Robust Control of Active Suspension to Improve Ride Comfort with Structural Constraints

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1 Introduction

In this study, the robust $H_2$ controller for the active suspension is proposed. The purpose of the study is to improve the ride comfort, satisfying the structural constraints. Based on ISO 2631[1], the frequency bands 4-8[Hz] for the vertical acceleration and 0.63-0.8[Hz] for the pitch angular acceleration are suppressed by using frequency shaping. They are constraints on the vertical force of the wheel, the suspension stroke constraint, and the control input constraints are considered. In addition, the robust stability for the perturbation of the front and rear wheel positions is guaranteed. The robust stability for the perturbation of the ride comfort, the transfer functions, $W_1(s)$, $W_2(s)$, and $W_3(s)$, are introduced for the frequency filter of the body acceleration $\ddot{x}_f$ and $\ddot{x}_r$, and the pitch acceleration $\ddot{\theta}$, respectively. The gain diagram of $W_1(s)$, $W_2(s)$, and $W_3(s)$ are shown in Figure 2. Then, the new state space equations of the front wheel position $x_{1f}$, the rear wheel position $x_{1r}$, the car body position $x_{2}$, and the pitch angle $\theta$ are derived as (1) to (3), respectively ($k = f, r$).

$$M_{1k}\ddot{x}_{1k} = k_{2k}(x_{2k} - x_{1k}) + c_{2k}(\dot{x}_{2k} - \dot{x}_{1k}) - k_{1k}(\ddot{x}_{1k} - \dot{x}_{0k}) - c_{1k}(\dot{x}_{1k} - \dot{x}_{0k}) - F_k$$

$$(M_f + M_{2f})\ddot{x}_f = -k_f(x_{1f} - x_{1f}) - c_f(\dot{x}_{2f} - \dot{x}_{1f}) - k_r(x_{2r} - x_{1r}) - c_r(\dot{x}_{2r} - \dot{x}_{1r}) + F_f + F_r$$

$$(M_f M_{2f})\ddot{\theta} = -k_f M_{2f}(x_{2f} - x_{1f}) - c_f M_{2f}(\dot{x}_{2f} - \dot{x}_{1f}) + k_r M_{2f}(x_{2r} - x_{1r}) + c_r M_{2f}(\dot{x}_{2r} - \dot{x}_{1r}) + M_r F_f - M_{2f} F_r$$

The motion equations of the front car body position $x_{2f}$, the rear car body position $x_{2r}$, and the disturbance vector $w(t)$ are defined as (5) to (7).

$$\begin{align*}
x(t) &= \begin{bmatrix} x_{2f} - x_{1f} & x_{2r} - x_{1r} & x_{1f} - x_{0f} & x_{1r} - x_{0r} \\
\end{bmatrix}^T \\
y(t) &= \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \\
\end{bmatrix} = \begin{bmatrix} \ddot{x}_{2f} & \ddot{x}_{2r} & \ddot{\theta} \end{bmatrix}^T \\
w(t) &= \begin{bmatrix} \dot{x}_{0f} & \dot{x}_{0r} \end{bmatrix}^T, u(t) = [F_f, F_r]^T
\end{align*}$$

2 Modeling

The half car model is shown in Figure 1. The motion equations of the front wheel position $x_{1f}$, the car body position $x_{2}$, and the pitch angle $\theta$ are derived as (1) to (3), respectively ($k = f, r$).

$$M_{1k}\ddot{x}_{1k} = k_{2k}(x_{2k} - x_{1k}) + c_{2k}(\dot{x}_{2k} - \dot{x}_{1k}) - k_{1k}(\ddot{x}_{1k} - \dot{x}_{0k}) - c_{1k}(\dot{x}_{1k} - \dot{x}_{0k}) - F_k$$

$$(M_f + M_{2f})\ddot{x}_f = -k_f(x_{1f} - x_{1f}) - c_f(\dot{x}_{2f} - \dot{x}_{1f}) - k_r(x_{2r} - x_{1r}) - c_r(\dot{x}_{2r} - \dot{x}_{1r}) + F_f + F_r$$

$$(M_f M_{2f})\ddot{\theta} = -k_f M_{2f}(x_{2f} - x_{1f}) - c_f M_{2f}(\dot{x}_{2f} - \dot{x}_{1f}) + k_r M_{2f}(x_{2r} - x_{1r}) + c_r M_{2f}(\dot{x}_{2r} - \dot{x}_{1r}) + M_r F_f - M_{2f} F_r$$

The state vector $x(t)$, the output vector $y(t)$, and the disturbance vector $w(t)$ are defined as (5) to (7).

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3 Controller synthesis

3.1 Frequency filter

The frequency filters for the body acceleration and the pitch angular acceleration are designed to improve the ride comfort. The transfer functions, $W_1(s)$, $W_2(s)$, and $W_3(s)$, are introduced for the frequency filter of the body acceleration $\ddot{x}_f$ and $\ddot{x}_r$, and the pitch acceleration $\ddot{\theta}$, respectively. The gain diagram of $W_1(s)$, $W_2(s)$, and $W_3(s)$ are shown in Figure 2. Then, the new state space equations of the front wheel position $x_{1f}$, the rear wheel position $x_{1r}$, the car body position $x_{2}$, and the pitch angle $\theta$ are derived as (1) to (3), respectively ($k = f, r$).

$$(M_f M_{2f})\ddot{\theta} = -k_f M_{2f}(x_{2f} - x_{1f}) - c_f M_{2f}(\dot{x}_{2f} - \dot{x}_{1f}) + k_r M_{2f}(x_{2r} - x_{1r}) + c_r M_{2f}(\dot{x}_{2r} - \dot{x}_{1r}) + M_r F_f - M_{2f} F_r$$

