A Continuous-Time Seat Allocation Model with Up-Down Resets

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1 Introduction

Most static revenue management models attempt to maximize the expected revenue in a single-period seat inventory model, when the booking limit for low fare demands is fixed. In these models, once the booking procedure has been stopped, then it is never reopened. On the other hand, there are several papers discussing the dynamic airline seat inventory control. Most of them are related to discrete time (see [3]).

In this thesis we consider a seat inventory model. It differs from the existing literature in a sense that the planning horizon is continuous. We assume two fare classes for a set of identical seats. Furthermore, booking limits for the fare class can be reset to upward or downward, depending on whether the amount of high fare demands is large enough or not after a certain period of time.

In section 2 we develop a general framework of the continuous-time seat inventory model and then derive the expected revenue function under such booking policies. In section 3 we develop the reset model to incorporate cancellations with and without the callable property. In section 4 we explore some properties of value functions by using numerical examples.

2 The Model of Seat Allocation Problems

Let T be the time epoch at which sale ends. Let X_t and Y_t be the number of high fare and low fare demands at time t, respectively, which satisfy

$$dX_t = \mu_1 dt + \sigma_1 dZ_1(t), 0 \le t \le T \tag{1}$$

and

$$dY_t = \mu_2 dt + \sigma_2 dZ_2(t), 0 \le t \le T, \tag{2}$$

where $Z_1(\cdot)$ and $Z_2(\cdot)$ are independent and standard Brownian motions. Also, we assume that $X_0 = Y_0 = 0$. Note that $E[X_t] = \mu_1 t$ and $Var[X_t] = \sigma_1^2 t$. Negative values of demand can be treated as cancellations.

The cumulative demands $\overline{X}^{t_2-t_1}$ from time t_1 to t_2 is given by

$$\overline{X}^{t_2-t_1} = \int_{t_1}^{t_2} X_s ds, \quad 0 \le t_1 \le t_2 \le T$$

and

$$\overline{Y}^{t_2-t_1} = \int_{t_1}^{t_2} Y_s ds, \quad 0 \le t_1 \le t_2 \le T.$$

$\overline{X}^{t_2-t_1}$ is normally distributed with mean $\mu_1(t_2^2 - t_1^2)/2$ and variance $\sigma_1^2(t_2^3 - 3t_1^2 t_2 + 2t_1^3)/3$. The expected value and variance of Y have much the same as that of X.

Furthermore, let C be the total number of seats and L the initial booking limit for low fare. Let p and q be the high fare and the low fare, respectively and t_0 be the reset time.

The class of booking policies is restricted within the narrow limits satisfying the following procedure:

(i) Choose L, α, β and γ satisfying $0 \le L \le C$, $\gamma \le 1$, $\alpha \le 1$ and $1 \le \beta \le \{C - \gamma(C-L)\}/L$.

(ii) Observe the cumulative booking requests from the high fare $\overline{X}^{t_0}$ at the reset time t_0, $0 \le t_0 \le T$.

(iii) Reset the initial limit, according to the following two cases;

Case 1
If $\overline{X}^{t_0} \ge \gamma(C-L)$, then the initial booking limit L should be reset downward to the level αL for $\alpha \le 1$.

Case 2
If $\overline{X}^{t_0} < \gamma(C-L)$, then the initial booking limit L should be reset upward to the level βL for $1 \le \beta \le \{C - \gamma(C-L)\}/L$.

Let $v_1(L, \overline{X}, \overline{Y})$ be the total revenue from the booking policy for case 1 and $v_2(L, \overline{X}, \overline{Y})$ the total revenue for case 2. Then, we have

$$\begin{aligned} &v_1(L, \overline{X}, \overline{Y}) \\ &= q \min\{M_Y(L) + \overline{Y}^{T-t_0}, \beta L\} \\ &+ p \min\{\overline{X}^{t_0} + \overline{X}^{T-t_0}, C - \beta L \wedge (M_Y(L) + \overline{Y}^{T-t_0})\} \end{aligned} \tag{3}$$

and

$$\begin{aligned} &v_2(L, \overline{X}, \overline{Y}) \\ &= q \min\{M_Y(\alpha L) + \overline{Y}^{T-t_0}, \alpha L\} \\ &+ p \min\{\overline{X}^{t_0} + \overline{X}^{T-t_0}, C - \alpha L \wedge (M_Y(\alpha L) + \overline{Y}^{T-t_0})\}, \end{aligned} \tag{4}$$

where $M_Y(L) = \min\{\overline{Y}^{t_0}, L\}$ and $M_Y(\alpha L) = \min\{\overline{Y}^{t_0}, \alpha L\}$.

