On the Notion of Forcing for The Complete Boundingness

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Abstract

We concentrate on a natural notion of forcing for the complete boundingness. We provide equivalences so that the notion (1) preserves the first uncountable cardinal; (2) preserves the stationary subsets of the first uncountable cardinal; (3) semiproper; (4) proper. It appears that there exist corresponding large cardinals and combinatorial principles.

Introduction

We consider the notion of forcing for the complete boundingness. The complete boundingness has been investigated by various people. The following are a few of the developments known to us.

- Generic Ultrapowers and the complete boundingness and various observations. ([B-M])
- The equivalence between the complete boundingness and what we call the Zapletal's Conjecture. ([Y])
- An iterated forcing construction for the complete boundingness starting from the least regular cardinal which has cofinally many measurables below. ([M])

We have been informed of the following.

- On the consistency of the complete boundingness with CH. ([S-L])
- The large cardinal hypothesis used is necessary. ([D-D])

The references [W] and [S] appear to be the origins of many things. In this note, we intend to give our account which took place in the years betweeen 1998-2000 intermittently.

§1. Preliminary 1

We review the set of sets which are hereditarily of size less than a given regular uncountable cardinal.

1.1 Definition. Let $\theta > \omega$ be a regular cardinal. We denote $H_\theta = \{x \mid |\mathrm{TC}(x)| < \theta\}$, where $\mathrm{TC}(x)$ denotes the transitive closure of x.

The following are basic closure properties of H_θ's.

1.2 Proposition. *Let $\theta > \omega$ be a regular cardinal.*

(1) $x \in H_\theta$ *iff* ($x \subset H_\theta$ *and* $|x| < \theta$).

(2) $|H_\theta| = 2^{<\theta}$.

(3) *If* $y \in x \in H_\theta$, *then* $y \in H_\theta$.

(4) *If* $y \subseteq x \in H_\theta$, *then* $y \in H_\theta$.

(5) ${}^{<\theta}H_\theta \subset H_\theta$.

(6) *If* $x, y \in H_\theta$, *then* $\{x, y\}, x \times y, \bigcup x \in H_\theta$.

(7) *If* $H_\theta \models$ "$y = P(x)$", *then* $y = P(x)$.

(8) *If* $y = TC(P(x)) = P(x) \cup TC(x)$ *is of size* $< \theta$, *then* $H_\theta \models$ "$y = P(x)$".

(9) $H_\theta \prec_{\Sigma_1} H_\chi$ *for all regular* χ *with* $\theta < \chi$.

(10) *If* $V = L$, *then* $L_\theta = H_\theta$.

We next summarize the definabilities of H_θ's.

1.3 Proposition. *Let both θ and χ be uncountable regular with $2^{<\theta} < \chi$.*

(1) $H_\theta \in H_\chi$.
(2) Let $y = H_\theta$, then $H_\chi \models$ "$y = \{x \mid \mid TC(x) \mid < \theta\}$".
(3) If $\theta \in M \prec H_\chi$, then $H_\theta \in M$ and so $H_\theta \cap M \prec H_\theta$.

We review H_θ in generic extensions.

1.4 Proposition. *Let P be a preorder and θ be uncountable regular with $P \in H_\theta$. We denote V^P for the class of P-names in V.*

(1) $\Vdash_P$ "$H_\theta^{V[\dot{G}]} = \{\tau_{\dot{G}} \mid \tau \in H_\theta \cap V^P\}$".
Actually we have
(2) *For any $\tau \in V^P$ there is $\sigma \in H_\theta \cap V^P$ s.t. $\Vdash_P$ "if $\tau \in H_\theta^{V[\dot{G}]}$, then $\tau = \sigma$".*

We next summerize elementary substructures in generic extensions.

1.5 Proposition. *Let P be a preorder, θ be uncountable regular and N be a countable elementary substructure with $P \in N \prec H_\theta$.*

(1) *For any formula $\varphi(v_1, \cdots, v_n)$, there is a formula $\varphi^*(x, y, v_1, \cdots, v_n)$ s.t.*
for all $p \in P$ and all $\tau_1, \cdots, \tau_n \in V^P$ $p \Vdash_P$ "$H_\theta^{V[\dot{G}]} \models$ "$\varphi(\tau_1, \cdots, \tau_n)$"" iff $H_\theta \models$ "$\varphi^*(p, P, \tau_1, \cdots, \tau_n)$".
(2) $\Vdash_P$ "$N[\dot{G}] = \{\tau_{\dot{G}} \mid \tau \in N \cap V^P\} \prec H_\theta^{V[\dot{G}]}$".
(3) $\Vdash_P$ "If $N \cup \{\dot{G}\} \subseteq \dot{M} \prec H_\theta^{V[\dot{G}]}$, then $N[\dot{G}] \subseteq \dot{M}$".

§2. Preliminary 2

We review the semiproperness of preorders.

2.1 Definition. A preorder P is *semiproper*, if for all regular cardinals θ with $P \in H_{\rho(P)^+} \in H_\theta$, where $\rho(P) = \mid \mathrm{TC}(P) \mid$, and all countable elementary substructures N with $P \in N \prec H_\theta$, the following holds.

For any $p \in P \cap N$ there is $q \le p$ s.t. for any $\tau \in V^P \cap N$ with $\Vdash_P$"$\tau \in \omega_1^V$", we have $q \Vdash_P$"$\tau \in N$".

Equivalently, $q \Vdash_P$"$N[\dot{G}] \cap \omega_1^V = N \cap \omega_1^V$".

2.2 Definition. Let $A \supseteq \omega_1$. A set $S \subseteq [A]^\omega$ is *semistationary*, if

$$\bigcup_{X \in S} \{Y \in [A]^\omega \mid X \subseteq_{\omega_1} Y\}$$

is stationary in $[A]^\omega$, where $X \subseteq_{\omega_1} Y$ means that $X \subseteq Y$ and $X \cap \omega_1 = Y \cap \omega_1$.