The motion equations of the front car body position $x_{2f}$, the rear car body position $x_{2r}$, and the disturbance vector $w(t)$ are defined as (5) to (7).

$$\begin{align*}
x(t) &= \begin{bmatrix} x_{2f} - x_{1f} & x_{2r} - x_{1r} & x_{1f} - x_{0f} & x_{1r} - x_{0r} \\
\end{bmatrix}^T \\
y(t) &= \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \\
\end{bmatrix} = \begin{bmatrix} \ddot{x}_{2f} & \ddot{x}_{2r} & \ddot{\theta} \end{bmatrix}^T \\
w(t) &= \begin{bmatrix} \dot{x}_{0f} & \dot{x}_{0r} \end{bmatrix}^T, u(t) = [F_f, F_r]^T
\end{align*}$$
To prevent the suspension bottoming out, the constraints are given as (11) by the constraints of suspension strokes $S_f$ and $S_r$.

\[
\begin{align*}
|x_2f - x_{1f}| &< S_f \\
|x_2r - x_{1r}| &< S_r 
\end{align*} \tag{11}
\]

The evaluation output $z_2(t)$ is derived by using (10) and (11) as follows.

\[
z_2(t) = C_{22}x(t)
= \left[ \begin{array}{c} \frac{x_{2z} - x_{1z}}{S_f} \\
\frac{x_{2r} - x_{1r}}{S_r} \\
\frac{k_1(x_{1z} - x_{20})}{(M_2z + M_1f)y} \\
\frac{k_1(x_{1r} - x_{20})}{(M_2r + M_1r)y} \end{array} \right]^T \tag{12}
\]

To prevent the saturation of the control inputs, the constraints are given as (13) by defining the constraint of input voltage as $U_{max_f}$ and $U_{max_r}$.

\[
\begin{align*}
|Ff| &< U_{max_f} \\
|Fr| &< U_{max_r} \tag{13}
\end{align*}
\]

### 3.3 Robust $H_2$ controller

The robust stability for the perturbation of the car body weight $M_{2f}$ and $M_{2r}$, are considered. However, there are the nonlinear terms of the uncertain parameters $M_{2f}$ and $M_{2r}$ in matrices $\hat{A}_f$, $\hat{B}_w$, $C_{21}$, $D_{21}$, and $C_{22}$. The nonlinear terms of the matrices are transformed by introducing $\alpha_f$, $\beta_f$, $\alpha_r$, and $\beta_r$.

\[
\begin{align*}
\alpha_f &= \frac{1}{M_{2f}}, \quad \beta_f = \frac{1}{M_{2f} + M_{1f}} = \frac{\alpha_f}{\alpha_f M_{1f} + 1}, \\
\alpha_r &= \frac{1}{M_{2r}}, \quad \beta_r = \frac{1}{M_{2r} + M_{1r}} = \frac{\alpha_r M_{1r} + 1}{\alpha_r}
\end{align*}
\]

Therefore, the LMI conditions of the Robust $H_2$ controller with constraint are given as Corollary 1[2].

**Corollary 1** If there exist $X$ and $Y$ satisfying following LMI conditions (14) to (17), the closed-loop system is stabilized with satisfying the structural constraints. Also, the upper bound of $H_2$ norm is less than $\gamma$ and the state feedback gain is $K = YX^{-1}$. Here $w_{max}$ is the upper bound of disturbance energy calculated as

\[
\int_0^\infty \| w(t) \|^2 dt.
\]

\[
\begin{align*}
\text{minimize } & \gamma^2 \text{ subject to } \\
\begin{bmatrix} He\{\hat{A}_iX + \hat{B}_{w_j}Y\} & M^T \\
M & -I \end{bmatrix} & < 0 \tag{14} \\
(M_{i,j} = W_z C_{21}zX + W_z D_{21}zY) \\
\begin{bmatrix} Z & \tilde{B}_w^T \\
\tilde{B}_w & X \end{bmatrix} & > 0, \ \text{Trace}(Z) < \gamma^2 \tag{15} \\
\begin{bmatrix} \frac{1}{w_{max}^2} & 0 \\
\frac{1}{w_{max}^2} & C_{22}zX \end{bmatrix}^T & > 0 \tag{16} \\
(i = 1, 2, 3, \ j = 1, 2, 3) \\
\begin{bmatrix} \frac{1}{w_{max}^2} U_{max_f}^2 & Y_1 \\
\frac{1}{w_{max}^2} U_{max_r}^2 & Y_2 \end{bmatrix}^T & X > 0 \tag{17}
\end{align*}
\]

Where $Y_1$ and $Y_2$ are the first row and the second row of the matrix $Y$, respectively.

### 4 Simulation and Experiment

Figure 4 and Figure 5 indicate the gain diagram of the front car body acceleration $\ddot{x}_2$ and the pitch angular acceleration $\ddot{\theta}$ when the front and rear weights of the car body are considered as $M_{2f} = 1.45[kg]$ and $M_{2r} = 1.45[kg]$. As can be seen from the Figure 4, the body acceleration is suppressed to about 82% at 5.61[Hz] by the proposed controller in comparison with the passive suspension. The pitch acceleration is suppressed to about 57% at 0.72[Hz] in Figure 5. It can be said that the ride comfort is improved by the proposed controller. Figure 6 indicates the vertical force of the front wheel and the front suspension stroke($M_{2f} = 1.45[kg]$, $M_{2r} = 1.45[kg]$). As can be seen

![Figure 4 Frequency Response of $\ddot{x}_2$](image)

![Figure 5 Frequency Response of $\ddot{\theta}$](image)

![Figure 6 Suspension Stroke and Vertical Force (Front)](image)