smallest set of formulas containing all the theorems in **K4** and Löb's axiom

$$L(p) = \Box(\Box p \supset p) \supset \Box p.$$

and closed under modus ponens, substitution and necessitation.

By a sequent, we mean the expression

$$\Gamma \to \Delta.$$

For brevity's sake, we write

$$A_1, \cdots, A_k, \Gamma_1, \cdots, \Gamma_\ell \to \Delta_1, \cdots, \Delta_m, B_1, \cdots, B_n$$

instead of

$$\{A_1, \cdots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma^\ell \to \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \cdots, B_n\}.$$

By $\mathsf{Sub}(\Gamma \to \Delta)$, we mean the set of subformulas of each formula in $\Gamma \cup \Delta$.

The sequent style system **GK4** for **K4** is defined from the following axioms and rules in the usual way.

Axioms of GK4

$$A \to A$$

$$\bot \to$$

Inference rules of GK4

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(T \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to T)$$

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi - \{A\} \to \Delta - \{A\}, \Lambda}(\text{cut})$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}(\supset\to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to\supset)$$

$$\frac{\Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\Box_{K4})$$

The following two lemmas can be proved in the usual way.

Lemma 1.2. $\rightarrow A \in \mathbf{GK4}$ *if and only if* $A \in \mathbf{K4}$.

Lemma 1.3. *If* $\Gamma \rightarrow \Delta \in \mathbf{GK4}$*, then there exists a cut-free proof figure for* $\Gamma \rightarrow \Delta$ *in* $\mathbf{GK4}$.

The sequent style system **GGL** for **GL** is the system obtained from **GK4** by replacing $(\Box_{K4})$ by the following inference rule:

$$\frac{\Box A, \Gamma, \Box\Gamma \rightarrow A}{\Box\Gamma \rightarrow \Box A}(\Box_{GL}).$$

The following lemma can also be proved in the usual way.

Lemma 1.4. $\rightarrow A \in \mathbf{GGL}$ *if and only if* $A \in \mathbf{GL}$.

Definition 1.5. A subfigure of a proof figure P is defined as follows:
(1) P is a subfigure of P,
(2) if $P = \dfrac{P_1}{\Gamma \rightarrow \Delta}$, then each subfigure of P_1 is a subfigure of P,
(3) if $P = \dfrac{P_1 \quad P_2}{\Gamma \rightarrow \Delta}$, then each subfigure of P_1 or P_2 is a subfigure of P.

A proof figure Q is called a proper subfigure of a proof figure P if Q is a subfigure of P and $Q \neq P$.

2 A property of Löb's axiom

In this section, we show a property of Löb's axiom.

Definition 2.1. The expression $\Box^n A$ is defined inductively as follows:
(1) $\Box^0 A = A$,
(2) $\Box^{k+1} A = \Box(\Box^k A)$.

The following lemma is important for the proof of cut-elimination theorem.

Lemma 2.2. $\Box^n L(A) \rightarrow L(A) \in \mathbf{GK4}$, for any $n \geq 0$.

Proof. By an induction on k, we can show

$$\Box^{k+1} L(A) \rightarrow \Box^k L(A) \in \mathbf{GK4}$$

for any $k \geq 0$. Using cut, possibly several times, we obtain the lemma. $\dashv$

Corollary 2.3. *For any $n \geq 0$,*

$$\Gamma \to \Delta \in \mathbf{GGL} \text{ if and only if } \Gamma \to \Delta \in \mathbf{GK4} + \Box^n L(p),$$

where $\mathbf{GK4} + \Box^n L(p)$ is the system obtained by adding $\to \Box^n L(A)$ to $\mathbf{GK4}$ as an axiom.