The following provides a model theoretic equivalence to the semiproperness.([S])

2.3 Thoerem. *Let P be a preorder. The following are equivalent.*

(1) *P is semiproper.*
(2) *P preserves not only ω_1 but also every semistationary set. Namely, if $S \subseteq [A]^\omega$ is semistationary, then $\Vdash_P$ "$\bigcup\{Y \in ([A]^\omega)^{V[\dot{G}]} \mid \exists X \in S\ X \subseteq_{\omega_1} Y\}$ is stationary in $([A]^\omega)^{V[\dot{G}]}$".*

As a corollary to the proof of the above theorem which is due to [S], we have

2.4 Corollary. *Let P be a preorder and let $\rho(P) = \mid TC(P) \mid$. The following are all equivalent.*

(1) *P is semiproper.*
(2) *P preserves ω_1 and every semistationary set in all $[A]^\omega$ with $\omega_1 \subseteq A$.*
(3) *P preserves ω_1 and every semistationary set in $[H_{\rho(P)^+}]^\omega$.*

(4) $\{N \prec H_{\rho(P)^+} \mid P \in N$ *and* $\forall p \in P \cap N \exists q \leq p$ q *is* (P, N)*-semi-generic*$\}$ *contains a club.*

(5) For all countable elementary substructure M *of* $H_{(2^{\rho(P)})^+}$ *with* $P \in M$ *and all* $p \in P \cap M$*, there is* $q \leq p$ *s.t.* q *is* (P, M)*-semi-generic.*

§3. The Principle CB

We consider the following combinatorial principle.

3.1 Definition. The *complete boundingness* (CB) holds, if for any $f : \omega_1 \longrightarrow \omega_1$, there is γ with $\omega_1 < \gamma < \omega_2$ and a sequence $\langle X_i \mid i < \omega \rangle$ of continuously increasing countable subsets of γ with $\bigcup\{X_i \mid i < \omega_1\} = \gamma$ s.t. for all $i < \omega_1$ the order type of X_i is greater than $f(i)$.

We first mention an equivalence to CB. The equivalence is the original to CB and due to [B-M] and [W].

3.2 Proposition. *The following are equivalent.*

(1) CB holds.

(2) NS_{ω_1} *is completely bounded. Namely, for any* $f : \omega_1 \longrightarrow \omega_1$ *there is* $\gamma \in (\omega_1, \omega_2)$*, a bijection* $\pi : \omega_1 \longrightarrow \gamma$ *and a club* $C \subseteq \omega_1$ *s.t. for all* $\delta \in C$ $f(\delta) < o.t.(\pi''\delta)$*.*

Among others, the following are observed in [B-M] and possibly in [W].

3.3 Theorem.

(1) If NS_{ω_1} *is saturated, then* NS_{ω_1} *is completely bounded.*

(2) If NS_{ω_1} *is completely bounded, then* $\diamondsuit_{\omega_1}$ *gets negated.*

Then [Y] improved a result due to [B-M] to the following.

3.4 Theorem. *The following are equivalent.*

(1) NS_{ω_1} *is completely bounded.*

(2) The Zapletal's Conjecture holds. Namely, for any club $C \subseteq \omega_1$*, the pull-back* $C^* = \{X \in [\omega_2]^\omega \mid o.t.(X) \in C\}$ *contains a club.*

(3) For any $f : \omega_1 \longrightarrow \omega_1$ *there are club many* $\gamma < \omega_2$*, bijections* $\pi : \omega_1 \longrightarrow \gamma$ *and clubs* $C \subseteq \omega_1$ *s.t. for all* $\delta \in C$ $f(\delta) < o.t.(\pi''\delta)$*.*

We add the following to record.

3.5 Theorem. *The following are equivalent.*

(1) CB hold.

(2) For any regular $\theta \geq \omega_2$ *and any countable* $Y \prec H_\theta$*, the canonical extension* $Z = \{f(Y \cap \omega_1) \mid f \in Y, f : \omega_1 \longrightarrow H_\theta\} \prec H_\theta$ *of* Y *satisfies* $Z \cap \omega_1 = o.t.(Y \cap \omega_2)$*.*

Proof. (1) implies (2): Fix θ, Y and Z. We first observe that the following always holds.

Claim 1. $o.t.(Y \cap \omega_2) \leq Z \cap \omega_1$*.*

Proof. Let $\pi : \mathrm{o.t.}(Y \cap \omega_2) \longrightarrow Y \cap \omega_2$ be the isomorphism. For any $\gamma \in Y \cap \omega_2$, take a sequence $\langle X_i \mid i < \omega_1 \rangle$ of continuously increasing countable subsets of γ s.t. $\bigcup\{X_i \mid i < \omega_1\} = \gamma$. Since $\gamma \in Y$, we may assume $\langle X_i \mid i < \omega_1 \rangle \in Y$. Let $f_\gamma : \omega_1 \longrightarrow \omega_1$ be defined by $f_\gamma(i) = \mathrm{o.t.}(X_i)$. We may assume $f_\gamma \in Y$. Note that

$$\pi\lceil \mathrm{o.t.}(\gamma \cap Y) : \mathrm{o.t.}(\gamma \cap Y) \longrightarrow \gamma \cap Y = X_{Y \cap \omega_1}.$$

This map is the isomorphism. So we have

$$\mathrm{o.t.}(\gamma \cap Y) = \mathrm{o.t.}(X_{Y \cap \omega_1}) = f_\gamma(Y \cap \omega_1) \in Z \cap \omega_1.$$

But o.t.$(Y \cap \omega_2) = \{\text{o.t.}(\gamma \cap Y) \mid \gamma \in Y \cap \omega_2\}$. Thus $\leq$ holds.

□

But by CB, we get

Claim 2. *o.t.*$(Y \cap \omega_2) \geq Z \cap \omega_1$.

Proof. Suppose $f(Y \cap \omega_1) < \omega_1$. We may assume $f \in Y$ with $f : \omega_1 \longrightarrow \omega_1$. By CB, we get $\omega_1 < \gamma < \omega_2$ and a sequence $\langle X_i \mid i < \omega_1 \rangle$ s.t. for all $i < \omega_1$ $f(i) <$ o.t.(X_i). Since $f \in Y$, we may assume $\gamma, \langle X_i \mid i < \omega_1 \rangle \in Y$. And so $f(Y \cap \omega_1) <$ o.t.$(X_{Y \cap \omega_1})$ and $X_{Y \cap \omega_1} = \bigcup\{X_i \mid i < Y \cap \omega_1\} = Y \cap \gamma$. Thus $f(Y \cap \omega_1) <$ o.t.$(Y \cap \gamma) <$ o.t.$(Y \cap \omega_2)$.

(2) implies (1): Given $f : \omega_1 \longrightarrow \omega_1$, construct a sequence of canonical extensions $\langle Y_i \mid i < \omega_1 \rangle$. By this we mean that

- $f \in Y_0 \prec H_{\omega_2}$.
- $Y_{i+1} = \{g(Y_i \cap \omega_1) \mid g \in Y_i \text{ and } g : \omega_1 \longrightarrow H_{\omega_2}\} \prec H_{\omega_2}$.
- For limit $i < \omega_1$, we take $Y_i = \bigcup\{Y_j \mid j < i\} \prec H_{\omega_2}$.