The expected total revenue $V(L)$ over the time interval $[0,T]$ is given by

$$V(L) = E\left[v_1(L,\overline{X},\overline{Y})1_{\{\overline{X}^{t_0}<\gamma(C-L)\}} + v_2(L,\overline{X},\overline{Y})1_{\{\overline{X}^{t_0}\geq\gamma(C-L)\}}\right].$$

2.1 Special Cases

Instead of exploring some conditions for concavity of V in L, we consider three special cases by adding special conditions to demand as follows;

a. A case of low fare demands large enough

b. A case when low fare demands are accepted until the reset time

c. A case when low fare demands are upgradable to high fare

2.1.1 A Case of Low Fare Demands Large Enough

Suppose that low fare demands are much bigger than the booking limit for low fare demands. Let $F_{\overline{X}^{t_0}}$ and $F_{\overline{X}^{T-t_0}}$ be the probability distribution of cumulative high fare demands at the reset time t_0 and the departure time T, respectively. The equations (3) and (4) can be rewritten as

$$v_1(L,\overline{X}) = q\beta L + p\min\{\overline{X}^{T-t_0} + \overline{X}^{t_0}, C-\beta L\}$$

and

$$v_2(L,\overline{X}) = q\alpha L + p\min\{\overline{X}^{T-t_0} + \overline{X}^{t_0}, C-\alpha L\}.$$

We have the expected revenue $V(L)$ as follows;

$$V(L) = pC - \int_0^{\gamma(C-L)} T_{x_1}(\beta L)dF_{\overline{X}^{t_0}}(x_1) - \int_{\gamma(C-L)}^{\infty} T_{x_1}(\alpha L)dF_{\overline{X}^{t_0}}(x_1),$$

where

$$T_{x_1}(s) = (p-q)s + p\int_0^{C-s-x_1} F_{\overline{X}^{T-t_0}}(x_2)dx_2.$$

Remark 1 *Both $E[v_1(L,\overline{X})]$ and $E[v_2(L,\overline{X})]$ are concave in L but $V(L)$ is not necessarily concave in L.*

We now investigate the booking limit L^* maximizing the expected revenue $V(L)$. Differentiating $V(L)$ with respect to L, we obtain

$$\begin{aligned}
&\gamma f_{\overline{X}^{t_0}}(\gamma(C-L))\{T_{\gamma(C-L)}(\alpha L) - T_{\gamma(C-L)}(\beta L)\} \\
&= \beta p P[\overline{X}^{t_0} + \overline{X}^{T-t_0} > C - \beta L \mid \overline{X}^{t_0} \leq \gamma(C-L)] \\
&+ \alpha p P[\overline{X}^{t_0} + \overline{X}^{T-t_0} > C - \alpha L \mid \overline{X}^{t_0} > \gamma(C-L)] \\
&- q\{\beta F_{\overline{X}^{t_0}}(\gamma(C-L)) + \alpha \overline{F}_{\overline{X}^{t_0}}(\gamma(C-L))\}. \quad (5)
\end{aligned}$$

The optimal booking limit L^* satisfying equation (5) can not be unique. Note that the model can be reduced to the traditional model when we assume $t_0 = T$ and $\alpha = \beta = \gamma = 1$. In this case, the optimal booking limit $\hat{L}$ can be rewritten as

$$\hat{L} = \max\left\{0 \leq L \leq C : \Pr[\overline{X}^T \leq C - L] \geq \frac{q}{p}\right\}.$$

2.1.2 Spill Rates

There are two possible interpretations of the term "spill rate" in the airline context. The first is that the spill rate is the expected proportion of flights on which some high fare reservations must be refused because of low fare bookings, called the *flight spill rate* S_1. The second is that the spill rate is the expected proportion of high fare reservations that must be refused out of the total number of such reservations, called the *passenger spill rate* S_2. It seems that the second be more meaningful since it relates more closely to the amount of high fare revenue lost.

3 A Seat Inventory Control of Callable Seats with Up-Down Resets.

In this section we treat with callable products in the airlines seat inventory model in which the airlines is allowed to cancel the booking seats by paying some compensation to the passengers booked in advance.