Lemma 2.4. *Let P be a proof figure for $\Gamma \to \Delta$ in $\mathbf{GK4} + \Box^{n+1} L(p)$. Then there exist formulas $A_1, \cdots, A_m$ such that*

$$\Box^{n+1} L(A_1), \cdots, \Box^{n+1} L(A_m), \Gamma \to \Delta \in \mathbf{GK4}.$$

Proof. We use an induction on the number $\#(P)$ of axioms of the form $\to \Box^{n+1} L(A)$ in P. If $\#(P) = 0$, then P is a proof figure for $\Gamma \to \Delta$ in $\mathbf{GK4}$. Suppose that $\#(P) > 0$ and the lemma holds for any P^* such that $\#(P^*) < \#(P)$. Then there exists an axiom $\to \Box^{n+1} L(A_1)$ in P for some A_1. For a subfigure Q of P, we define $h(Q)$ as follows:

(1) $h(A \to A) = \dfrac{A \to A}{\Box^{n+1} L(A_1), A \to A}$,

(2) $h(\bot \to) = \dfrac{\bot \to}{\Box^{n+1} L(A_1), \bot \to}$,

(3) $h(\to \Box^{n+1} L(A)) = \dfrac{\to \Box^{n+1} L(A)}{\Box^{n+1} L(A_1) \to \Box^{n+1} L(A)}$, where $A \neq A_1$,

(4) $h(\to \Box^{n+1} L(A_1)) = \Box^{n+1} L(A_1) \to \Box^{n+1} L(A_1)$,

(5) $h(\dfrac{P_1 \quad \cdots \quad P_k}{\Gamma \to \Delta}) = \dfrac{h(P_1) \quad \cdots \quad h(P_k)}{\Box^{n+1} L(A_1), \Gamma \to \Delta}$ if the inference rule that introduces $\Gamma \to \Delta$ is not $(\Box_{K4})$,

(6) $h(\dfrac{P_1}{\Box\Gamma \to \Box A}) = \dfrac{\dfrac{\dfrac{h(P_1)}{\text{using } (T \to), \text{ possibly several times}}}{L(A_1), \Gamma, \Box^{n+1} L(A_1), \Box\Gamma \to A}}{\Box^{n+1} L(A_1), \Box\Gamma \to \Box A}$ if the inference rule that introduces $\Box\Gamma \to \Box A$ is $(\Box_{K4})$.

Note that $h(P)$ is a proof figure for

$$\Box^{n+1} L(A_1), \Gamma \to \Delta$$

satisfying $\#(h(P)) < \#(P)$. Using the induction hypothesis, we obtain the lemma. $\dashv$

3 Cut-elimination

In this section, we prove the following theorem using the cut-free system $\mathbf{GK4}$ and Lemma 2.4.

Lemma 3.1. *If* $\Gamma \to \Delta \in \mathbf{GGL}$*, then there exists a cut-free proof figure for* $\Gamma \to \Delta$ *in* $\mathbf{GGL}$*.*

To prove the theorem above, we provide some preparations.

Definition 3.2. By $\mathbf{GGL}^*$, we mean the system obtained from $\mathbf{GGL}$ by adding the inference rule $(\Box_{K4})$ in $\mathbf{GK4}$.

Definition 3.3. Let P be a cut-free proof figure in $\mathbf{GGL}^*$. We define $dep_\Box(P)$ as follows:
(1) $dep_\Box(A \to A) = dep_\Box(\bot \to) = 0$,
(2) $dep_\Box(\dfrac{P_1 \quad \cdots \quad P_n}{\Gamma \to \Delta})$
$= \begin{cases} dep_\Box(P_1) + 1 & \text{if } I \text{ is } (\Box_{K4}) \text{ or } (\Box_{GL}) \\ \max\{dep_\Box(P_1), \cdots, dep_\Box(P_n)\} & \text{otherwise} \end{cases}$
where I is the inference rule that introduces $\Gamma \to \Delta$ in $\dfrac{P_1 \quad \cdots \quad P_n}{\Gamma \to \Delta}$.