Then for any $i < \omega_1$, we have

$$f(i) \in Y_{i+1} \cap \omega_1 = \text{o.t.}(Y_i \cap \omega_2).$$

We put $X_i = Y_i \cap \omega_2$ and $\gamma = \bigcup\{X_i \mid i < \omega_1\}$. It is easy to see that $\gamma < \omega_2$ and this $\langle X_i \mid i < \omega_1 \rangle$ works.

□

§4. The Notion of Forcing $Q(f, \kappa)$ for CB

We introduce a natural partially ordered set to force CB. In the following, we typically consider either $\kappa = \omega_2$ or κ is a measurable cardinal.

4.1 Definition. Let $f : \omega_1 \longrightarrow \omega_1$ and κ be a regular cardinal with $\kappa \geq \omega_2$. We define $p = \langle X_i^p \mid i \leq \alpha^p \rangle \in Q(f, \kappa)$, if

- p is a sequence of continuously increasing countable subsets of κ of length $\alpha^p + 1 < \omega_1$.
- For all $i \leq \alpha^p$ $f(i) <$ o.t.(X_i^p).

For $p, q \in Q(f, \kappa)$, we set $q \leq p$, if $q \supseteq p$.

We first mention a density. The proof is a simplified version due to Y. Yoshinobu.

4.2 Lemma. *For any $p \in Q(f, \kappa)$, any countable subset X of κ and any $\alpha < \omega_1$, we have $q \leq p$ s.t. if $\alpha^p < \alpha$, then $\alpha^q = \alpha$ and $X \subseteq X_\alpha^q$.*

Proof. Given p, X and α with $\alpha^p < \alpha$, take a countable subset Y of κ so that $X \cup X_{\alpha^p}^p \subseteq Y$ and o.t.$(Y) > \sup\{f(i) \mid \alpha^p < i \leq \alpha\}$. Let $q = p \cup \{(i, Y) \mid \alpha^p < i \leq \alpha\}$. This q works.

□

We consider equivalent conditions on $Q = Q(f, \kappa)$ so that

(1) Q preserves ω_1.
(2) Q preserves every stationary subset of ω_1.
(3) Q is semiproper.
(4) Q is proper.

It turns out that we have a beautiful picture including CB and the Weak Chang's Conjecture.

§5. $Q(f,\kappa)$ May Preserve ω_1

5.1 Proposition. *Let $Q = Q(f,\kappa)$. The following are equivalent.*

(1) Q preserves ω_1.

(2) Q is σ-Baire

(3) $S(f,\kappa) = \{X \in [\kappa]^\omega \mid\ X \cap \omega_1 < \omega_1$ and for all $i \leq X \cap \omega_1$ $f(i) <$ o.t.$(X)\}$ is stationary.

Proof. (1) implies (2): Suppose Q preserved ω_1. Let $\langle D_n \mid n < \omega\rangle$ be a sequence of open dense subsets of Q and $p \in Q$. By the density we take a sequence $\langle \dot{X}_i \mid i < \omega_1\rangle$ of Q-names so that

- $\Vdash_Q$"$\bigcup \dot{G} = \langle \dot{X}_i \mid i < \omega_1^V\rangle$".

We construct a sequence of Q-names $\langle \dot{p}_n \mid n < \omega\rangle$ s.t.

- $p \Vdash_Q$"$\dot{p}_n \leq p$ and $\dot{p}_n \in D_n \cap \dot{G}$".

Since Q preserves ω_1, we have $\Vdash_Q$"$\sup\{\alpha^{\dot{p}_n} \mid n < \omega\} < \omega_1$". Since $\Vdash_Q$"$\forall \beta < \omega_1\ \langle \dot{X}_i \mid i \leq \beta\rangle \in Q$", we have $p \Vdash_Q$"$\exists q \leq p\ q \in \bigcup\{D_n \mid n < \omega\}$". So we have $q \leq p$ with $q \in \bigcup\{D_n \mid n < \omega\}$.

(2) implies (3): Let $h :\ {}^{<\omega}\kappa \longrightarrow \kappa$. We want to find $X \in S(f,\kappa)$ s.t. X is h-closed. To this end take a sufficiently large regular cardinal θ and a countable elementary substructure N with $f, \kappa, Q, h \in N \prec H_\theta$. Notice that in every generic extension $V[\dot{G}]$ via Q, we have

- $N[\dot{G}] \cap \kappa$ is h-closed.
- For all $i < N[\dot{G}] \cap \omega_1$ $f(i) \in N[\dot{G}] \cap \omega_1$.
- $\bigcup \dot{G} = \langle \dot{X}_i \mid i < \omega_1\rangle \in N[\dot{G}]$.

Since $\dot{X}_{N[\dot{G}]\cap\omega_1} = N[\dot{G}] \cap \kappa$,

- $f(N[\dot{G}] \cap \omega_1) <$ o.t.$(N[\dot{G}] \cap \kappa)$.

Since Q is σ-Baire,

$$\omega_1 \in X = N[\dot{G}] \cap \kappa \in V.$$

We observe this X works. To see $X \in S(f,\kappa)$, take $i < X \cap \omega_1$. Then $f(i) \in X \cap \omega_1 < \omega_1$. So we have $f(i) <$ o.t.(X). We also have $f(X \cap \omega_1) <$ o.t.(X). To see X is h-closed, notice that $h \in N[\dot{G}] \prec H_\theta^{V[\dot{G}]}$ and so $h``{}^{<\omega}(N[\dot{G}] \cap \kappa) \subseteq N[\dot{G}] \cap \kappa$. Namely, X is h-closed.

(3) implies (1): Suppose $\alpha < \omega_1$ and $p \Vdash_Q$"$\dot{g} : \alpha \longrightarrow \omega_1^V$". We want to find $q \leq p$ and $\beta < \omega_1$ s.t. $q \Vdash_Q$"$\dot{g}``\alpha \subseteq \beta$". By (3), we may take a sufficiently large regular cardinal θ and a countable elementary substructure N with

- $p, \dot{g}, Q, \alpha \in N \prec H_\theta$.
- $N \cap \kappa \in S(f,\kappa)$.

And so,

- $f(N \cap \omega_1) <$ o.t.$(N \cap \kappa)$.

We construct a (Q,N)-generic sequence $\langle p_n \mid n < \omega\rangle$ with $p_0 \leq p$. Then we have

- $\sup\{\alpha^{p_n} \mid n < \omega\} = N \cap \omega_1$.
- $\bigcup\{X^{p_n}_{\alpha^{p_n}} \mid n < \omega\} = N \cap \kappa$.

Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(N \cap \omega_1, N \cap \kappa)\}$.