3.1 Optimal Seat Allocation with Cancellation

In the model with reset downward, we assumed that the airlines pays no penalty cost for low fare passengers denied for booking. In this section, we relax this assumption. Let h denote the compensation cost $(h > q)$ due to boarding refused. The low fare passenger may be canceled at t_0 and then may be paid the compensation. Hence, we have the total revenue in reset downward as follows;

$$\begin{aligned}
&\overline{v}_2(L,\overline{X},\overline{Y}) \\
&= v_2(L,\overline{X},\overline{Y}) - (h-q)(M_Y(L) - \alpha L)^+. \quad (6)
\end{aligned}$$

The total revenue in reset upward is the same form of equation (3), that is,

$$\overline{v}_1(L,\overline{X},\overline{Y}) = v_1(L,\overline{X},\overline{Y}).$$

Defining $\overline{V}(L)$ as the expected total revenue with cancellation over the time interval $[0,T]$, $\overline{V}(L)$ can be given by

$$\overline{V}(L) = E\Big[\overline{v}_1(L,\overline{X},\overline{Y})1_{\{\overline{X}^{t_0}<\gamma(C-L)\}} + \overline{v}_2(L,\overline{X},\overline{Y})1_{\{\overline{X}^{t_0}\geq\gamma(C-L)\}}\Big]. \quad (7)$$

3.2 The Model with Callable Property

In this section, we consider another type of low fare tickets that has a callable property to reduce the cost of compensation. Low fare passengers who agree to grant the call option are paid a prespecified recall price if their seats are recalled. We call those passengers "callable passengers". This callable property is introduced by Gallego et al. [2]. There are three types of tickets.

- Type 1 : High fare ticket
  This type ticket is not recalled by the airlines. The fare p is the highest among the tickets.
- Type 2 : Low fare ticket with a callable property
  This ticket has a priority to be recalled for reset downward and the airlines pays the recall price d, $q \leq d \leq p$ to the customer.
- Type 3 : Low fare ticket with no compensation
  When the booking limit is reset downward, this ticket is recalled after the booking of all callable passengers are canceled. And they receive the recall price q. Therefore, there is no compensation since the ticket price is equal to the recall price.

Thus, the tickets 2 and 3 share the same property in the case of reset upward. To count the number of the callable passengers, we define

$$D_i = \begin{cases} 1 & \text{if the } i\text{th customer makes the decision to grant the call,} \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $\{D_1, D_2, \cdots\}$ are independent and identically distributed with mean $ED_i = \delta$ the probability of granting the call to the airlines. If B seats are booked, then $H(B) = \sum_{i=1}^{B} D_i$ seats are confirmed and is binomially distributed with mean $B\delta$.

In this case a sequence of operations occurs as follows:

(i) Choose L, α, β, γ, and announce the recall price d.

(ii) Observe the cumulative booking requests from the high fare $\overline{X}^{t_0}$ at the reset time t_0,

  (a) If $\overline{X}^{t_0} \geq \gamma(C-L)$, then L should be reset downward by recalling the tickets. And pay the compensation to the number of passengers given by $\min\{H(\overline{Y}^{t_0} \wedge L), \alpha L\}$ passengers.

  (b) If $\overline{X}^{t_0} < \gamma(C-L)$, then L should be reset upward. The option is not exercised.

If $\overline{X}^{t_0} < \gamma(C-L)$, the total revenue $\tilde{v}_1(L,\overline{X},\overline{Y})$ takes the same form of (3);

$$\tilde{v}_1(L,\overline{X},\overline{Y}) = v_1(L,\overline{X},\overline{Y}). \quad (8)$$

If $\overline{X}^{t_0} \geq \gamma(C-L)$, we have

$$\begin{aligned}\tilde{v}_2(L,\overline{X},\overline{Y}) &= v_2(L,\overline{X},\overline{Y}) \\ &\quad -(d-q)\min\{H(M_Y(L)), (M_Y(L)-\alpha L)^+\}. \quad (9)\end{aligned}$$

When $d = h$ and $\delta = 1$ in equation (9), we get equation (6). The expected total revenue $\tilde{V}(L)$ is given by

$$\tilde{V}(L) = E\Big[\tilde{v}_1(L,\overline{X},\overline{Y})1_{\{\overline{X}^{t_0}<\gamma(C-L)\}} + \tilde{v}_2(L,\overline{X},\overline{Y})1_{\{\overline{X}^{t_0}\geq\gamma(C-L)\}}\Big].$$

These arguments lead to the following result.