Lemma 3.4. *Let* P *be a cut-free proof figure for*

$$\Box^n\Pi, \Gamma \to \Delta$$

in $\mathbf{GGL}^*$*. If* $dep_\Box(P) < n$ *and* $\Pi \cap \mathsf{Sub}(\Gamma \to \Delta) = \emptyset$*, then there exists a cut-free proof figure for* $\Gamma \to \Delta$ *in* $\mathbf{GGL}^*$*.*

Proof. We use an induction on P. If $\Pi = \emptyset$, the lemma is obvious. Suppose that $\Pi \neq \emptyset$ and the lemma holds for any proper subfigure of P. Then P is not axiom, and hence, there exists an inference rule I that introduces the end sequent of P. Here we only show the case that I is either $(\Box_{K4})$ or $(\Box_{GL})$ since the other case can be shown by the induction hypothesis. The inference rule I is of the form:

$$\frac{\Lambda, \Box^{n-1}\Pi, \Box^n, \Pi, \Gamma', \Box\Gamma' \to A}{\Box^n\Pi, \Box\Gamma' \to \Box A}$$

where $\Lambda \in \{\{\Box A\}, \emptyset\}$ and $\Box\Gamma' \to \Box A$ is $\Gamma \to \Delta$. Let P_1 be the proof figure that introduces the upper sequent of I. Since I is either $(\Box_{K4})$ or $(\Box_{GL})$, we have $dep_\Box(P_1) < dep_\Box(P) < n$, and thereby, $dep_\Box(P_1) < n - 1$. Also we have $(\Pi \cup \Box\Pi) \cap \mathsf{Sub}(\Gamma \to \Delta) = \emptyset$. Since $\mathsf{Sub}(\Lambda, \Gamma', \Box\Gamma' \to A) \subseteq \mathsf{Sub}(\Gamma \to \Delta)$, we have $(\Pi \cup \Box\Pi) \cap \mathsf{Sub}(\Lambda, \Gamma', \Box\Gamma' \to A) = \emptyset$. Using the induction hypothesis, there exists a cut-free proof figure for $\Lambda, \Gamma', \Box\Gamma' \to A$ in $\mathbf{GGL}^*$. Using $(\Box_{K4})$ or $(\Box_{GL})$, we obtain the lemma. $\dashv$

By $\mathcal{P}(\Box A)$, we mean the set of each cut-free proof figure P in $\mathbf{GGL}^*$ such that the inference rule introducing the end sequent of P is either $(\Box_{K4})$ or $(\Box_{GL})$ and its principal formula in the succedent is $\Box A$.

Definition 3.5. We define a mapping $h_{\Box C}$ on the set of cut-free proof figures in $\mathbf{GGL}^*$ as follows:

(1) $h_{\Box C}(A \to A) = \dfrac{A \to A}{\Box C, A \to A}$,

(2) $h_{\Box C}(\bot \to) = \dfrac{\bot \to}{\Box C, \bot \to}$,

(3) $h_{\Box C}(\dfrac{P_1 \quad \cdots \quad P_n}{\Gamma \to \Delta}) = \dfrac{h_{\Box C}(P_1) \quad \cdots \quad h_{\Box C}(P_n)}{\Box C, \Gamma \to \Delta}$ if $\dfrac{P_1 \quad \cdots \quad P_n}{\Gamma \to \Delta} \notin \mathcal{P}(\Box D)$ for any $\Box D$,

(4) $h_{\Box C}(\dfrac{P_1}{\Box\Gamma \to \Box A}) = \dfrac{\dfrac{\dfrac{h_{\Box C}(P_1)}{\text{using } (T \to) \text{, possibly several times}}}{\Box A, C, \Gamma, \Box C, \Gamma \to A}}{\Box C, \Gamma \to \Box A}$ if $\dfrac{P_1}{\Box\Gamma \to \Box A} \in \mathcal{P}(\Box A)$ for each $A \neq C$,

(5) $h_{\Box C}(\dfrac{P_1}{\Box\Gamma \to \Box C}) = \dfrac{\dfrac{\Box C \to \Box C}{\text{using } (T \to) \text{, possibly several times}}}{\Box C, \Gamma \to \Box C}$ if $\dfrac{P_1}{\Box\Gamma \to \Box C} \in \mathcal{P}(\Box C)$.

By $\#_{\Box}(P)$, we mean the sum of the number of inference rule $(\Box_{K4})$ in P and the number of inference rule $(\Box_{GL})$ in P.