Then $q \leq p$ is (Q,N)-generic. Hence $q \Vdash_Q$"$\dot{g}``\alpha \subseteq N[\dot{G}] \cap \omega_1^V = N \cap \omega_1^V < \omega_1^V$".

□

To complete the picture at this level, let us recall

5.2 Definition. The *Weak Chang's Conjecture* holds, if there is no $f : \omega_1 \longrightarrow \omega_1$ s.t. for all $\beta < \omega_2$, $\{i < \omega_1 \mid f_\beta(i) < f(i)\}$ contains a club, where f_β denotes the β-th canonical function.

The following is known.

5.3 Lemma. *The following are equivalent.*

(1) The Weak Chang's Conjecture holds.

(2) For any regular cardinal $\theta \geq \omega_2$ and any $p \in H_\theta$, there is $\delta < \omega_1$ s.t. sup$\{o.t.(N \cap \omega_2) \mid p \in N \prec H_\theta$, N is countable and $N \cap \omega_1 = \delta\} = \omega_1$.

The following completes the picture at this level. The implication (1) implies (2) is due to [Y].

5.4 Proposition. *The following are equivalent.*

(1) For all $f : \omega_1 \longrightarrow \omega_1$, $Q(f, \omega_2)$ preserves ω_1.

(2) The Weak Chang's Conjecture holds.

Proof. (1) implies (2): Suppose $f : \omega_1 \longrightarrow \omega_1$. We assume $S(f, \omega_2) \subseteq \{X \in [\omega_2]^\omega \mid X \cap \omega_1 < \omega_1$ and $f(X \cap \omega_1) < \text{o.t.}(X)\}$ is stationary. So for each $X \in S(f, \omega_2)$, there is $\beta \in X$ s.t. $\text{o.t.}(\beta \cap X) = f(X \cap \omega_1)$. By applying the Pressing Down Lemma, we get $\beta < \omega_2$ and a stationary set $T \subseteq S(f, \omega_2)$ s.t. for any $X \in T$, we have $\beta \in X$ and $f(X \cap \omega_1) = \text{o.t.}(X \cap \beta)$. Hence $\{\gamma < \omega_1 \mid f(\gamma) = f_\beta(\gamma)\}$ is stationary. Hence $\{i < \omega_1 \mid f_\beta(i) < f(i)\}$ does not contain any club.

(2) implies (1): Let $f : \omega_1 \longrightarrow \omega_1$. We want to show $S(f, \omega_2)$ is stationary. To this end let $\pi : {}^{<\omega}\omega_2 \longrightarrow \omega_2$. We need to find $X \in S(f, \omega_2)$ which is π-closed. By the Weak Chang's Conjecture, if we take $\theta = \omega_3$ and $c = (f, \pi)$, then there is $\delta < \omega_1$ s.t.

$$\sup\{\text{o.t.}(N \cap \omega_2) \mid c \in N \prec H_{\omega_3} \text{ and } N \cap \omega_1 = \delta\} = \omega_1.$$

We calculate $f(\delta) < \omega_1$ and may choose N so that

- $c \in N \prec H_{\omega_3}$.
- $N \cap \omega_1 = \delta$ and $\text{o.t.}(N \cap \omega_2) > f(\delta)$.

Let $X = N \cap \omega_2$. Then X is π-closed and $f(X \cap \omega_1) = f(\delta) < \text{o.t.}(N \cap \omega_2) = \text{o.t.}(X)$. So we have $X \in S(f, \omega_2)$.

□

So we have the exact consistency strength concerning the preservation of ω_1.

5.5 Corollary. *The following are equiconsistent.*

(1) For all $f \in {}^{\omega_1}\omega_1$, $Q(f, \omega_2)$ preserves ω_1.

(2) There is an almost $< \omega_1$-Erdős cardinal.

Proof. See [D-P] or [D-L].

□

5.6 Proposition. *If the Weak Chang's Conjecture holds, then there is no simplified $(\omega_1, 1)$-morasses. And so ω_2 is strongly inaccessible in L.*

Proof. Suppose to the contrary, we had a simplified $(\omega_1, 1)$-morass

$$\langle \theta_i, F_{ij} \mid i < j \leq \omega_1 \rangle$$

s.t.

- $\theta_0 = 1$ and $\theta_{\omega_1} = \omega_2$.
- $0 < \theta_i < \omega_1$.
- For $i < j < \omega_1$, we have $\mid F_{ij} \mid \leq \omega$.
- For any $f \in F_{ij}$, $f : \theta_i \longrightarrow \theta_j$ is $\in$-preserving.
- $F_{ii+1} = \{id_{\theta_i}, f_{ii+1}\}$, where there is $\sigma_i < \theta_i$ and $\theta_{i+1} = \theta_i + (\theta_i - \sigma_i)$.

  And,
  $$f_{ii+1}(\alpha) = \begin{cases} \alpha & \text{if } \alpha < \sigma_i, \\ \theta_i + (\alpha - \sigma_i) & \text{o.w.} \end{cases}$$
- For limit $j \leq \omega_1$, $\theta_j = \bigcup\{f``\theta_i \mid f \in F_{ij}\}$.
- For $i < j < k \leq \omega_1$, we have $F_{jk} \circ F_{ij} = F_{ik}$.
- For limit $j \leq \omega_1$, $i_1, i_2 < j$ and $f_1 \in F_{i_1 j}, f_2 \in F_{i_2 j}$, there is k with $i_1, i_2 < k < j$ and there are $g_1 \in F_{i_1 k}$, $g_2 \in F_{i_2 k}$ and $h \in F_{kj}$ s.t. $f_1 = h \circ g_1$ and $f_2 = h \circ g_2$.

The following is well-known (see [V]).

5.7 Lemma. *If $f, g \in F_{ij}$, $\alpha, \beta \in \theta_i$ and $f(\alpha) = g(\alpha)$, then $\alpha = \beta$ and $f\lceil\alpha = g\lceil\alpha$.*

Now we define $f : \omega_1 \longrightarrow \omega_1$ by $f(i) = \theta_i$.