Lemma 2 *If $h \geq d$, the revenue realized with callable ticket is greater than or equal to the corresponding revenue without the callable ticket for any feasible values of L. That is,*

$$\tilde{V}(L) = \overline{V}(L) + R(L)$$

and

$$\begin{aligned}R(L) &= \overline{F}_{\overline{X}^{t_0}}(x_1)E[(h-d)(M_Y(L)-\alpha L)^+ \\ &\quad +(d-q)\{(M_Y(L)-\alpha L)^+ - H(M_Y(L))\}^+] \\ &\geq 0.\end{aligned}$$

3.3 A Case of Low Fare Demands Large Enough with Callable Property

We use the normal approximation to the binomial distribution $H(B)$ to compute simply. Thus, $H(B)$ is normal distribution with mean δB and variance $\delta(1-\delta)B$. Let Z denote the probability distribution. The equations (8) and (9) can be rewritten as

$$\tilde{v}_1(L,\overline{X}) = q\beta L + p\min(\overline{X}^{T-t_0} + \overline{X}^{t_0}, C-\beta L)$$

and

$$\tilde{v}_2(L,\overline{X}) = q\alpha L + p\min(\overline{X}^{T-t_0} + \overline{X}^{t_0}, C-\alpha L) - (d-q)\min\{H(L), L-\alpha L\}.$$

We have the expected revenue $\tilde{V}(L)$ as follows;

$$\tilde{V}(L) = V(L) - (d-q)\overline{F}_{\overline{X}^{t_0}}(\gamma(C-L)) \times \left\{(1-\alpha)L - \int_0^{L-\alpha L} Z(z)dz\right\}. \quad (10)$$

4 Numerical Examples

In this section, we present numerical results of optimal booking policies in several examples.

4.1 Results of Section 2

Suppose that the high fare demand process is given by equation (1). We assume that the airplane has the capacity $C = 300$ and we fix $q = 100$, $t_0 = 90$, $T = 120$, $\alpha = 0.9$, $\beta = 1.1$, $\gamma = 0.4$ and $p = 350$. In Table 1, we compare the revenue function of our model with the one of the classical model ($\alpha = \beta = \gamma = 1$, $t_0 = T$).

4.2 Results of Section 3

Let $\mu_1^{t_0}$ and $\sigma_1^{t_0}$ be the expected value and standard deviation of high fare demands in $[0, t_0]$, and μ_1^T and σ_1^T be the expected value and standard deviation of high fare demands in $[t_0, T]$. If low fare passenger's reservation price P_L is uniformly distributed between $[q, p]$, the probability δ is

$$\delta = \left(\frac{d-q}{p-q}\right)^+.$$

Suppose that the model parameters are given by $C = 300$, $q = 100$, $t_0 = 90$, $T = 120$, $\alpha = 0.9$, $\beta = 1.1$, $\gamma = 0.3$, $h = 150$, $p = 350$, and $d = 150$. In Table 2, we compare the revenue function of callable model (10) with the cancellation model (7).

5 Conclusion

In this thesis, we have formulated the seat allocation model with upward-downward resets for the initial booking limit. Future research in this area could be a dynamic model in which there are n times of the reset opportunities available.

Table 1: The impact of revenue performance in section 2. (C : the classical model, R : the reset model, D1 : (μ_1,σ_1)=(0.004, 0.02), D2 : (μ_1, σ_1)=(0.01, 0.04), D3 : (μ_1, σ_1)=(0.016, 0.07)).

X_t	C R	$\lfloor\hat{L}\rfloor$ $\lfloor L^*\rfloor$	$V(\hat{L})$ $V(L^*)$	%	$\hat{S}_1$ S_1	$\hat{S}_2$ S_2
D1	C	263	35,452.74		29.5	9.8
	R	243	35,447.14	0.00	14.8	1.1
D2	C	211	44,419.82		28.8	7.6
	R	224	45,819.43	3.15	25.1	2.9
D3	C	155	52,579.02		28.7	8.3
	R	173	55,115.03	4.82	27.6	5.0

Table 2: The impact of revenue performance in section 3 and comparison of models. (R : the reset model, CN : with cancellation model, CL : with callable model, D1:{($\mu_1^{t_0}$, $\sigma_1^{t_0}$)=(0.003, 0.01), (μ_2^T, σ_2^T)=(0.008, 0.03)}, D2:{($\mu_1^{t_0}$, $\sigma_1^{t_0}$)=(0.007, 0.02), (μ_1^T, σ_1^T)=(0.016, 0.1)}, D3:{($\mu_1^{t_0}$, $\sigma_1^{t_0}$)=(0.011, 0.03), (μ_1^T, σ_1^T)=(0.024, 0.17)}, OBL : Optimal booking limit).

X_t	Case	OBL	Expected Revenue	%
	R	233	37,980.66	
D1	CN	232	37,921.20	
	CL	232	37,926.03	0.01
	R	194	46,004.06	
D2	CN	188	45,694.43	
	CL	193	45,900.25	0.45
	R	150	54,412.58	
D3	CN	144	54,074.00	
	CL	150	54,516.60	0.82

References

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