Lemma 3.6. *Let P be a cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GGL}^*$. If there exists a subfigure $Q \in \mathcal{P}(\Box C)$ of P, then $h_{\Box C}(P)$ is a cut-free proof figure for $\Box C, \Gamma \to \Delta$ such that $\#_{\Box}(P) > \#_{\Box}(h_{\Box C}(P))$ and $dep_{\Box}(P) \geq dep_{\Box}(h_{\Box C}(P))$.*

Proof. By an induction on P.

Lemma 3.7. *Let P be a cut-free proof figure for*

$$\Box^{2n+2}\Pi, \Gamma \to \Delta$$

in $\mathbf{GGL}^*$, where n is the number of elements in $\{C \mid \Box C \in \mathsf{Sub}(\Gamma \to \Delta)\}$. Then there exists a cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GGL}^*$.

Proof. We use an induction on $\#_{\Box}(P) + \omega(dep_{\Box}(P))$. We note that

$$\Box^{n+1}\Pi \cap \mathsf{Sub}(\Gamma \to \Delta) = \emptyset$$

and the end sequent of P can be expressed as

$$\Box^{n+1}(\Box^{n+1}\Pi), \Gamma \to \Delta.$$

If $dep_{\Box}(P) < n+1$, then by Lemma 3.4, we obtain the lemma. Suppose that $dep_{\Box}(P) \geq n+1$ and the lemma holds for any proper subfigure of P. Then there exists a sequence

$$P_1, \cdots, P_{n+1}, \cdots, P_{dep_{\Box}(P)}$$

of subfigures of P satisfying

(1) $P_i \in \mathcal{P}(\Box C_i)$ for some C_i,

(2) P_{i+1} is a proper subfigure of P_i.

We note $C_i \in \mathsf{Sub}(\Gamma \to \Delta)$ for $i \leq n+1$. So, there exist i and j such that $C_i = C_j$ and $1 \leq i < j \leq n+1$. On the other hand, P_i is of the form

$$\frac{P_i'}{\Box\Gamma_i' \to \Box C_i}$$

Using Lemma 3.6, $h_{\Box C_i}(P_i')$ is a cut-free proof figure for $\Box C_i, \Gamma_i', \Box\Gamma_i' \to C_i$ such that $\#_\Box(P_i') > \#_\Box(h_{\Box C_i}(P_i'))$ and $dep_\Box(P_i') \geq dep_\Box(h_{\Box C_i}(P_i'))$.

By P', we mean the figure obtained from P by replacing P_i' by $h_{\Box C_i}(P_i')$. Since

$$\frac{\Box C_i, \Gamma_i', \Box\Gamma_i' \to C_i}{\Box\Gamma_i' \to \Box C_i}$$

is an inference rule in $\mathbf{GGL}^*$, P' is a cut-free proof figure for the end sequent of P such that $\#_\Box(P) > \#_\Box(P')$ and $dep_\Box(P) \geq dep_\Box(P')$. Using the induction hypothesis, we obtain the lemma. $\dashv$

Lemma 3.8. *Let P be a cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GGL}^*$. Then there exists a cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GGL}$.*

Proof. By replacing each inference rule ($\Box_{K4}$) in P by

$$\cfrac{\cfrac{\Gamma, \Box\Gamma \to A}{\Box A, \Gamma, \Box\Gamma \to A}}{\Box\Gamma \to \Box A}$$

we obtain a cut-free proof figure in $\mathbf{GGL}$. $\dashv$

Proof of Theorem 3.1. Let n be the number of elements in $\{C \mid \Box C \in \mathsf{Sub}(\Gamma \to \Delta)\}$. By Corollary 2.3 and Lemma 2.4, there exist formulas $A_1, \cdots, A_m$ such that

$$\Box^{2n+2}L(A_1), \cdots, \Box^{2n+2}L(A_m), \Gamma \to \Delta \in \mathbf{GK4}.$$

Using Lemma 1.3, there exists a cut-free proof figure for the above sequent in $\mathbf{GK4}$, and hence, in $\mathbf{GGL}^*$. Using Lemma 3.7, there exists a cut-free proof figure for $\Gamma \to \Delta$ in $\mathbf{GGL}^*$. Using Lemma 3.8, we obtain the theorem. $\dashv$

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