Claim. *$S(f, \omega_2) = \{X \in [\omega_2]^\omega \mid X \cap \omega_1 < \omega_1$ and $\forall i \leq X \cap \omega_1\ f(i) < o.t.(X)\}$ is not stationary.*

Proof. Define

$$C = \{X \in [\omega_2]^\omega \mid \forall \xi, \eta \in X \text{ with } \xi < \eta \exists i < X \cap \omega_1 < \omega_1 \exists \bar{\xi}, \bar{\eta} < \theta_i \exists f \in F_{i\omega_1}\ f(\bar{\xi}) = \xi, f(\bar{\eta}) = \eta\}.$$

We observe that this C is a club disjoint from $S(f, \omega_2)$. To see C is a club, we mention the unboundedness of C. Given $Y \in [\omega_2]^\omega$, take a sufficiently large regular cardinal θ and a countable elementary substructure N with $Y, \langle \theta_i, F_{ij} \mid i < j \leq \omega_1 \rangle \in N \prec H_\theta$. Let $X = N \cap \omega_2$. Then we have $Y \subset X$. If $\xi, \eta \in X$, then there is $i < \omega_1$, $f \in F_{i\omega_1}$, $\bar{\xi}, \bar{\eta} \in \theta_i$ s.t. $f(\bar{\xi}) = \xi$ and $f(\bar{\eta}) = \eta$. By the elementarity, we may assume $i < X \cap \omega_1$.

We next mention that if $X \in C$, then $\theta_{X\cap\omega_1} \geq$ o.t.(X). And so $f(X \cap \omega_1) \geq$ o.t.(X). Hence $X \notin S(f, \omega_2)$. To show this, we define $p : X \longrightarrow \theta_{X\cap\omega_1}$ by $p(\xi) = \bar{\xi}$, where there is $g_\xi \in F_{X\cap\omega_1\omega_1}$ $g_\xi(\bar{\xi}) = \xi$. It is easy to show that this p is $\in$-preserving.

If there is no $(\omega_1, 1)$-morasses in V, then ω_2 is strongly inaccessible (see [D]).

□

§6. $Q(f, \kappa)$ May Preserve Every Stationary Subset of ω_1

6.1 Proposition. *The following are equivalent.*

(1) $Q(f, \kappa)$ preserves every stationary subset of ω_1.

(2) $S(f, \kappa) = \{X \in [\kappa]^\omega \mid X \cap \omega_1 < \omega_1$ and for all $i \leq X \cap \omega_1\ f(i) < o.t.(X)\}$ is projectively stationary. Namely, for any stationary subset T of ω_1, we have $\{X \in S(f, \kappa) \mid X \cap \omega_1 \in T\}$ is stationary.

Proof. (1) implies (2): Let T be a stationary subset of ω_1. We need to show that $\{X \in S(f, \kappa) \mid X \cap \omega_1 \in T\}$ is stationary. To this end let $\pi : {}^{<\omega}\kappa \longrightarrow \kappa$ be given. We want to find $X \in S(f, \kappa)$ s.t. $X \cap \omega_1 \in T$ and X is π-closed. Take any $Q(f, \kappa)$-generic filter G over V and we argue in $V[G]$. Let $\bigcup G = \langle X_i \mid i < \omega_1 \rangle$ be a sequence forced. Back in V, take a sufficiently large regular cardinal θ and a sequence of countable elementary substructures $\langle N_i \mid i < \omega_1 \rangle$ of H_θ s.t. N_0 contains f, π and relevant names. Then in $V[G]$, we form the sequence of countable elementary substructures $\langle N_i[G] \mid i < \omega_1 \rangle$ of $H_\theta^{V[G]}$. Since $G \in N_0[G]$, we have $\langle X_i \mid i < \omega_1 \rangle \in N_0[G]$. Since we assume T remains stationary and $\{\delta < \omega_1 \mid N_\delta[G] \cap \omega_1 = N_\delta \cap \omega_1 = \delta\}$ is a club, we may take $\delta < \omega_1$ s.t. $N_\delta[G] \cap \omega_1 = \delta \in T$. Notice that we have $X_\delta = N_\delta[G] \cap \kappa$. And so $f(\delta) <$ o.t.$(N_\delta[G] \cap \kappa)$. Let $X = N_\delta[G] \cap \kappa \in V$. Then it is easy to check that this X works.

(2) implies (1): Since $S(f,\kappa)$ is projectively stationary, it is stationary. Hence $Q(f,\kappa)$ is σ-Baire. Let $T \subseteq \omega_1$ be stationary, $p \in Q(f,\kappa)$ and $p \Vdash_{Q(f,\kappa)}$"$\dot{C} \subseteq \omega_1$ be a club". We want to find $q \leq p$ and $\delta \in T$ s.t. $q \Vdash_{Q(f,\kappa)}$"$\delta \in \dot{C}$". To this end take a sufficiently large regular cardinal θ and a countable elementary substructure N s.t.

- $p, \dot{C}, Q(f,\kappa), f \in N \prec H_\theta$.
- $\delta = N \cap \omega_1 \in T$.
- $N \cap \kappa \in S(f,\kappa)$.

And so,

- $f(\delta) <$ o.t.$(N \cap \kappa)$.

Let $\langle p_n \mid n < \omega \rangle$ be a $(Q(f,\kappa), N)$-generic sequence with $p_0 \leq p$. Then

- $\sup\{\alpha^{p_n} \mid n < \omega\} = \delta$.
- $\bigcup\{X^{p_n}_{\alpha^{p_n}} \mid n < \omega\} = N \cap \kappa$.

Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(\delta, N \cap \kappa)\}$. Then it is easy to see that $q \leq p$ and q is $(Q(f,\kappa), N)$-generic. And so $q \Vdash_{Q(f,\kappa)}$"$\dot{C} \in N[\dot{G}]$ and $\delta = N \cap \omega_1 = N[\dot{G}] \cap \omega_1 \in \dot{C}$".

□

The complete picture at this level is as follows. Though we may not know the exact consistency strength of this level.

6.2 Proposition. *The following are equivalent.*

(1) For all $f \in {}^{\omega_1}\omega_1$, $Q(f,\omega_2)$ preserves every stationary subset of ω_1.

(2) For all $f \in {}^{\omega_1}\omega_1$, $S(f,\omega_2)$ is projectively stationary.

(3) For any constant c and any regular cardinal $\theta \geq \omega_2$ with $c \in H_\theta$, $\{\delta < \omega_1 \mid sup\{o.t.(N \cap \omega_2) \mid c \in N \prec H_\theta$ and $N \cap \omega_1 = \delta\} = \omega_1\}$ contains a club. (A strong form of the Weak Chang's Conjecture.)

Proof. We have seen (1) iff (2).

(2) implies (3): Suppose not. Then there must be a regular cardinal θ and a constant $c \in H_\theta$ s.t. $A = \{\delta < \omega_1 \mid \sup\{\text{o.t.}(N \cap \omega_2) \mid c \in N \prec H_\theta \text{ and } N \cap \omega_1 = \delta\} = \omega_1\}$ does not contain any club. So we may define a function $f : \omega_1 \longrightarrow \omega_1$ s.t. for $\delta \in \omega_1 - A$, $f(\delta) = \sup\{\text{o.t.}(N \cap \omega_2) \mid c \in N \prec H_\theta \text{ and } N \cap \omega_1 = \delta\} < \omega_1$. Since $S(f,\omega_2)$ is projectively stationary, we may take a countable elementary substructure N s.t.

- $c \in N \prec H_\theta$.
- $\delta = N \cap \omega_1 \in \omega_1 - A$.
- $f(\delta) <$ o.t.$(N \cap \omega_2)$.

But o.t.$(N \cap \omega_2) \in \{\text{o.t.}(M \cap \omega_2) \mid c \in M \prec H_\theta \text{ and } M \cap \omega_1 = \delta\}$ and so o.t.$(N \cap \omega_2) \leq f(\delta)$. This is a contradiction.

(3) implies (2): Let $T \subseteq \omega_1$ be stationary and $\pi : {}^{<\omega}\omega_2 \longrightarrow \omega_2$ be given. We want to find $X \in S(f,\omega_2)$ s.t. $X \cap \omega_1 \in T$ and X is π-closed. Let $\theta = \omega_3$ and $c = (\pi, f)$. Then by (3), we have $\delta \in T$ s.t. $\sup\{\text{o.t.}(N \cap \omega_2) \mid c \in N \prec H_{\omega_3} \text{ and } N \cap \omega_1 = \delta\} = \omega_1$.

We calculate $f(\delta) < \omega_1$ and fix a countable elementary substructure N s.t.

- $f(\delta) <$ o.t.$(N \cap \omega_2)$.
- $c \in N \prec H_{\omega_3}$.
- $N \cap \omega_1 = \delta$.

Let $X = N \cap \omega_2$. Then it is easty to check that $X \in S(f,\omega_2)$ works.

□

§7. $Q(f,\kappa)$ May Be Semiproper

7.1 Proposition. *The following are equivalent.*

(1) $Q(f,\kappa)$ is semiproper.

(2) For all regular cardinals $\theta \geq (2^\kappa)^+$ and all countable elementary substructures N^* of H_θ with $\kappa, f \in N^*$, there is a countable elementary substructure M^* of H_θ s.t. $N^* \subseteq_{\omega_1} M^*$ and $f(M^* \cap \omega_1) <$ o.t.$(M^* \cap \kappa)$.

(3) For any club $D \subseteq [H_{\kappa^+}]^\omega$ there is a club $C \subseteq [H_{\kappa^+}]^\omega$ s.t. for any $X \in C$, there is $Y \in D$ s.t. $X \subseteq_{\omega_1} Y$ and $Y \cap \kappa \in S(f,\kappa)$.

Proof. (1) implies (2): Let $Q = Q(f,\kappa)$. Suppose Q is semiproper. In particular Q preserves ω_1 and so Q is σ-Baire.

Claim 1. $E = \{N \in [H_{\kappa^+}]^\omega \mid \exists M \in [H_{\kappa^+}]^\omega\ N \subseteq_{\omega_1} M \prec H_{\kappa^+}\ f(M \cap \omega_1) < \text{o.t.}(M \cap \kappa)\}$ contains a club.

Proof. Suppose not. We write $H = H_{\kappa^+}$ for short. Let

$$S = \{N \in [H]^\omega \mid N \prec H \text{ and } N \notin E\}.$$

Hence S is stationary. Now let G be any Q-generic filter over V. Then S remains semistationary in $V[G]$. Let $\langle X_i \mid i < \omega_1\rangle = \bigcup G$. It is routin to see that

$$C = \{M \in [H]^\omega \mid M \prec H \text{ and } X_{M\cap\omega_1} = M \cap \kappa\}$$

is a club. So we have countable sets $N, M \in V$ s.t.

- $N \in S$ and $N \subseteq_{\omega_1} M \prec H$.
- $X_{M\cap\omega_1} = M \cap \kappa$.

And so

- $f(M \cap \omega_1) < \text{o.t.}(M \cap \kappa)$.

Hence $N \in E$. However, this contradicts to $N \in S$.

□

Claim 2. Let θ be a regular cardinal with $\theta \geq (2^\kappa)^+$. If $N^* \prec H_\theta$ is countable with $\kappa, f \in N^*$, then there is a countable M^* s.t. $N^* \subseteq_{\omega_1} M^* \prec H_\theta$ and $f(M^* \cap \omega_1) < \text{o.t.}(M^* \cap \kappa)$.

Proof. Since $H_{\kappa^+} \in H_\theta$, we have

$$H_\theta \models \text{“}E = \{N \in [H_{\kappa^+}]^\omega \mid \exists M \in [H_{\kappa^+}]^\omega\ N \subseteq_{\omega_1} M \prec H_{\kappa^+}\ f(M \cap \omega_1) < \text{ o.t.}(M \cap \kappa)\} \text{ contains a club.”}$$

So we may assume that a club as such belongs to N^*. Hence $N = N^* \cap H_{\kappa^+}$ is in the club. Therefore we may take a countable $M \prec H_{\kappa^+}$ s.t. $N \subseteq_{\omega_1} M$ and $f(M \cap \omega_1) < \text{o.t.}(M \cap \kappa)$. Let

$$M^* = \{g(s) \mid g \in N^*,\ g : {}^{<\omega}\kappa \longrightarrow H_\theta,\ s \in {}^{<\omega}(M \cap \kappa)\}$$

Then since every function from ${}^{<\omega}\kappa$ to κ belongs to H_{κ^+}, we have

- $N^* \subseteq_{\omega_1} M^* \prec H_\theta$.
- $M \cap \kappa = M^* \cap \kappa$.

And so

- $f(M^* \cap \omega_1) < \text{o.t.}(M^* \cap \kappa)$.

□

(2) implies (3): Let $D \subseteq [H_{\kappa^+}]^\omega$ be a given club. Take a sufficiently large regular cardinal θ. Let C be a club in $[H_{\kappa^+}]^\omega$ contained in $\{N^* \cap H_{\kappa^+} \mid \kappa, D, f \in N^* \prec H_\theta\}$. Now if $X \in C$, then there is $N^* \prec H_\theta$ s.t. $X = N^* \cap H_{\kappa^+}$ and $\kappa, D, f \in N^* \prec H_\theta$. By (2), we have M^* s.t. $N^* \subseteq_{\omega_1} M^*$ and $M^* \cap \kappa \in S(f,\kappa)$. Since $D \in N^* \subseteq M^*$ and D is a club, we have $M^* \cap H_{\kappa^+} \in D$. Let $Y = M^* \cap H_{\kappa^+}$. Then $Y \in D$, $X \subseteq_{\omega_1} Y$ and $Y \cap \kappa \in S(f,\kappa)$.

(3) implies (1): Let θ be a regular cardinal so that $Q(f,\kappa), H_{\kappa^+} \in H_\theta$. Take a club $D \subseteq \{M^* \cap H_{\kappa^+} \mid M^* \prec H_\theta\}$. By (2) we have a club C in $[H_{\kappa^+}]^\omega$ as such. Let $E = \{N^* \prec H_\theta \mid N^* \cap H_{\kappa^+} \in C\}$. This E is a club in $[H_\theta]^\omega$.

Claim. If $N^* \in E$, then there is M^* s.t. $N^* \subseteq_{\omega_1} M^* \prec H_\theta$ and $M^* \cap \kappa \in S(f,\kappa)$.

Proof. Suppose $N^* \in E$. Then $N = N^* \cap H_{\kappa^+} \in C$. So there is $M \in D$ s.t. $N \subseteq_{\omega_1} M$ and $M \cap \kappa \in S(f,\kappa)$. Let

$$M^* = \{g(s) \mid g \in N^*,\ g : {}^{<\omega}\kappa \longrightarrow H_\theta,\ s \in {}^{<\omega}(M \cap \kappa)\}.$$

Since every function from ${}^{<\omega}\kappa$ to ω_1 is in H_{κ^+}, we may check that

- $N^* \subseteq_{\omega_1} M^* \prec H_\theta$.
- $M \cap \kappa \subseteq M^*$.
- $N^* \cap \omega_1 = N \cap \omega_1 = M \cap \omega_1 = M^* \cap \omega_1$.

And so

- $M^* \cap \kappa \in S(f,\kappa)$.

□

Now for any $N^* \in E$ with $Q = Q(f,\kappa) \in N^*$ and $p \in N^* \cap Q$, we take M^* as claimed. We want to find $q \le p$ s.t. q is (Q, N^*)-semi-generic. To this end we may take any (Q, M^*)-generic sequence $\langle p_n \mid n < \omega\rangle$ with $p_0 \le p$. Then by the density, we know

- $N^* \cap \omega_1 = M^* \cap \omega_1 = \sup\{\alpha^{p_n} \mid n < \omega\}$.
- $\kappa \cap M^* = \bigcup\{X^{p_n}_{\alpha^{p_n}} \mid n < \omega\}$.

Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M^* \cap \omega_1, M^* \cap \kappa)\}$. Then $q \le p$ is (Q, M^*)-generic and so q is (Q, N^*)-semi-generic. Since E is a club, we are done.

□

7.2 Note. The Weak Reflection Principle implies a statement with similar form as (3) at ω_2 (see p.669 in [W]). Namely, suppose for any stationary $S \subseteq [\omega_2]^\omega$, there is α with $\omega_1 < \alpha < \omega_2$ s.t. $S \cap [\alpha]^\omega$ is stationary. Then for any club $D \subseteq [\omega_2]^\omega$, there is a club $C \subseteq [\omega_2]^\omega$ s.t. for any $X \in C$, there is $Y \in D$ s.t. $X \subseteq_{\omega_1} Y$ and $X \neq Y$.

7.3 Corollary. *(1) If κ is measurable, then $Q(f,\kappa)$ is semiproper for all f.*

(2) If the Strong Chang's Conjecture holds, then $Q(f,\omega_2)$ is semiproper for all f.

Proof. Let us recall the Strong Chang's Conjecture. For all sufficiently large regular cardinals θ and any countable elementary substructure N of H_θ, there is a countable elementary substructure M of H_θ s.t. $N \subseteq_{\omega_1} M$ and $N \cap \omega_2 \neq M \cap \omega_2$. In either case, we know that countable elementary substructures are streched while preserving the intersection with ω_1.

□

7.4 Note. (1) The Reflection Principle implies The Strong Chang's Conjecture (p.59 in [B]).

(2) If we Levy collapse a measurable cardinal, then The Strong Chang's Conjecture holds (p.603 and also see p.615 in [S]).

§8. $Q(f,\kappa)$ May Be Proper

We now finish the picture at the highest level.

8.1 Proposition. *The following are equivalent.*

(1) $Q(f,\kappa)$ is proper.

(2) For all regular cardinals $\theta \geq \mid TC(Q(f,\kappa)) \mid^+$ and all countable elementary substructures N^* of H_θ with $\kappa, f \in N^*$, we have $f(N^* \cap \omega_1) <$ o.t.$(N^* \cap \kappa)$.

(3) $S(f,\kappa)$ contains a club.

Proof. (1) implies (2): Let θ be a regular cardinal with $\theta \geq\mid \mathrm{TC}(Q(f,\kappa)) \mid^+$. Take any countable elementary substructure $N^* \prec H_\theta$ with $\kappa, f \in N^*$. Notice that we have $Q(f,\kappa) \in N^*$. We must observe $f(N^* \cap \omega_1) <$ o.t.$(N^* \cap \kappa)$. To do so we may take any $q \in Q(f,\kappa)$ which is $(Q(f,\kappa), N^*)$-generic and any $Q(f,\kappa)$-generic filter G with $q \in G$. Let $\bigcup G = \langle X_i \mid i < \omega_1 \rangle \in N^*[G]$. Notice that $X_{N^*[G]\cap\omega_1} = N^*[G] \cap \kappa$ holds. Since q is $(Q(f,\kappa), N^*)$-generic, we have $N^* \cap \kappa = N^*[G] \cap \kappa$. Since $f(N^* \cap \omega_1) <$ o.t.$(X_{N^*\cap\omega_1})$ and $X_{N^*\cap\omega_1} = N^* \cap \kappa$, we are done.

(2) implies (3): Take a sufficiently large regular cardinal θ and a club $D \subseteq \{N^* \cap \kappa \mid \kappa, f \in N^* \prec H_\theta\}$. By (2), if $X \in D$, then we have $f(X \cap \omega_1) <$ o.t.(X).

(3) implies (1): Let θ be a sufficiently large regular cardinal. Suppose N^* is a countable elementary substructure of H_θ s.t. $\kappa, f \in N^*$. Notice that we have $Q(f,\kappa), S(f,\kappa) \in N^*$. Since $S(f,\kappa)$ contains a club, we have $N^* \cap \kappa \in S(f,\kappa)$. Hence $f(N^* \cap \omega_1) <$ o.t.$(N^* \cap \kappa)$. Now take any $p \in Q(f,\kappa) \cap N^*$ and construct any $(Q(f,\kappa), N^*)$-generic sequence below p. The sequence has a lower bound. The bound is a $(Q(f,\kappa), N^*)$-generic condition. Since there are club many N^*, we conclude $Q(f,\kappa)$ is proper.

□

In the following, the equivalences (2)-(5) are due to [Y].

8.2 Theorem. *The following are equivalent.*

(1) For any $f : \omega_1 \longrightarrow \omega_1$, $Q(f,\omega_2)$ is proper.

(2) CB holds.

(3) For any $f : \omega_1 \longrightarrow \omega_1$, $\{X \in [\omega_2]^\omega \mid f(X \cap \omega_1) <$ o.t.$(X)\}$ contains a club.

(4) For any club $C \subseteq \omega_1$, $\{X \in [\omega_2]^\omega \mid$ o.t.$(X) \in C\}$ contains a club.

(5) For any club $C \subseteq \omega_1$, there is γ with $\omega_1 < \gamma < \omega_2$ and a sequence of continuously increasing countable subsets $\langle X_i \mid i < \omega_1 \rangle$ of γ s.t. for all $i < \omega_1$, we have o.t.$(X_i) \in C$.

Proof. We know (1) iff (3).

(3) implies (2): Let $f : \omega_1 \longrightarrow \omega_1$. Let $g(i) = \sup\{f(j) \mid j \leq i\}$. Take a sufficiently large regular cardinal θ. We then choose a continously increasing countable elementary substructures $\langle N_i \mid i < \omega_1 \rangle$ s.t.

- $N_i \prec H_\theta$.
- $f(i) \leq g(i) \leq g(N_i \cap \omega_1) <$ o.t.$(N_i \cap \omega_2)$.

Let $X_i = N_i \cap \omega_2$ for each $i < \omega_1$. Since $\bigcup\{N_i \cap \omega_1 \mid i < \omega_1\} = \omega_1$, we have $\bigcup\{N_i \cap \omega_2 \mid i < \omega_1\} = \gamma < \omega_2$ for some γ. Hence this γ and the X_i's work.

(2) implies (1): Let θ be a sufficiently large regular cardinal. Let N be a countable elementary substructure of H_θ with $Q(f,\omega_2) \in N$. Since we assume CB, we may assume that there is γ, $\langle X_i \mid i < \omega_1 \rangle \in N$

s.t. for all $i < \omega_1$ $f(i) <$ o.t.(X_i). Let $p \in Q(f, \omega_2) \cap N$. Since $f(N \cap \omega_1) <$ o.t.$(X_{N\cap\omega_1})$ and $X_{N\cap\omega_1} = \bigcup\{X_i \mid i < N \cap \omega_1\} \subseteq N \cap \gamma \subset N \cap \omega_2$, we have $f(N \cap \omega_1) <$ o.t.$(N \cap \omega_2)$. Hence any $(Q(f, \omega_2), N)$-generic sequence below p would have a lower bound. And any lower boud $q \leq p$ would be $(Q(f, \omega_2), N)$-generic.

(3) implies (4): Let θ be a sufficiently large regular cardinal. By (3) we know that if N is countable and $N \prec H_\theta$, then for any $f \in {}^{\omega_1}\omega_1 \cap N$, we have $f(N \cap \omega_1) <$ o.t.$(N \cap \omega_2)$. Therefore if $M = \{f(N \cap \omega_1) \mid f \in N\}$, then o.t.$(N \cap \omega_2) = M \cap \omega_1$ holds. Here $\kappa = \omega_2$ is crutial in $Q(f, \kappa)$. Let $C \subseteq \omega_1$ be a club. If $C \in N \subset M$, then $M \cap \omega_1 \in C$ and so o.t.$(N \cap \omega_2) \in C$. Therefore $\{X \in [\omega_2]^\omega \mid$ o.t.$(X) \in C\}$ contains a club induced by the $N \cap \omega_2$'s.

(4) implies (3): Given $f : \omega_1 \longrightarrow \omega_1$, let $C = \{i < \omega_1 \mid \forall j < i \ f(j) < i\}$. Then C is a club. By (4) $\{X \in [\omega_2]^\omega \mid$ o.t.$(X) \in C\}$ contains a club. But if o.t.$(X) \in C$, $X \cap \omega_1 < \omega_1$ and $\omega_1 \in X$, then $X \cap \omega_1 <$ o.t.(X) and so $f(X \cap \omega_1) <$ o.t.(X).

(4) implies (5): Let θ be a sufficiently large regular cardinal. Let $C \subseteq \omega_1$ be a club. If N is countable and $C \in N \prec H_\theta$, then we may assume that o.t.$(N \cap \omega_2) \in C$. Now start to construct a sequence of countable elementary substructures $\langle N_i \mid i < \omega_1\rangle$ of H_θ with $C \in N_0$. Since $\bigcup\{N_i \cap \omega_1 \mid i < \omega_1\} = \omega_1$, we have $\bigcup\{N_i \cap \omega_2 \mid i < \omega_1\} = \gamma < \omega_2$ for some γ. Let $X_i = N_i \cap \omega_2$. Then the X_i's and γ work.

(5) implies (2): Let $f : \omega_1 \longrightarrow \omega_1$. We consider $g(i) = \sup\{f(j) \mid j \leq i\}$ and $C = \{i < \omega_1 \mid \forall j < i \ g(j) < i\}$ which is a club. By (5), we have γ and $\langle X_i \mid i < \omega_1\rangle$ s.t. for all $i < \omega_1$ o.t.$(X_i) \in C$. Since $\bigcup\{X_i \mid i < \omega_1\} = \gamma$ and $\omega_1 < \gamma$, $D = \{i < \omega_1 \mid \omega_1 \in X_i$ and $X_i \cap \omega_1 = i\}$ is a club. Let $\langle i_\alpha \mid \alpha < \omega_1\rangle$ enumerate D. For $\alpha < \omega_1$, we have $\alpha \leq i_\alpha$ and $i_\alpha = X_{i_\alpha} \cap \omega_1 <$ o.t.(X_{i_α}). So we have $f(\alpha) \leq g(\alpha) \leq g(i_\alpha) = g(X_{i_\alpha} \cap \omega_1) <$ o.t.(X_{i_α}). Hence γ and $\langle X_{i_\alpha} \mid \alpha < \omega_1\rangle$ work.

□

8.3 Corollary. *The following is consistent. CB holds and $\square_{\omega_1}$ holds. Hence $Q(f, \omega_2)$ are all proper (and so semiproper), yet the Chang's Conjecture (and so the Strong Chang's Conjecture) fails.*

Proof. Force $\square_{\omega_1}$ with the initial segments over any model where CB holds. Since the notion of forcing is ω_2-Baire, CB gets preserved.

□

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